O‘ZBEKISTON RESPUBLIKASI
OLIY VA O‘RTA MAXSUS TA‘LIM VAZIRLIGI

MUXAMMAD AL-XORAZMIY NOMIDAGI
TOSHKENT AXBOROT TEXNOLOGIYALARI
UNIVERSITETI

S. S. SADADDINOVA

CALCULUS

(MATEMATIKA)

O‘zbekiston Respublikasi
Oliy va o‘rta maxsus ta’lim vazirligi tomonidan
darslik sifatida tavsiya etilgan.

1-qism

Toshkent – 2021

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Ushbu darslik texnika, pedagogika, moliya hamda iqtisodiyot oliy o‘quv yurtlarining “Calculus (Matematika)”, “Differensial hisob va uning tatbiqlari”, “Matematik analiz”, “Sonli usullar”, “Iqtisodiyotda matematika” kurslari materiallarini o‘z ichiga oladi. Jumladan, funksiyalar va matematik modellar, limitlar nazariyasi, funksiyaning uzluksizligi, turli xil ko‘rinishdagi funksiyalarning hosilalarini topish, funksiyalarning yuqori tartibli hosila va differensiallari, funksiya asimptotalarini aniqlash, funksiyalarni approksimatsiyalash, optimallashtirish usullari, funksiyalarni integrallash mavzulari keltirilgan. Shuningdek, har bir mavzuning texnikada, iqtisodiyotda, tibbiyotda, ijtimoiy olam va boshqa ko‘plab sohalardagi amaliy tatbiqlari asoslab berilgan.

Kitobda hozirgi zamon hisoblash matematikasi asoslarining yutuqlari o‘z aksini topgan.

Darslik barcha muhandis – texnika va iqtisodiyot bakalavriyat ta’lim yo‘nalishlari, shu jumladan Muhammad al – Xorazmiy nomidagi Toshkent axborot texnologiyalari universiteti va uning 5 ta hududiy filiallari hamda pedagogika universitetlari talabalari uchun mo‘ljallangan, shuningdek, litsey va o‘rta maktabning yuqori sinf o‘quvchilari, kichik ilmiy hodimlar va professor-o‘qituvchilar ham foydalanishlari mumkin.

UO‘K:
KBK:

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SO‘Z BOSHI

Mazkur darslikni yozishda O‘zbekiston Respublikasi Prezidentining “O‘zbekiston Respublikasi oliy ta’lim tizimini 2030 yilgacha rivojlantirish konsepsiyasini tasdiqlash to‘g‘risida” 2019 yil 8 oktabrdagi ПФ-5847 sonli, “2017 - 2021 yillarda O‘zbekiston Respublikasini rivojlantirishning beshta ustuvor yo‘nalishi bo‘yicha Harakatlar strategiyasini «Ilm, ma’rifat va raqamli iqtisodiyotni rivojlantirish yili»da amalga oshirishga oid davlat dasturi to‘g‘risida” 2020 yil 2 martdagi ПФ -5953 sonli farmonlari, 2019 yil 9 iyuldagi matematika ta’limi va fanlarini yanada rivojlantirishga oid qarori hamda O‘zbekiston Respublikasi Prezidenti Administratsiyasining 2020 yil 9 yanvardagi PA1/1-20 sonli topshirig‘ida belgilangan vazifalari asos qilib olingan.

“Calculus (Matematika) 1-qism” deb nomlanuvchi ushbu darslik kredit ta’lim tizimining “Calculus (Matematika)” fan dasturining bir qismini qamrab olgan va bu darslikni hozirgi iqtisodiy sharoitda talabalar uchun amaliy qo‘llanma bo‘ladi, deb ishonamiz. Zarur matematik bilimlarga ega bo‘lgan talaba uchun amaliy masalalarni tushunish qiyinchilik tug‘dirmaydi. Asosiy maqsadimiz talaba uchun faqat matn tushunarli bo‘lib qolmasdan, kitobdan olgan bilimlarini amaliyotda qo‘llay bilishiga erishishdir.

Matematik bilimlar asosan tasavvuriy ko‘rinishda bo‘ladi. Ushbu kitob yordamida talaba o‘rgangan bilimlarini tekshirib ko‘rishi va bu bilan o‘z-o‘zini baholashi, kitobni o‘qib zavq olishi, vatanimizga bo‘lgan muhabbati yanada mustahkamlanishi, yanada ko‘proq ilm izlashga intilishi kerak. Ana shundagina maqsadimizga erishgan bo‘lamiz.

Biz real hayotda uchraydigan misol va masalalarni olishga harakat qildik. Jumladan, iqtisodiy va moliyaviy kattaliklar, ishlab chiqarishdagi o‘sish, sog‘liqni saqlash, ekologik tadqiqotlar, gidrometereologiya ma‘lumotlari asosida, turizm haqida, tarixiy obidalar, kishilar hayoti, psixologiyasi, anatomiyasi, inson miyasining axborotni qabul qilish omillari haqida, ma‘lumotlarni uzatish tizimlari qatnashgan misol va masalalar tuzdik. Bu bilan matematikaning qanchalik qo‘llanilish sohalari keng ekanligini ko‘rsatishga harakat qildik.

Matematika fani bo‘yicha zamonaviy axborot texnologiyalar asosidagi animatsiyalar, grafik materiallar, diagramma va jadvallar yordamida talabaning tasavvurini boyitishga hamda fanga qiziqtirishga urindik.

Bizning uslub: Nostandart yondoshuv. Yangi tushunchani tahlil qilishda, bu tuhunchani ko‘rgazmalar vositasida shunday tasvirladikki, talaba oldingi hosil qilingan malakalariga asoslangan holda isbotlarsiz ham tushuna oladi. Jumladan,

1. Funksiya uzluksizligini tushuntirish uchun mavzugacha turli ko‘rinishdagi funksiyalarning grafiklari, real voqealarning diagrammalarini berib bordikki, funksiya uzluksizligi mavzusiga yetib borganda talabalar funksiyalar uzilishga ega bo‘lishi ham mumkinligini bilib olishgan bo‘lishadi.
2. Funksiya hosilasi ta‘rifini berishda undan oldin o‘rta qiymat tushunchasini kiritdik. Bu yondoshuv hosilani geometrik nuqtai nazardan tushuntirishga qaraganda ancha maqbul deb o‘ylaymiz.
3. Optimallashtirish masalalarida standart yo‘nalishdan bormay, biroz chekinish qildik. Oldin funksiya qiymatlarini topib, jadval tuzishni va jadval asosida grafik chizishni o‘rgatdik. So‘ngra chizma asosida maksimal va minimal qiymatlarni baholash, bu qiymatlar nimani anglatishini tasavvur qilishiga undadik. Bunday uslub talabaning masala mohiyatiga chuqur yetib borishini ta‘minlaydi…

Ushbu fanni o‘rgatish doirasida talabalarning turli saviyada bo‘lishlarini ham inobatga oldik. Ayniqsa, **oliy ta’lim muassasalarida kredit ta’lim tizimiga o‘tilishi masofaviy ta’limni tashkil etishni ham nazarda tutadi**, shu nuqtai nazardan darslikning 1-bobida funksiyalar nazariyasi bo‘limini yoritdik. Funksiyani nima ekanini bilmasdan turib, funksiyalar ustida bajariladigan amallarni, funksiyalar qayerda ishlatilishi haqida tasavvurga ega bo‘lmasdan, nima uchun hosilasi yoki integralini hisoblash kerakligini tushuna olmaymiz. Axir voqea-hodisalarning barchasini funksiya sifatida tasvirlash mumkin-ku.

Darslikni 2 qismga ajratdik. Ushbu qo‘lingizdagi 1-qismi bo‘lib, u beshta bobdan iborat.

1-BOB. FUNKSIYALAR, GRAFIKLAR, MODELLAR. KIRISH.

2-BOB. DIFFERENSIALLASH.

3-BOB. DIFFERENSIALNING TATBIQLARI.

4-BOB. KO‘RSATKICHLI VA LOGARIFMIK FUNKSIYALARNING HOSILALARI.

5-BOB. INTEGRALLASH.

Darslikni yozishda akademik litseyda ta’lim berishdagi 10 yillik va universitetdagi faoliyatimizda orttirgan 15 yillik pedagogik tajribamizga suyandik.

Albatta, kitob kamchiliklardan xoli emas. Keyingi nashrlarimizda kuzatilgan kamchiliklarni bartaraf qilib boramiz. Asosiysi kitob ayni paytida jahon ta‘limi bilan integratsiyalashuv jarayonida yozilgan bo‘lib, talabalar uchun kerakli amaliy qo‘llanma sifatida xizmat qiladi.

Ushbu kitobni o‘rganib, o‘zining tanqidiy fikr va mulohazalarini bildiradigan barcha ilm kishilariga oldindan o‘z minnatdorchiligimizni bildiramiz.

Muallif
f.-m.f.n. Sanobar Sabirovna Sadaddinova

Kitobda quyidagi o‘lchov birliklari ishlatilgan:

Uzunlik o‘lchovlari

1 dyuym	0.0833 ft	2.54 sm				
1 fut (ft)	12 dyuym	30.48 sm	0.3048 m			
1 mil	1760 yard	742000 sm	7420 m	7.420 km		
1 yard	3 ft	91.44 sm				

Hajm (suyuqlik) o‘lchovlari

1 fl		0.028 litr
1 barrel		158.998 litr
1 gallon		3.785 litr
1 kvart		0.94625 litr

Yuza (maydon) o‘lchovlari

1 kv.mil	640 akr	258.99 ga	
1 akr	4047 m^2	4840 kv. yard	43560 ft^2
1 kv. Yard	0.836 m^2		

Massa (og‘irlik) o‘lchovlari

1 misqol	4.25 gr	100 ta arpa (bug‘doy) doni og‘irligi
1 funt	453.6 gr	

Harorat o‘lchovlari

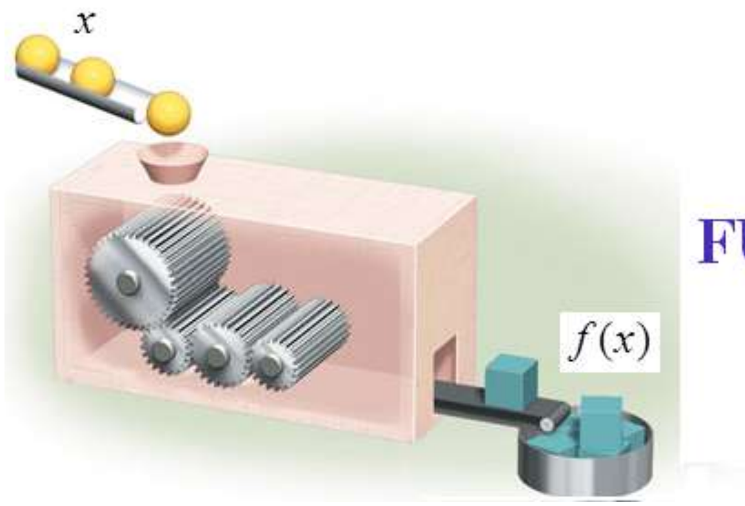
0^0 C	32^0 F
5^0 C	41^0 F
10^0 C	50^0 F

Selsiy o‘lchov birligidan Farengeyt o‘lchov birligiga o‘tish:

$$F = \frac{9}{5}C + 32$$

Farengeyt o‘lchov birligidan Selsiy o‘lchov birligiga o‘tish:

$$C = \frac{5(F - 32)}{9}$$



I BOB. FUNKSIYALAR GRAFIKLAR MODELLAR

(KIRISH)

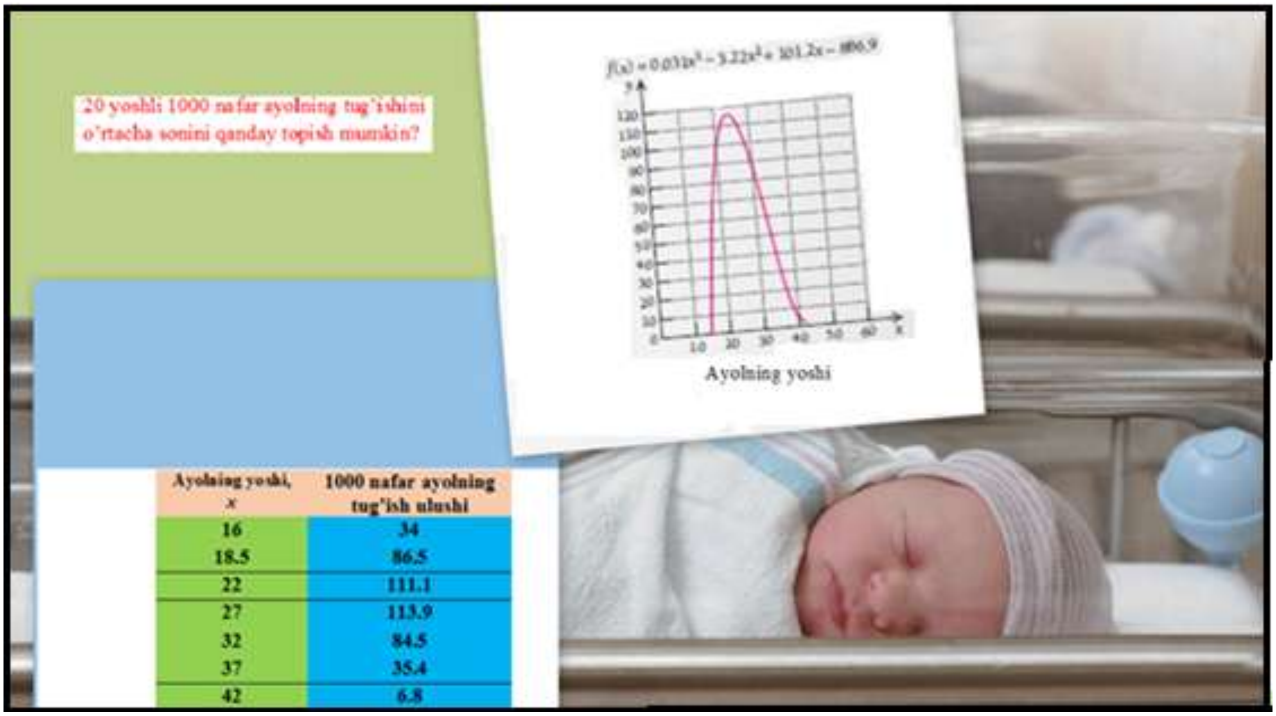
Mazkur bobda nimalar o‘rganiladi?

I bobni “Calculus (Matematika)” faniga kirish qismi deyish mumkin. Bu bobda funksiyalar va ularning grafik (diagrammalari)lari, aniqlanish hamda o‘zgarish sohalari, funksiyalarning belgilanishi, sohalardagi tatbiqlari o‘rganiladi. Chunki, har qanday jarayonni, voqea - hodisani, hattoki gapni ham funksiya sifatida tasvirlash mumkin.

Misol uchun, O‘zbekistonning birinchi prezidenti I.A.Karimovning **“Ma’naviyat moddiylikdan ming chandon ortiqdir”**, degan ibratli gapini funksiya ko‘rinishida $y = 1000x$ deb yozish mumkin.

Funksiyalarni to‘liq tushunib olish bilan mutaxassislikka doir bilimlarimiz ham shakllanib boradi.

Shuningdek, talab va taklif, umumiy qiymat, umumiy foyda, sof foyda, matematik modellashtirish, empirik egri chiziq yasash kabi tushunchalarni o‘rganamiz.



1.1. Grafiklar va tenglamalar

Hisob fani nimani o‘rganadi? Kursning boshida bu savol umumiy bo‘lib ko‘rinishi mumkin. Hozircha bu savolga sodda shaklda javob beramiz.

Energetik ichimlik qutisini tasavvur qiling.
Oddiy algebraik masala:

Qutining eni, bo‘yi va balandligining uzunliklari yig‘indisi 207 mm. Agar qutining balandligi enidan 3 marta katta va bo‘yi enidan 7 mm uzun bo‘lsa, uning o‘lchamlarini toping.

Yechilishi: ► Berilgan ma‘lumotlar asosida quyidagicha tenglamalar sistemasini tuzamiz va uni yechamiz:
$$\begin{cases} w+l+h=207 \\ h=3w \\ l=w+7 \end{cases}$$

Javob: Qutining eni $w=40$ mm, bo‘yi $l=47$ mm, balandligi $h=120$ mm ga teng.

Balki ishlab chiqaruvchi bu masalani bir necha marta yechib ko‘rishiga to‘g‘ri kelgandir.

Energetik qutining hajmi 200 sm^3 (qutida **200sm^3=6.75fl.oz** deb yozib qo‘yiladi) bo‘lishi kerak. Agar qutining balandligi enidan 2 marta katta bo‘lsa, uning to‘la sirti minimal bo‘lishi mumkinmi?

Mana endi berilgan masala hisob fani masalasiga aylandi.

1-jadval.

Qutining balandligi enidan 2 marta katta, V=200 sm^3			**Qutining to‘la sirti** $S_{to'la} = 2wh + 2lh + 2wl$
Eni, *w* sm	**Balandligi, *h* sm**	**Bo‘yi, *l* sm**	
5 sm	10 sm	4 sm	220 sm^2
4 sm	**8 sm**	**6.25 sm**	**Minimum = 214 sm^2**
3 sm	6 sm	11.1 sm	236 sm^2
2 sm	4 sm	25 sm	316 sm^2

Ushbu masalani yechish usullaridan biri quyidagicha:

hajmi 200 sm^3 bo‘lgan qutining balandligi enidan 2 marta katta bo‘lishi kerak. Shu bilan birga to‘la sirti eng kichik bo‘lishi kerak, ya‘ni kam mahsulot sarflab, hajmi 200 sm^3 bo‘lgan quti yasash mumkinmi? Buning uchun bir nechta o‘lchamlarni tanlash kerak. Agar siz katta formatdagi jadvallar bilan ishlaydigan dasturiy ta‘minotga ega bo‘lsangiz va unda ishlay olsangiz, 1-jadvalning birinchi va ikkinchi ustunlaridagi qiymatlarni o‘zgartirib, 1-jadvalni kengaytirishingiz mumkin. Qutining eni, balandligi, bo‘yini erkin tanlab, to‘la sirtlarni hosil qilamiz.

$$S_{to'la} = 2wh + 2lh + 2wl \,,$$

bunda balandlik $h = 2w$; hajm $V = w \cdot h \cdot l$ ga teng.

1-jadvaldagi qiymatlardan eng kichigi $S_{to'la} = 214\,\text{sm}^2$ masalaning yechimi bo‘ladi.

Ammo, bu yechim eng kichik, yagona yechim ekaniga qanday ishonch hosil qilamiz? Balki, boshqa undan kichik bo‘lgan to‘la sirt qiymatlari ham bordir. Bu savolga javob berish uchun hisob fani tushunchalariga ehtiyoj sezamiz. 3-bobda maksimal-minimal qiymatlarni topishga doir masalalarni o‘rganamiz. Ana o‘sha bobda buni bilib olamiz.

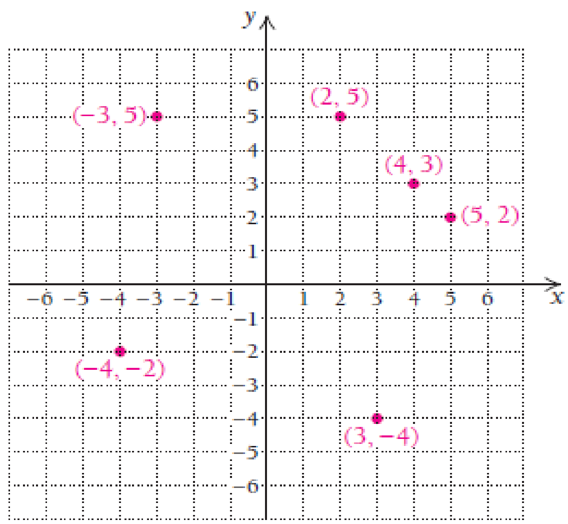
2-chizma. Diagrammaning ko‘rinishi.

Diagrammalarni tadqiq qilish – hisob fanining eng muhim vazifalaridan hisoblanadi. Diagramma va grafiklar munosabatlarni vizuallashtirish imkonini beradi. Masalan, diagramma asosida O‘zbekiston aholisining yillar davomida uzoq umr ko‘rishining o‘zgarishini baholash, yangi tug‘ilgan chaqaloqlarning yashab ketish ko‘rsatkichini (2-chizma) va shunga o‘xshash iqtisodiy, ijtimoiy sohalarda kuzatiladigan o‘zgarishlar ko‘rsatkichlarini baholash mumkin.

Keyinroq o‘rganadigan yana bitta mavzu – bu bitta koordinata o‘qidagi qiymatning o‘zgarishi ikkinchi koordinataga qanday ta‘sir ko‘rsatishini aniqlash.

1.1.1. Berilgan sonlar juftligini koordinata sistemasida tasvirlash

Tekislikdagi har bir nuqta biror (x, y) sonlar juftligidan iborat.



Birinchi songa nuqtaning **1-koordinatasi,** ikkinchi songa esa nuqtaning **2-koordinatasi** deyiladi. Birgalikda bu sonlar **nuqtaning koordinatalari** deyiladi. Chizmada keltirilgan vertikal Y chiziqqa **ordinata o‘qi**, gorizontal X chiziqqa esa **abssissa o‘qi** deyiladi.

1.1.2. Tenglamaning grafigi

Ikki o‘zgaruvchili tenglamaning yechimi – bu ma‘lum sonlar juftligini tenglamadagi o‘zgaruvchilar o‘rniga qo‘yganda to‘g‘ri tenglikning hosil bo‘lishidir.

Misol uchun (2;-1) nuqta $2x^2 + y = 7$ tenglamaning yechimi bo‘ladi, chunki x ni 2 bilan, y ni -1 bilan almashtirsak, to‘g‘ri tenglik hosil bo‘ladi:

$$2x^2 + y = 7$$
$$2 \cdot 2^2 + (-1) = 7$$
$$7=7.$$

Ta‘rif. Tenglamaning grafigi – bu tenglama yechimlari deb ataluvchi $(x; y)$ sonlar juftliklarini tasvirlovchi chizmadir.

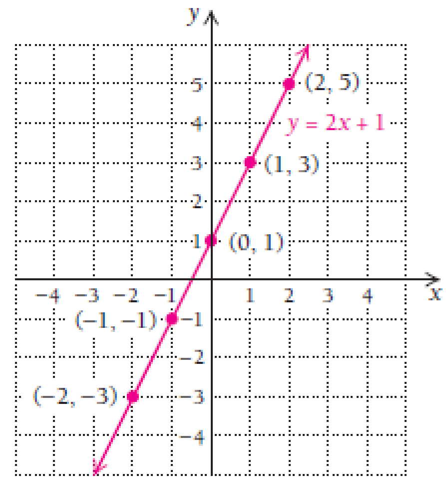
Tenglamaning grafigini chizishda tenglamani qanoatlantiradigan yetarlicha ko‘p miqdordagi sonlar juftligini aniqlashimiz mumkin. Grafik to‘g‘ri chiziqdan, egri chiziqdan yoki boshqa biror chiziqlar birlashmasidan iborat bo‘lishi mumkin.

1-misol. $y = 2x + 1$ tenglama grafigini chizing.

Yechilishi: ► Dastlab tenglamani to‘g‘ri tenglikka aylantiradigan sonlar juftliklarini aniqlaymiz va jadvalga yozamiz. Sonlar juftliklarini aniqlash uchun x ning o‘rniga ixtiyoriy bitta sonni tanlab, so‘ngra y ni topish mumkin. Aytaylik, x uchun -2 ni tanladik, uni $y = 2x + 1$ tenglamaga qo‘yamiz va $y = 2x + 1 = 2 \cdot (-2) + 1 = -3$ ni hosil qilamiz. Demak, $(-2; -3)$

juftlik tenglamaning yechimi ekan. Xuddi shuningdek, x uchun musbat sonlarni ham tanlashimiz mumkin. 0 ni tanlash ham mumkin, uni $y=2x+1$ tenglamaga qo‘yamiz va $y=2x+1=2\cdot0+1=1$ ni hosil qilamiz. Demak, (0;1) juftlik ham tenglamaning yechimi bo‘ladi. Jadvaldagi barcha qiymatlarni millimetrli qog‘ozda tasvirlaymiz.

x	y	(x, y)
-2	-3	$(-2,-3)$
-1	-1	$(-1,-1)$
0	1	$(0,1)$
1	3	$(1,3)$
2	5	$(2,5)$



1) Ixtiyoriy x ni tanlang
2) y ni hisoblang
3) $(x;y)$ juftlikni hosil qiling
4) Nuqtani tasvirlang

Tasvirdan ko‘rish mumkinki, agarda biz cheksiz ko‘p nuqtalarni topsak va ularni koordinata tekisligida tasvirlasak, shu nuqtalarning geometrik o‘rnidan iborat – uzluksiz to‘g‘ri chiziqni hosil qilishimiz mumkin ekan. ◄

1-vazifa: $y=x-1$ tenglamaning grafigini mustaqil chizing.

2-misol. $3x+5y=10$ tenglama grafigini chizing.

Yechilishi: ► Xuddi 1-misolga o‘xshab, x ning o‘rniga ixtiyoriy sonni tanlab, so'ngra y qiymatni topish mumkin. Lekin bu gal hisoblashlarni soddalashtirish maqsadida tenglamani y ga nisbatan yechamiz:

$3x+5y=10$

$3x+5y-3x=10-3x$ tenglikning har ikki tomonidan $3x$ ni ayiramiz;

$5y=10-3x$ soddalashtiramiz;

$\frac{1}{5}\cdot 5y=\frac{1}{5}\cdot(10-3x)$ tenglikning har ikki tomonini $\frac{1}{5}$ ga ko'paytiramiz yoki 5 ga bo'lamiz;

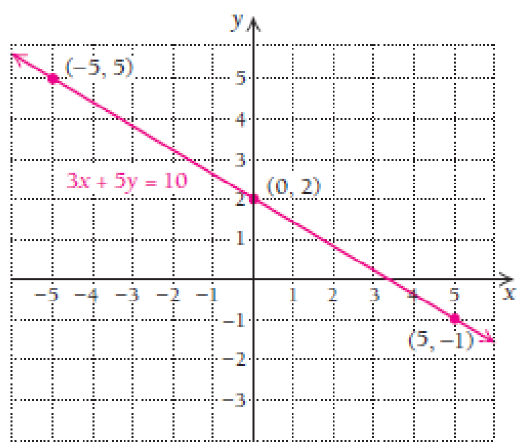
$y=\frac{1}{5}\cdot 10-\frac{1}{5}\cdot 3x$ distributivlik qonunini qo'llaymiz;

$y=2-\frac{3}{5}x$ standart ko'rinishga o'tkazamiz:

$$y=-\frac{3}{5}x+2$$

Endi tenglama grafigini yasash uchun x ning o'rniga 5 ga karrali bo'lgan uchta sonni tanlaymiz: 0, 5 va -5. Chunki ixtiyoriy sonni tanlasak, y ning qiymati kasr son bo'lib qolishi mumkin. Soddalik uchun shunday qilamiz.

x	y	(x, y)
0	2	(0, 2)
5	−1	(5, −1)
−5	5	(−5, 5)



2-vazifa: $3x-5y=10$ tenglamani yechib, so'ngra grafigini chizing (2-misolga qarang).

1-va 2-misollarda chiziqli tenglamalarni ko‘rib chiqdik. Endi siz bilan kvadratik tenglamalar grafigini yasashni o‘rganamiz.

$ax^2+bx+c=0$ ko‘rinishdagi tenglamaga **to‘la kvadrat tenglama** deyiladi. $y=ax^2+bx+c$ ko‘rinishdagi formulaga **umumiy ko‘rinishdagi kvadrat funksiya** deyiladi, bunda a,b,c lar haqiqiy sonlar bo‘lib, ularga **koeffitsiyentlar** deyiladi.

Kvadrat funksiya grafigi nuqtalarning geometrik o‘rnidan iborat bo‘lib, unga **parabola** deyiladi.

3-misol. $y=x^2-1$ tenglama grafigini yasang.

Yechilishi: ► Grafikni yasash uchun nuqtalarni aniqlash kerak. Bunda tenglamadagi x ning o‘rniga -2, -1, 0, 1, 2 qiymatlarni berib, y ning qiymatlarini hisoblab topamiz:

$y(-2)=(-2)^2-1=3 \quad \longrightarrow \quad (-2,3)$ nuqtani aniqladik.

$y(-1)=(-1)^2-1=0 \quad \longrightarrow \quad (-1,0)$

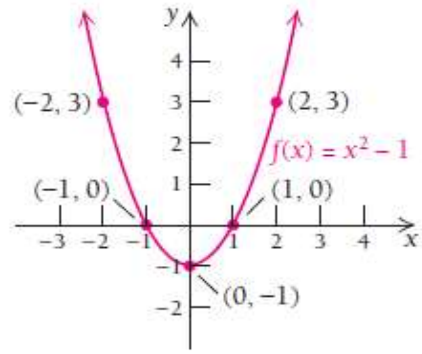
$y(0)=0^2-1=-1 \quad \longrightarrow \quad (0,-1)$

$y(1)=1^2-1=0 \quad \longrightarrow \quad (1,0)$

$y(2)=2^2-1=3 \quad \longrightarrow \quad (2,3)$.

Topilgan qiymatlarni jadvalda tasvirlash ham mumkin. Nuqtalarni dekart koordinata tekisligida belgilaymiz va ularni tutashtiramiz, natijada grafikni yasaymiz.

x	$f(x)$	$(x, f(x))$
-2	3	$(-2, 3)$
-1	0	$(-1, 0)$
0	-1	$(0, -1)$
1	0	$(1, 0)$
2	3	$(2, 3)$



◄

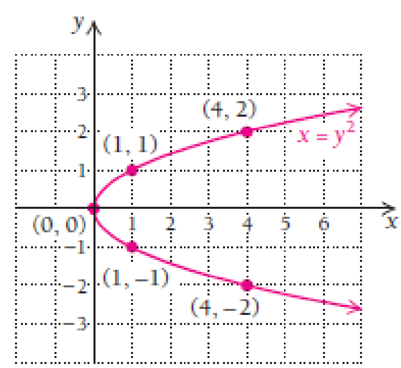
3-vazifa: $y = 3 - x^2$ **tenglamaning grafigini mustaqil chizing.**

4-misol. $x = y^2$ tenglama grafigini yasang.

Yechilishi: ►Ushbu tenglamada x erksiz o‘zgaruvchi, y ni esa erkli o‘zgaruvchi deb qarash mumkin. Shuning uchun y ga ixtiyoriy qiymat berib, x ni hisoblaymiz.

x	y	(x, y)
4	-2	$(4, -2)$
1	-1	$(1, -1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
4	2	$(4, 2)$

1) Ixtiyoriy y ni tanlang
2) x ni hisoblang
3) $(x; y)$ juftlikni tuzing
4) Nuqtalarni yasang



◄

4-vazifa: $x = 1 - y^2$ **tenglamaning grafigini chizing.**

1.1.3. Matematik modellar

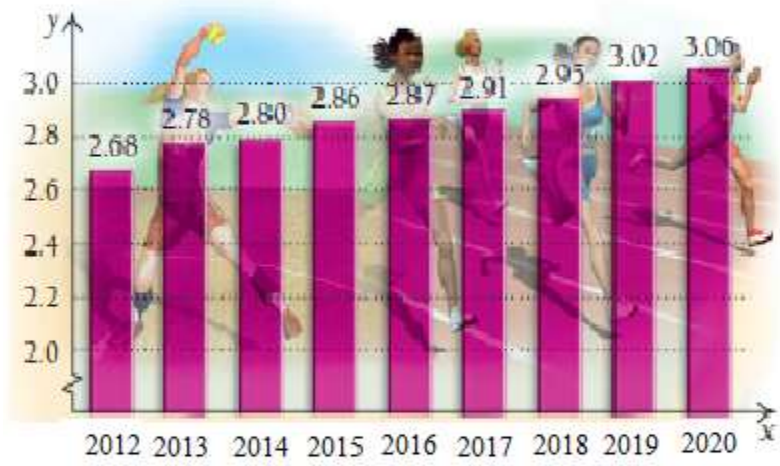
Real hayotda ro‘y beradigan voqea-hodisalarni matematik tilda bayon qilinishiga **matematik model** deyiladi. Algebrada funksiyalar model sifatida qaraladi. Matematik model yordamida voqea-hodisalarning natijalari qanday bo‘lishi haqida bashorat qilish mumkin. Agar bashorat noaniq chiqsa yoki tajriba natijalari bilan modeldan olingan natijalar mos kelmasa, u holda modelni o‘zgartirish yoki undan voz kechish kerak.

Har qanday modelni qo‘shimcha yangi ma‘lumotlarni kiritib, qayta tuzish mumkin. Matematik model ko‘pincha davomli jarayonni ifodalaydi. Masalan, aholining o‘sish tezligini aniq bashorat qiluvchi matematik modellar mavjud. Matematik modelni tuzish 6 bosqichda amalga oshiriladi.

Matematik model tuzish algoritmi:

1. Real hayotiy masalani tanlang.
2. Ma‘lumotlar yig‘ing.
3. Ma‘lumotlarni tahlil qiling.
4. Modelni quring.
5. Modelni tekshiring va takomillashtiring.
6. Tushuntiring va bashoratlang.

5-misol. 2012 – 2020 yillarda o‘rta maktab qizlarning yengil atletika sport turiga qatnashish diagrammasi. Y o‘qda yengil atletikaga qatnashgan o‘quvchi qizlar soni (million birlikda).



$N = 0.042t + 2.71$ modeldan foydalanib, 2025 yilda sportning yengil atletika turiga o‘rta maktabdan nechta o‘quvchi qiz qatnashishini bashoratlang. Bunda $t-$parametr 2012 yildan keyingi yillar soni va $N-$qatnashuvchilar soni (million birlikda).

Yechilishi: ► 2012 yildan 2025 yilgacha 13 yil bor, shuning uchun $t = 13$ deb olamiz.

$$N = 0.042t + 2.71 = 0.042 \cdot 13 + 2.71 = 3.256.$$

Ushbu modelga ko‘ra, 2025 yilda sportning yengil atletika turiga o‘rta maktabdan taxminan 3 mln 256 ming o‘quvchi qiz qatnashar ekan.

Ko‘pchilik modellar singari 5-misoldagi model ham aniq emas. Misol uchun, $t = 1$ deb olsak, u holda $N = 2.752$ qiymat hosil bo‘ladi, ya‘ni real 2.78 qiymatdan ozroq farq qiluvchi sonni olamiz. Agar baholash nuqtai nazaridan qarasak, model haqiqatga yaqin, ya‘ni **model adekvat**.

Kubik model $N = 0.001x^3 - 0.014x^2 + 0{,}087x + 2.69$ ni qarash mumkin. Bu modelda ham $t = 1$ desak, u holda $N \approx 2.76$ qiymatni olamiz. Lekin $t = 13$ deb olsak, 5-misolning javobidan juda katta farq qiluvchi $N = 3.652$ son hosil bo‘ladi. Demak, bu model adekvat emas ekan. Chunki bu modelning bashoratiga ko‘ra, 2025 yilda 1-modeldagidan qariyb 400 000 ortiq o‘quvchi qizlar yengil atletikaga qatnashadigan bo‘lib chiqdi. Modelni haqiqat bilan yaqinligini aniqlash uchun yillar kesimida tekshirish kerak va bunda modelni juda ehtiyotkorlik bilan tuzish kerak.◄

Modellar ichida juda aniq va muhim bo‘lgan **murakkab foizlarni darajaga oshirish modeli** mavjud. Faraz qilaylik, P dollarni yiliga i foiz stavkasi bo‘yicha investitsiyaga kiritdik. Birinchi yil oxirida A_1 dollar olinadi:

$$A_1 = P + Pi$$

$$A_1 = P(1+i) = P \cdot r$$

Bunda $r = 1+i$ belgilash kiritamiz. Ikkinchi yil boshida bizda $P \cdot r$ dollar bor edi, ikkinchi yil oxirida esa $A_2 = A_1 \cdot r = (P \cdot r)r$

$$A_2 = P \cdot r^2 = P(1+i)^2$$

ga teng bo‘ladi.

Demak, uchinchi yil boshida bizda $P \cdot r^2$ dollar bor, yil oxirida esa A_3 bo‘ladi:

$$A_3 = A_2 \cdot r = (P \cdot r^2)r = P \cdot r^3 = P(1+i)^3.$$

1-teorema: Agar P miqdordagi mablag‘ yiliga i foiz foyda keltiradigan ishga yo‘naltirilgan bo‘lsa, u holda t yildan keyin bu mablag‘ quyidagi tenglik bo‘yicha o‘sadi:

$$A = P(1+i)^t.$$

6-misol. Tadbirkorlik: murakkab foiz. Faraz qiling, Fibonachchi Investitsiya fondi 1000$ ni yiliga 5% foyda keltiradigan ishga sarfladi. 2 yildan keyin bu mablag‘ qancha bo‘ladi?

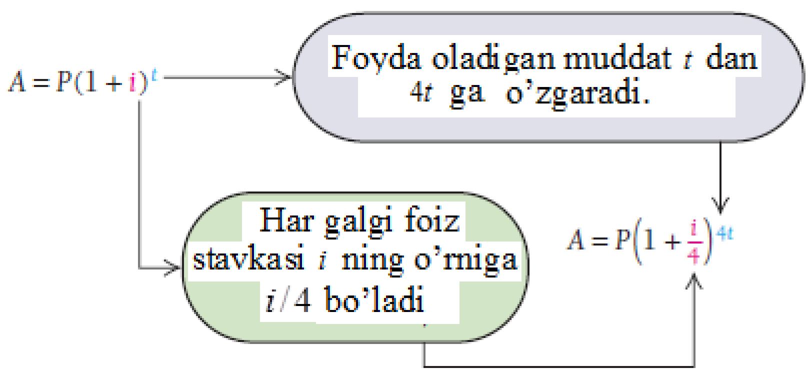
Yechilishi: ► $A = P(1+i)^t$ formuladan foydalanamiz. $P = 1000$, $i = 0.05$ va $t = 2$ belgilashlarni kiritamiz:

$$A = 1000(1+0.05)^2 = 1000 \cdot 1.05^2 = 1000 \cdot 1.1025 = 1102.50 \ \$$$

2 yildan so‘ng 1102.50 $ mablag‘ to‘planar ekan. ◄

5-vazifa. Tadbirkorlik: murakkab foiz. Faraz qiling, Fibonachchi Investitsiya fondi 1000$ ni yiliga 6% bilan investitsiya qildi. 2 yildan keyin bu mablag‘ qancha bo‘ladi?

Yiliga 4 marta, ya’ni kvartaliga foiz qo‘shiladigan daromad formulasini yuqoridagi tenglik asosida keltirib chiqaramiz:



Har kvartalda foiz qo‘shilgani uchun umumiy foiz 4 ga bo‘lib qo‘shiladi. Chunki, 1 yil 4 kvartaldan iborat.

Quyidagi teorema o‘rinli:

2-teorema: Agar P miqdordagi mablag‘ yilning ma‘lum davrida i foiz daromad qiladigan ishga sarflangan bo‘lsa, u holda t yildan keyin bu mablag‘ quyidagiga teng bo‘ladi: $A = P\left(1+\frac{i}{n}\right)^{nt}$.

7-misol. Tadbirkorlik: murakkab foiz. Faraz qiling, sarmoyador 1000$ ni yilning har kvartalida 5% foyda oladigan ishga tikdi. 3 yildan keyin bu mablag‘ qancha bo‘ladi?

Yechilishi: ► $A = P\left(1+\frac{i}{n}\right)^{nt}$ formuladan foydalanamiz. Bunda P=1000 \$, $i = 5\% = 0.05$, $n = 4$, $t = 3$ qiymatlarni formulaga qo‘yamiz.

$$A = P\left(1+\frac{i}{n}\right)^{nt} = 1000\cdot\left(1+\frac{0.05}{4}\right)^{4\cdot 3} = 1000\cdot(1+0.0125)^{12} = 1160.754518 \approx 1160.75\,\$$$

Demak, 3 yildan keyin sarmoyador 1000 $ ini 1160.75 $ qilib olar ekan.◄

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-24 misollarda tenglamalar grafigini chizing:

1. $y = x+5$ **2.** $y = -2x$ **3.** $y = x-2$

4. $y = \frac{1}{3}x-4$ **5.** $y = -\frac{2}{3}x$ **6.** $y = -\frac{5}{3}x+2$

7. $x+y=5$ **8.** $x-y=2$ **9.** $6x-2y=4$

10. $3y+6x=-9$ **11.** $5x-6y=12$ **12.** $2y+5x=10$

13. $y = x^2 - 3$ **14.** $y = x^2 - 4$ **15.** $x = y^2 - 2$

16. $x = y^2 + 5$ **17.** $x = y^2 - 1$ **18.** $x = y^2 + 3$

19. $y = |x|$ **20.** $y = |3 - x|$ **21.** $y = 5 - x^2$

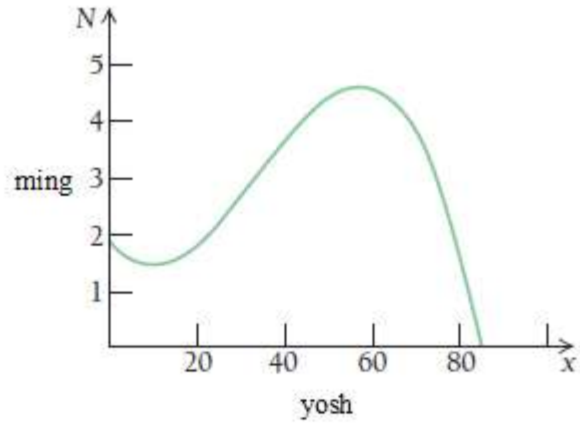
22. $y = x^3 - 1$ **23.** $x = y^2 - 1$ **24.** $y - 7 = x^3$

25-34 masalalar mavzuning tatbiqlariga oid:

25. Rekordlar hisoboti. Tadqiqotlarga ko‘ra, yugurish musobaqalarining jahon rekordini chiziqli tenglama asosida modellashtirish mumkin. Xususan, R jahon rekordi x yilda km/minut hisobida modellashtiriladi. $R = -0.00582x + 15.3476$ modeldan foydalanib, 1954, 2008, 2016 va 2020 yillar uchun jahon rekordlarini baholang.

26. Tibbiyot. Ibuprofen dorisi og‘riqni qoldirish uchun buyuriladi. Quyidagi $A = 0.5t^4 + 3.45t^3 - 96.65t^2 + 347.7t$; $0 \le t \le 6$
model asosida qondagi ibuprofen miqdorini modellashtirish mumkin. 400 mg dori ichgan bemor qonida t soat o‘tgach ibuprofen A miqdorini baholang. 400 mg dori ichgan bemor qonida 2 soat o‘tgach ibuprofen A miqdori qancha bo‘ladi?

27. Eshitish qobiliyati past bo‘lgan kishilar. Ming kishi orasida x yoshdagi kishilarning N nafari quyidagi diagramma orqali approksimatsiyalanadi:



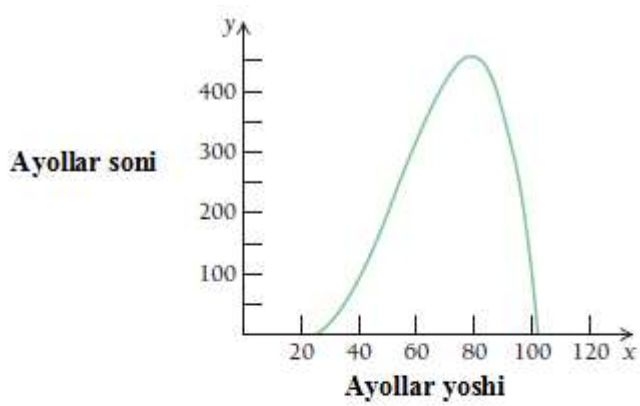
Diagrammadan foydalanib, quyidagi savollarga javob bering:

a) 20, 40, 50 va 60 yoshli o‘zbeklarda eshitish qobiliyati past kishilar sonini approksimatsiyalang;

b) Qanday yoshdagi eshitish qobiliyati past bo‘lgan kishilar soni 4 mingga teng?

c) Diagrammani o‘rganib, eshitish qobiliyati past bo‘lgan kishiarning soni qaysi yoshda eng ko‘p bo‘lishini aniqlang.

28. Tibbiyot: Ko‘krak bezi rakining yoshga mosligi. Quyidagi diagramma 100000 ayol uchun x yosh funksiyasi sifatida approksimatsiyalanadi. Bunda x o‘zgaruvchi 25-102 yosh oralig‘ida o‘zgaradi.



a) 40 yoshli ayollarning ko‘krak saratoni bilan kasallanishi nimaga teng?

b) Qaysi yoshda 100 000 ayoldan 400 nafarida bu kasallikka uchrashi mumkin?

c) Diagrammani o‘rganib, qaysi yoshdagi ayollarda bu kasallik ko‘p uchrashini aniqlang.

29. Murakkab foiz. Investor “Aloqabank”dan 100000 $ ga 2.8 % lik deposit sertifikatini sotib oldi.

a) 1 yildan so‘ng investitsiya bahosi qancha bo‘ladi?

b) yarim yilda-chi?

c) bir kvartalda-chi?

d) bir kunda-chi?

30. Murakkab foiz. Investor “Aloqabank”dan 300 000 $ ga 2.2 % lik deposit sertifikatini sotib oldi.

a) 1 yildan so‘ng investitsiya bahosi qancha bo‘ladi?

b) yarim yilda-chi?

c) bir kvartalda-chi?

d) bir kunda-chi?

31. Murakkab foiz. Investor 30 000 $ ga 4 % lik obligatsiyalarni sotib oldi.

a) 1 yildan so‘ng investitsiya bahosi qancha bo‘ladi?

b) yarim yilda-chi?

c) bir kvartalda-chi?

d) bir kunda-chi?

32. Murakkab foiz. Investor 1000 $ ga 5 % lik obligatsiyalarni sotib oldi. a) 1 yildan so‘ng investitsiya bahosi qancha bo‘ladi?

b) yarim yilda-chi?

c) bir kvartalda-chi?

d) bir kunda-chi?

Ssuda bo‘yicha har oylik to‘lovni aniqlash. Agar P dollar qarz olingan bo‘lib, n oy davomida har oy oxirida M oylik to‘lov qilinayotgan bo‘lsin.

$$M = P\frac{\frac{i}{12}\left(1+\frac{i}{12}\right)^n}{\left(1+\frac{i}{12}\right)^n - 1}$$

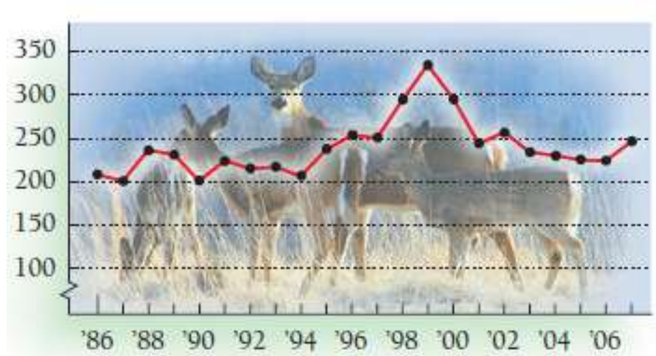
Bunda i – yillik foiz stavkasi, n – oylik to‘lovlarning umumiy soni.

33. Avtosalon 18000 $ lik avtomobilni 3 yil muddatga 6.4% li kreditga beradi. Har oylik to‘lovni toping.

34. O‘zqurilishsanoatbank uy-joy sotib olish uchun 100 000 $ ni 30 yilga 4.8 % bilan ipotekaga beradi. U holda oylik to‘lov qancha bo‘ladi?

35. Pensiya jamg‘armasi. Saida 30 yoshidan boshlab, pensiya jamg‘armasiga pul o‘tkazadi. Agar u har yili oyligidan 1200 $ hisobga o‘tkazsa va bu jamg‘arma har yili 8 % ga ko‘payib borsa, 60 yoshga yetganda Saidaning hisobida qancha pul bo‘ladi? ($A = P(1+i)^t$ formuladan foydalanasiz.)

36. Kanada ohulari. 1986 yildan 2006 yilgacha bo‘lgan davrda Kanada davlati qo‘riqxonasida ohular soni quyidagi diagramma bilan approksimatsiyalanadi:



a) Qaysi yillarda ohular soni 250 000 dan ko‘p bo‘lgan?

b) Qaysi yillarda ohular soni 200 000 ga teng bo‘lgan?

c) Qaysi yillarda ohular soni eng ko‘p (maksimum) bo‘lgan?

d) Qaysi yillarda ohular soni eng kam (minimum) bo‘lgan?

(Manba: Kanada, ichki baliqchilik va yovvoyi tabiat bo‘limi.)

1.2. Funksiyalar va matematik modellar

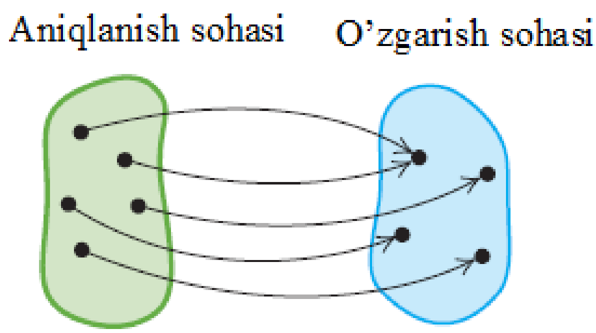
Funksiya tushunchasi matematikada juda muhim tushunchalardan biri hisoblanadi. Funksiya ikkita to‘plam orasidagi munosabatni aniqlaydigan maxsus ko‘rinishdir.

Misol keltiramiz:

1. Telefon klavyaturasidagi har bir belgiga son yozib qo‘yiladi.
2. Do‘kondagi har bir qo‘l telefonining modeliga uning bahosi yozilgan.
3. Har bir haqiqiy songa mos uning kubi yozilgan.

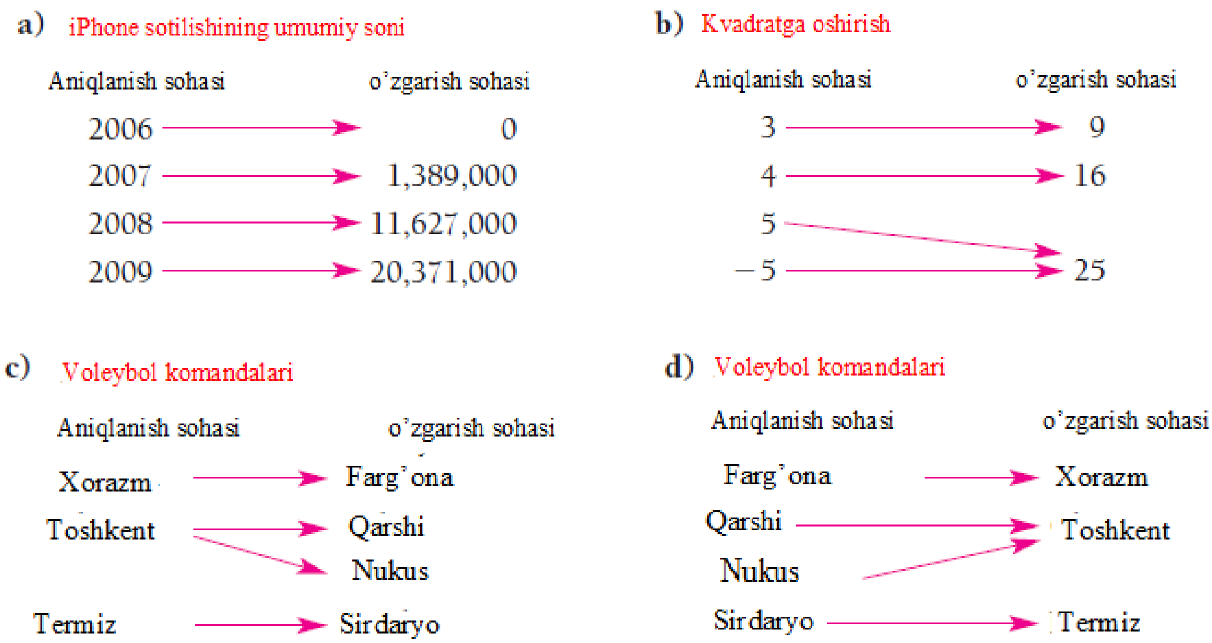
Ushbu misollardagi birinchi to‘plamga **aniqlanish sohasi**, ikkinchi to‘plamga **o‘zgarish sohasi** deyiladi.

Aniqlanish sohasidan olingan har bir elementga o‘zgarish sohasidan faqat bitta element mos qo‘yiladi. Bu moslikka **funksiya** deyiladi.



Ta‘rif. Funksiya – aniqlanish sohasi deb ataluvchi birinchi to‘plam elementlari bilan o‘zgarish sohasi deb ataluvchi ikkinchi to‘plam elementlari orasidagi moslikdir. Bunda aniqlanish sohasidan olingan har bir elementga o‘zgarish sohasidan faqat bitta element mos qo‘yiladi.

1-misol. Berilgan diagrammalarning qaysi birlari funksiya bo‘ladi?



Yechilishi: ►

a) - moslik **funksiya bo‘ladi**, chunki aniqlanish sohasining har bir elementiga o‘zgarish sohasining faqat bitta elementi mos qo‘yilgan.

b) -moslik ham **funksiya**, chunki aniqlanish sohasining har bir elementiga o‘zgarish sohasining faqat bitta element mos qo‘yilgan. Shu bilan birga aniqlanish sohasining 2 ta elementiga o‘zgarish sohasidan bitta 25 soni mos qo‘yilgan.

c) -moslik **funksiya emas**, chunki aniqlanish sohasidagi bitta Toshkent komandasi o‘zgarish sohasidagi Qarshi va Nukusga mos qo‘yilgan.

d) -moslik **funksiya**, chunki aniqlanish sohasining har bir elementiga o‘zgarish sohasining faqat bitta element mos qo‘yilgan. Shuningdek aniqlanish sohasining 2 ta elementiga o‘zgarish sohasidan bitta Toshkent mos qo‘yilgan.◄

2-misol. Quyidagilarning qaysilari funksiya bo‘la oladi?

	Aniqlanish sohasi	Moslik	O‘zgarish sohasi
a)	Oila	Oila a‘zosining vazni	Musbat sonlar to‘plami
b)	Butun sonlar to‘plami $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$	Har bir sonning kvadrati	Nomanfiy sonlar to‘plami $\{0, 1, 4, 9, 16, 25, \ldots\}$
c)	Barcha viloyatlar to‘plami	Har bir viloyatdagi deputat	O‘zbekiston deputatlari to‘plami

Yechilishi:►

a) **- funksiya bo‘ladi,** chunki har bir kishiga faqat bitta vazn to‘g‘ri keladi.

b) **- funksiya bo‘ladi,** chunki har bir butun sonning faqat bitta kvadrati mavjud.

c) **- funksiya bo‘lmaydi,** chunki bitta viloyatdan bir nechta deputat saylangan.◄

Funksiya ta‘rifiga ko‘ra, funksiyani sonlar juftligi sifatida qarash mumkin. Bunda juftlikning 1-koordinatasi aniqlanish sohasining elementi, 2-koordinatasi esa o‘zgarish sohasiga tegishli bo‘ladi. Agar 1-misolning b) shartidagi funksiyani f deb belgilasak, u quyidagi ko‘rinishda bo‘ladi:

$$f = \{(3; 9), (4; 16), (5; 25), (-5; 25)\},$$

Aniqlanish sohasi: $D(f) = \{3; 4; 5; -5\}$, o‘zgarish sohasi: $E(f) = \{9; 16; 25\}$.

1.2.1. Funksiya qiymatini hisoblash

Matematikada qaraladigan ko‘pchilik funksiyalar tenglamalar bilan beriladi. Misol uchun: $y = 2x + 3$ yoki $y = x^2 - 4$ kabi.

Berilgan $y = 2x + 3$ funksiyaning grafigini chizish uchun shu tenglamani qanoatlantiradigan sonlar juftliklarini aniqlash kerak. Bunda ixtiyoriy tanlab olingan x qiymatga mos funksiya qiymati hisoblanadi.

$x = 4$ da $y = 2x + 3 = 2 \cdot 4 + 3 = 11$ ga teng. Grafik (4; 11) nuqtadan o‘tadi.

$x = -5$ da $y = 2x + 3 = 2 \cdot (-5) + 3 = -7$ bo‘ladi. Grafik (-5; -7) nuqtadan o‘tadi.

$x = 0$ da $y = 2x + 3 = 2 \cdot 0 + 3 = 3$ ga teng. Grafik (0; 3) nuqtadan o‘tadi.

$y = 2x + 3$ tenglamada x ning qiymatlari **kirish** (aniqlanish sohasining elementlari) deyiladi. y ning qiymatlari **natija yoki chiqish** (o‘zgarish sohasining elementlari) deyiladi. $y = 2x + 3$ tenglama x ning funksiyasi sifatida $f(x) = 2x + 3$ ko‘rinishda yoziladi. Shunga asosan funksiya qiymatlarini ham quyidagicha yozish mumkin:

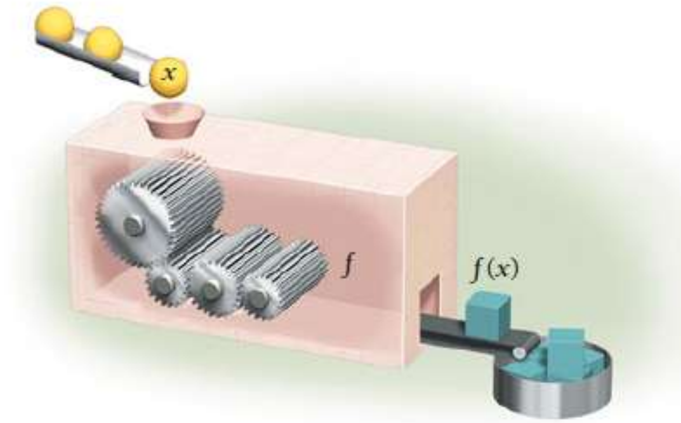
$f(4) = 2 \cdot 4 + 3 = 11$;

$f(-5) = 2 \cdot (-5) + 3 = -7$;

$f(0) = 2 \cdot 0 + 3 = 3$.

Funksiyani mashina deb faraz qilaylik, kirishiga x elementni beramiz, chiqishida $f(4) = 11$ funksiya qiymati hosil bo‘ladi. Mashina ichida $f(x) = 2x + 3$ ga mos $2 \cdot 4 + 3$ amal hisoblanadi.

Funksiya $f(x) = 2x + 3$	
Kirish	Chiqish
4	11
-5	-7
0	3
t	$2t + 3$
$a + h$	$2(a + h) + 3$



Esda saqlang! Funksiya vaqt emas va hech qachon undagi yo‘l hisobga olinmaydi.

3-misol. Kvadratga oshiradigan funksiya $f(x) = x^2$ bo‘lsin. $f(-3),\ f(1),\ f(k),\ f(\sqrt{k}),\ f(1+t),\ f(x+h)$ funksiya qiymatlarini hisoblang.

Yechilishi: ► $f(-3) = (-3)^2 = 9$;

$f(1) = 1^2 = 1$;

$f(k) = k^2$;

$f(\sqrt{k}) = (\sqrt{k})^2 = k$;

$f(1+t) = (1+t)^2 = 1 + 2t + t^2$;

$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$. ◄

1-vazifa. $f(x) = 2x^2 - 1$ **funksiyaning** $f(0),\ f(-1),\ f(h),\ f(1+h),\ f(x-h)$ **qiymatlarini mustaqil hisoblang.**

4-misol. $f(x) = 3x^2 - 2x + 8$ funksiya berilgan. Uning $f(0),\ f(-5),\ f(7a)$ qiymatlarini hisoblang.

Yechilishi: ►Funksiya qiymatini hisoblashning bir usuli quyidagicha, agar formula berilgan bo‘lsa, uni qolip deb o‘ylash kerak:

$$f(\square) = 3\square^2 - 2\square + 8.$$

Ushbu kirishdagi qiymat uchun natijani topishda o‘ylaymiz: “chapdagi bo‘sh o‘ringa nima qo‘yilishidan qat’iy nazar, o‘ngdagi bo‘sh o‘ringa ham shu qiymat qo‘yiladi”:

$f(0) = 3\cdot 0^2 - 2\cdot 0 + 8 = 8;$

$f(-5) = 3\cdot(-5)^2 - 2\cdot(-5) + 8 = 3\cdot 25 + 10 + 8 = 93;$

$f(7a) = 3\cdot(7a)^2 - 2\cdot(7a) + 8 = 3\cdot 49a^2 - 14a + 8 = 147a^2 - 14a + 8.$◄

2-vazifa. $f(x) = 3x^2 + 2x - 9$ funksiyaning $f(4),\ f(-5),\ f(0),\ f(a),\ f(5a)$ qiymatlarini mustaqil hisoblang.

5-misol. Berilgan funksiya kirishdan kirish kvadratini ayiradi: $f(x) = x - x^2$. Funksiyaning $f(4),\ f(x+h),\ \dfrac{f(x+h) - f(x)}{h}$ qiymatlarini hisoblang.

Yechilishi: ►

1) $f(4) = 4 - 4^2 = -12;$

2) $f(x+h) = (x+h) - (x+h)^2 = x + h - (x^2 + 2xh + h^2) = x + h - x^2 - 2xh - h^2;$

3) $$\frac{f(x+h) - f(x)}{h} = \frac{x + h - x^2 - 2xh - h^2 - f(x)}{h} = \frac{x + h - x^2 - 2xh - h^2 - (x - x^2)}{h} =$$

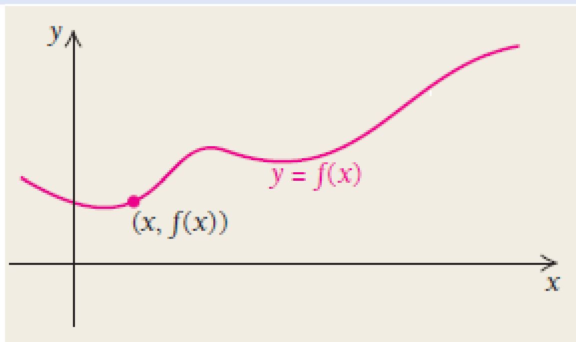
$$= \frac{x + h - x^2 - 2xh - h^2 - x + x^2}{h} = \frac{h - 2xh - h^2}{h} = \frac{h(1 - 2x - h)}{h} = 1 - 2x - h, \quad h \neq 0.$$◄

3-vazifa. $f(x) = 3x - x^2$ funksiyaning $f(4),\ f(x+h),\ \dfrac{f(x+h) - f(x)}{h}$ qiymatlarini mustaqil hisoblang.

1.2.2. Funksiyalarning grafiklari

Kvadratga oshiradigan funksiya $f(x) = x^2$ ni qarang. Kirishdagi 3 chiqishdagi 9 bilan juftlik hosil qiladi: (3, 9).

Ta'rif. f **funksiyaning grafigi** – chizma bo'lib, u barcha $(x; f(x))$ kirish-chiqish juftliklarini tasvirlaydi. Agar funksiya tenglama bilan berilgan bo'lsa, funksiyaning grafigi shu tenglamaning grafigidir: $y = f(x)$.

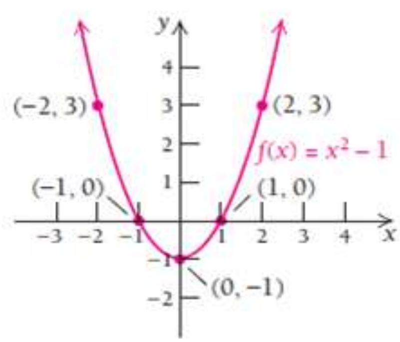


6-misol. $f(x) = x^2 - 1$ funksiya grafigini chizing.

Yechilishi: ►

x	$f(x)$	$(x, f(x))$
−2	3	(−2, 3)
−1	0	(−1, 0)
0	−1	(0, −1)
1	0	(1, 0)
2	3	(2, 3)

1) Ixtiyoriy x ni tanlang
2) y ni hisoblang
3) $(x; y)$ juftlikni hosil qiling
4) Nuqtani tasvirlang

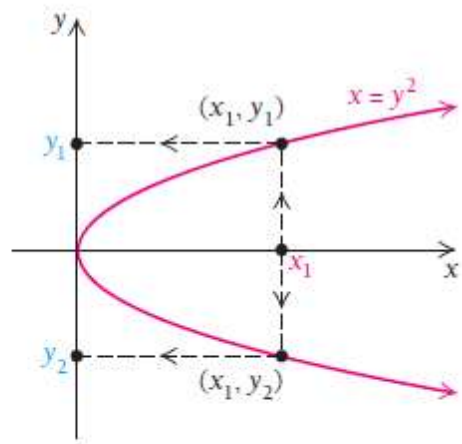


Grafikni chizish uchun jadvaldan kirish-chiqish juftliklarini olib, koordinata tekisligida tasvirlaymiz va egri chiziqni to‘ldirib boramiz.◄

4-vazifa. $f(x)=3-x^2$ funksiya grafigini mustaqil yasang.

1.2.3. Funksiyani vertikal chiziq yordamida tekshirish[1]

Keling endi berilgan grafikka qarab, bu diagramma funksiyaning grafigi bo‘la oladimi yoki yo‘qmi? - degan savolga javob izlaymiz.



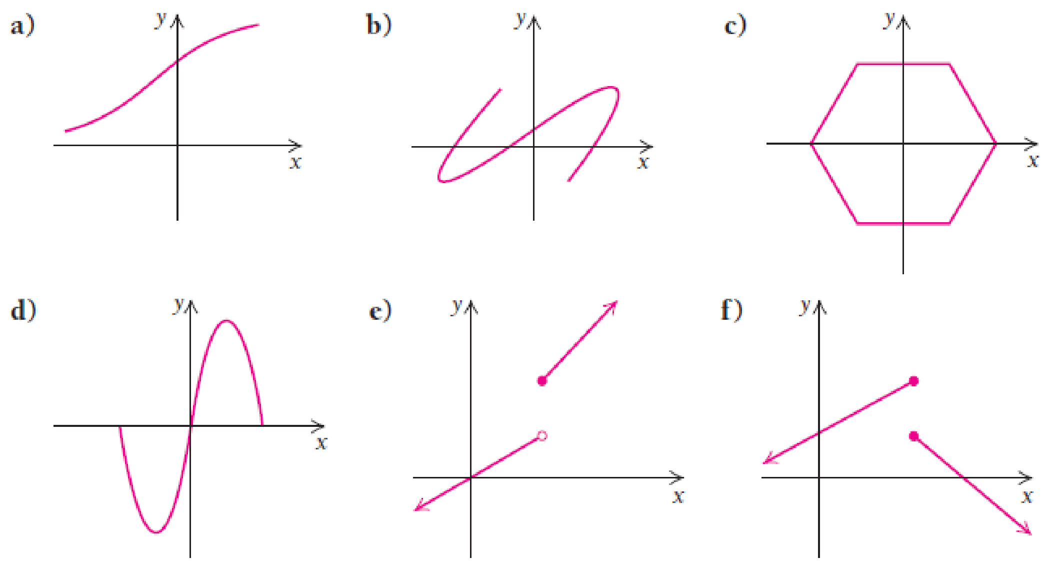
Chizmada bitta kirishga ikkita chiqish mos qo‘yilgan. Bilamizki, funksiya bo‘lishi uchun har bir kirishga faqat bitta chiqish mos qo‘yiladi. Bundan ko‘rinadiki, ushbu diagramma funksiya bo‘la olmas ekan. Yana shuni ham bilib qo‘yishimiz kerakki, **agar vertikal chiziq grafikni bittadan ortiq joyidan kesib o‘tsa, bu grafik funksiya emas.**

---

1. M. L. Bittinger, D. J. Ellenbogen, S. A. Surgent “Calculus and its Applications”, USA, Springer, 10-th edition, 2012. 729 p. (P.18)

Agar grafikning ixtiyoriy joyidan o‘tkazilgan vertikal chiziq uni faqat bitta joyidan kesib o‘tsa, bu grafik funksiya bo‘ladi.

7-misol. Quyidagi grafiklarning qaysi biri funksiya bo‘ladi?



Yechilishi: ►

a) - grafik **funksiya bo‘ladi**, chunki uning ixtiyoriy nuqtasidan o‘tkazilgan vertikal chiziq uni faqat bitta joyidan kesadi;

b) - grafik **funksiya bo‘lmaydi**, chunki uning ixtiyoriy nuqtasidan o‘tkazilgan vertikal chiziq uni bittadan ortiq joyidan kesadi;

c) - grafik **funksiya emas**;

d) - grafik **funksiya bo‘ladi**;

e) - grafik **funksiya bo‘ladi**;

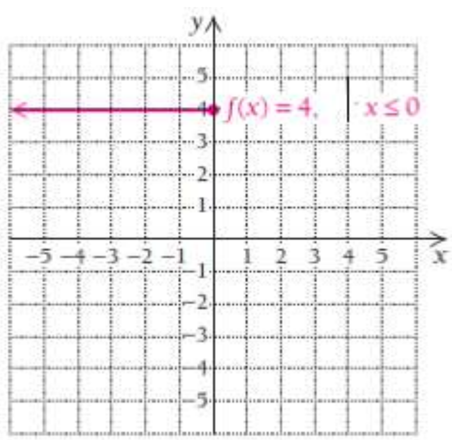
f) - grafik **funksiya emas**. ◄

1.2.4. Bo‘lakli aniqlangan funksiyalar

Ba‘zan funksiyalar bo‘lakli aniqlanadi. Shuning uchun ularning turli aniqlanish sohalari turli formulalar bilan beriladi. 7-misoldagi e) grafik bo‘lakli aniqlangan funksiyadir. Bo‘lakli aniqlangan funksiyaning grafigini chizish uchun odatda aniqlanish sohasi bo‘ylab, chapdan o‘ngga tomon harakatlanamiz, ya’ni gorizontal o‘qning har bir qismidagi x qiymat uchun funksiya bo‘lagi aniqlanganmi? degan savolga javob izlaymiz.

8-misol. $$f(x)=\begin{cases}4, & \text{agar } x\le 0 \text{ bo'lsa;}\\ 3-x^2, & \text{agar } 0<x\le 2 \text{ bo'lsa;}\\ 2x-6, & \text{agar } x>2 \text{ bo'lsa.}\end{cases}$$ bo‘lakli aniqlangan funksiyada **1-bo‘lak:** barcha 0 dan kichik yoki 0 ga teng bo‘lgan x lar uchun natija 4 ga teng; **2-bo‘lak:** barcha 0 dan katta va 2 dan kichik yoki teng bo‘lgan x lar uchun natija $3-x^2$ ga teng; **3-bo‘lak:** barcha 2 dan katta x lar uchun natija $2x-6$ ga teng. Funksiya grafigini yasang.

Yechilishi: ► 1) X o‘qi bo‘ylab, chapdan o‘ngga harakatlanamiz.

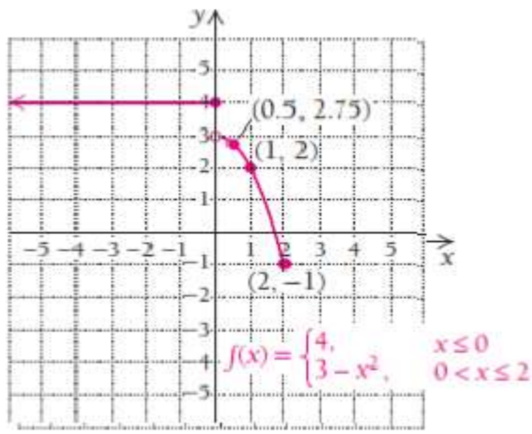


x lar 0 dan kichik yoki 0 ga teng bo‘lganda funksiya har doim 4 ga teng, ya‘ni $(-\infty;0]$ oraliqda funksiya gorizontal chiziqdan iborat. Bo‘yalgan nuqta – bu nuqta ham oraliqqa tegishli ekanini bildiradi.

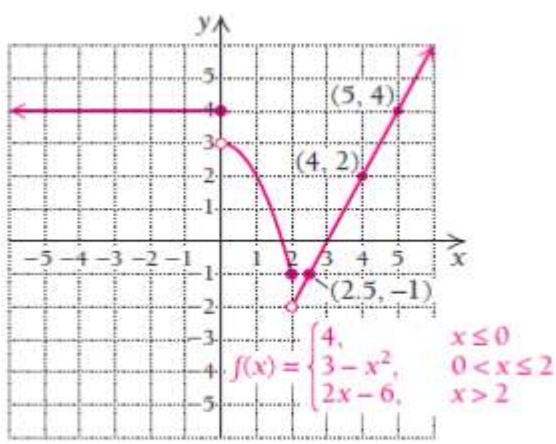
$$f(-5)=4, \qquad f(-1)=4, \qquad f(0)=4.$$

2) Ikkinchi bo‘lakdagi funksiyaning aniqlanish sohasi $(0;2]$ oraliq bo‘lib, shu oraliqda funksiya $f(x)=3-x^2$ ga teng.

$$f(0.5)=3-0.5^2=2.75, \qquad f(1)=3-1^2=2, \qquad f(2)=3-2^2=-1.$$



3) Uchinchi bo‘lakdagi funksiyaning aniqlanish sohasi $(2;\infty)$ oraliq bo‘lib,



shu oraliqda funksiya $f(x)=2x-6$ ga teng.

$$f(2.5) = 2 \cdot 2.5 - 6 = -1,$$

$$f(4) = 2 \cdot 4 - 6 = 2,$$

$$f(5) = 2 \cdot 5 - 6 = 4. \ ◄$$

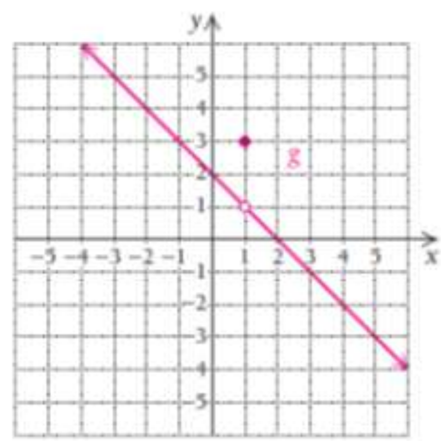
5-vazifa. Bo‘lakli aniqlangan funksiya grafigini mustaqil yasang:

$$f(x) = \begin{cases} 4, \ \text{agar} \ \ x \le 0 \ \text{bo'lsa;} \\ 4 - x^2, \ \text{agar} \ \ 0 < x \le 2 \ \text{bo'lsa;} \\ 2x - 6, \ \text{agar} \ \ x > 2 \ \text{bo'lsa.} \end{cases}$$

9-misol. $f(x) = \begin{cases} 3, \ \ \text{agar} \ \ x = 1 \\ 2 - x, \ \text{agar} \ \ x \ne 1 \end{cases}$ bo‘lakli aniqlangan funksiya grafigini yasang.

Yechilishi: ► Sonlar juftliklarini jadval yordamida aniqlab olamiz va koordinata tekisligiga belgilaymiz.

x	$g(x)$	$(x, g(x))$
-3	$-(-3) + 2$	$(-3, 5)$
0	$-0 + 2$	$(0, 2)$
1	3	$(1, 3)$
2	$-2 + 2$	$(2, 0)$
3	$-3 + 2$	$(3, -1)$



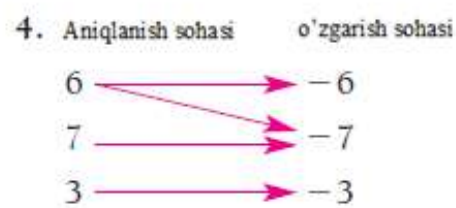
◄

6-vazifa. Bo‘lakli aniqlangan funksiya grafigini mustaqil yasang:

$$f(x) = \begin{cases} 1, \ \ \text{agar} \ \ x = -2 \\ 2 - x, \ \text{agar} \ x \ne -2 \end{cases}$$

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-6 misollarda keltirilgan diagrammalar funksiya bo‘ladimi?



5-16 misollarda berilgan ma‘lumotlarning qaysilari funksiya bo‘la oladi?

	Aniqlanish sohasi	Moslik	O‘zgarish sohasi
5.	iPodlar to‘plami	Har bir iPodning gigabaytlardagi xotirasi	Sonlar to‘plami
6.	iPodlar to‘plami	Har bir iPod egasi	Odamlar to‘plami
7.	iPodlar to‘plami	Har bir iPoddagi qo‘shiqlar to‘plami	Sonlar to‘plami
8.	iPodlar to‘plami	Har bir iPoddagi M. Rizayevaning qo‘shiqlari soni	Sonlar to‘plami
9.	Barcha haqiqiy sonlar to‘plami	Har bir sonni kvadratga oshirib, so‘ngra 8 ni qo‘shish	8 ga teng va undan katta bo‘lgan barcha musbat sonlar to‘plami

10.	Barcha haqiqiy sonlar to‘plami	Har bir sonni 4-darajaga oshirish	Barcha nomanfiy son to‘plami
11.	Ayollar to‘plami	Har bir odam biologik ona	Ayollar to‘plami
12.	Erkaklar to‘plami	Har bir odam biologik ota	Erkaklar to‘plami
13.	Tumandagi daha (kvartal)lar to‘plami	Yo‘llar kesishmasi	Ulangan ko‘chalar to‘plami
14.	Darsliklar to‘plami	Har bir kitobning juft raqamli sahifasi	Sahifalar to‘plami
15.	Geometrik shakllar to‘plami	Har bir shaklning yuzasi	Yuzalarning qiymatlari to‘plami
16.	Geometrik shakllar to‘plami	Har bir shaklning perimetri	Uzunlik qiymatlari to‘plami

17. Funksiya ixtiyoriy x sonini 4 ga ko‘paytiradi va hosil bo‘lgan sondan 5 ni ayiradi.

a) Funksiyaning analitik ko‘rinishini yozing.

b) Jadvalni to‘ldiring:

x	3	3.1	3.01	3.001
$f(x)$				

c) Funksiyaning $f(-3),\ f(1),\ f(k),\ f(1+t),\ f(x+h)$ qiymatlarini hisoblang.

18. Funksiya ixtiyoriy x sonini 4 ga ko‘paytiradi va hosil bo‘lgan songa 3 ni qo‘shadi.

a) Funksiyaning analitik ko‘rinishini yozing.

b) Funksiyaning $f(-3),\ f(1),\ f(k),\ f(1+t),\ f(x+h)$ qiymatlarini hisoblang.

c) Jadvalni to‘ldiring:

x	5	5.1	5.01	5.001
$f(x)$				

19. Berilgan funksiya kirishni kvadratga oshiradi va undan 3 ni ayiradi: $g(x)=x^2-3$. Funksiyaning $g(-1),\ g(0),\ g(1),\ g(5),\ g(a+h),\ \dfrac{g(a+h)-g(a)}{h}$ qiymatlarini hisoblang.

20. Berilgan funksiya kirishni kvadratga oshiradi va unga 4 ni qo‘shadi: $g(x)=x^2+4$. Funksiyaning $g(-3),\ g(0),\ g(1),\ g(7),\ g(v),\ g(a+h),\ \dfrac{g(a+h)-g(a)}{h}$ qiymatlarini hisoblang.

21. $g(x)=\dfrac{1}{(x+1)^2}$ funksiya berilgan.

a) Funksiyaning $g(-3),\ g(0),\ g(1),\ g(5),\ g(a),\ g(a+h),\ \dfrac{g(a+h)-g(a)}{h}$

qiymatlarini hisoblang.

b) $g(x)=\dfrac{1}{x^2+2x+1}$ funksiya kirishda nima vazifa bajaradi. Tushuntiring.

22. $f(x)=\dfrac{1}{(x-5)^2}$ funksiya berilgan.

a) Funksiyaning $f(-1),\ f(0),\ f(1),\ f(3),\ f(a),\ f(x+h),\ \dfrac{f(x+h)-f(x)}{h}$

qiymatlarini hisoblang.

b) $f(x)=\dfrac{1}{x^2-10x+25}$ funksiya kirishda nima vazifa bajaradi. Javobingizni tushuntiring.

23- 34 misollarda funksiya grafigini yasang:

23. $f(x)=2x-7$

24. $f(x)=3x-1$

25. $f(x)=-4x$

26. $f(x)=-3x$

27. $f(x)=x^2+3$

28. $g(x)=x^2-3$

29. $f(x)=1-x^2$

30. $f(x)=x^3+1$

31. $f(x)=\dfrac{x^2}{2}$

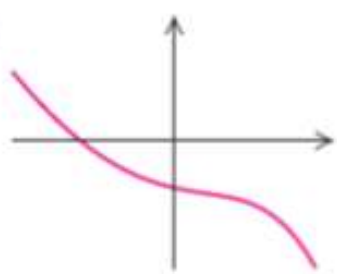
32. $f(x)=\dfrac{x^3}{3}$

33. $g(x)=6-x^2$

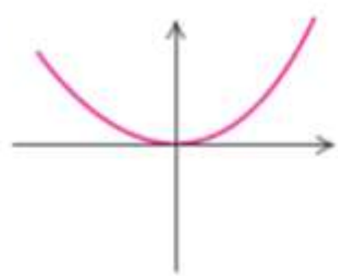
34. $g(x)=\dfrac{1}{3}x^2+1.$

35- 48 misollarda grafiklarni vertikal chiziq yordamida tekshirib, funksiya bo‘lishi yoki bo‘lmasligini aniqlang:

35.



36.



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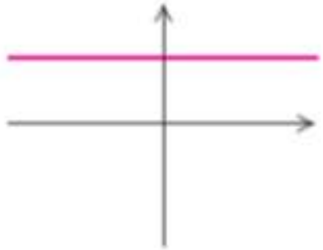
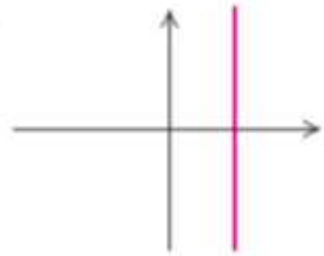
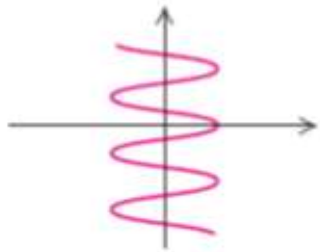


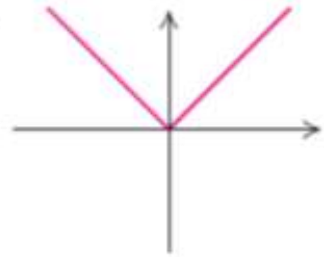
38.



39.



40.



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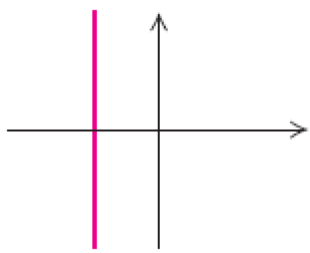
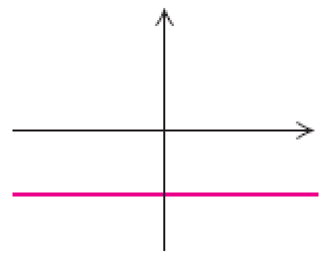
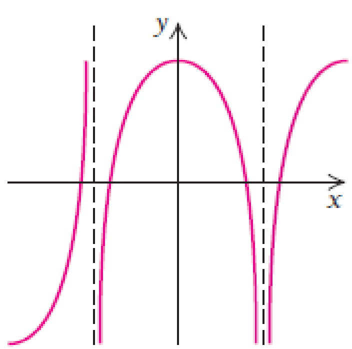


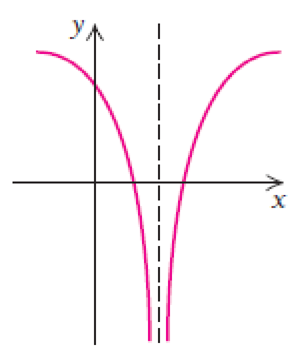
42.



43.



44.



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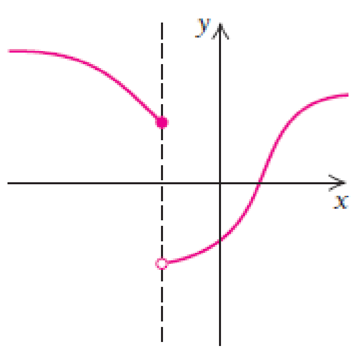
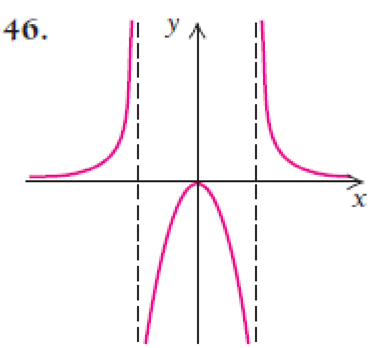


46.



47. a) $x = y^2 - 2$ tenglama grafigini chizing

b) $x = y^2 - 2$ tenglama funksiya bo‘ladimi?

48. a) $x = y^2 - 3$ tenglama grafigini chizing.

b) $x = y^2 - 3$ tenglik funksiya bo‘ladimi?

49. $f(x) = x^2 - 3x$ funksiyaning $\dfrac{f(x+h) - f(x)}{h}$ qiymatini aniqlang.

50. $g(x) = x^2 + 4x$ funksiyaning $\dfrac{g(x+h) - g(x)}{h}$ qiymatini aniqlang.

51-54 misollarda bo‘lakli aniqlangan funksiya qiymatlarini hisoblang:

$$f(x) = \begin{cases} 1 - 2x, & \text{agar } x < 0 \text{ bo'lsa} \\ 17, & \text{agar } x = 0 \text{ bo'lsa} \\ x^2 - 3, & \text{agar } 0 < x < 4 \text{ bo'lsa} \\ \dfrac{x}{2} + 1, & \text{agar } x \geq 4 \text{ bo'lsa} \end{cases}$$

51. Funksiyaning $f(-1)$ va $f(1)$ qiymatlarini aniqlang.

52. Funksiyaning $f(-3)$ va $f(3)$ qiymatlarini aniqlang.

53. Funksiyaning $f(0)$ va $f(10)$ qiymatlarini aniqlang.

54. Funksiyaning $f(-5)$ va $f(5)$ qiymatlarini aniqlang.

55-64 misollarda bo‘lakli aniqlangan funksiyalar grafiklarini chizing:

55. $f(x) = \begin{cases} 2, & \text{agar } x < 0 \\ -2, & \text{agar } x \geq 0 \end{cases}$

56. $f(x) = \begin{cases} 1, & \text{agar } x < 3 \\ -1, & \text{agar } x \geq 3 \end{cases}$

57. $f(x)=\begin{cases}6, & \text{agar } x=-2\\ x^2, & \text{agar } x\neq -2\end{cases}$

58. $f(x)=\begin{cases}3, & \text{agar } x=1\\ x^3, & \text{agar } x\neq 1\end{cases}$

59. $h(x)=\begin{cases}-x, & \text{agar } x<0\\ 4, & \text{agar } x=0\\ x+1, & \text{agar } x>0\end{cases}$

60. $g(x)=\begin{cases}1-2x, & \text{agar } x<1\\ 4, & \text{agar } x=1\\ x^2+1, & \text{agar } x>1\end{cases}$

61. $f(x)=\begin{cases}0{,}5x-1, & \text{agar } x<2\\ 1, & \text{agar } x=2\\ x-3, & \text{agar } x>2\end{cases}$

62. $f(x)=\begin{cases}x^2-1, & \text{agar } x<0\\ 1, & \text{agar } x=0\\ 2x-3, & \text{agar } x>0\end{cases}$

63. $f(x)=\begin{cases}4, & \text{agar } x=-2\\ x^2-1, & \text{agar } x\neq -2\end{cases}$

64. $f(x)=\begin{cases}-6, & \text{agar } x=-3\\ 5-x^2, & \text{agar } x\neq -3\end{cases}$

Murakkab foiz. Omonat kassada t yilga har kvartalda 6 % li jamg'arib boriladigan omonat $A(t)$ ga teng. Agar jamg'armaga P dollar qo'yilgan bo'lsa, daromad

$$A(t)=P\left(1+\frac{0.06}{4}\right)^{4t}$$

formula bilan hisoblanadi.

65-66 misollarda murakkab foiz formulasidan foydalaning:

65. Agar omonat kassaga har kvartalda 6 % jamg'arib borish uchun 500 $ qo'yilgan bo'lsa, 2 yildan so'ng qancha mablag' to'planadi?

66. Agar omonat kassaga har kvartalda 6 % jamg'arib borish uchun 800 $ qo'yilgan bo'lsa, 3 yildan so'ng qancha mablag' to'planadi?

Kimyoterapiya. Kimyoterapiyada nur dozasini hisoblashda bemor tanasining yuzasi zarur. Bemorning s tana yuzasi m^2 birlikda $s=\sqrt{\frac{hw}{3600}}$ formula bilan hisoblanadi. Bunda w – bemorning vazni, kg birlikda; h – bemorning bo‘yi, sm.

67-68 misollarda kimyoterapiya formulasidan foydalaning:

67. Bemorning bo‘yi 170 sm. Quyidagilar asosida bemorning tana yuzasini taqribiy hisoblang:

a) Bemor vazni 70 kg;

b) Bemor vazni 100 kg;

c) Bemor vazni 50 kg.

68. Bemorning vazni 70 kg. Quyidagilar asosida bemorning tana yuzasini taqribiy hisoblang:

a) Bemorning bo‘yi 150 sm;

b) Bemorning bo‘yi 180 sm.

Stress sabablarini baholash. Psixologiyada turli hayotiy voqea-hosidalar guruhiga qarab stress holati sonli baholanadi. Jadvalda turli hodisalar ularning ta‘sir darajasiga qarab, 1 dan 100 gacha oraliqda baholangan.

69.

Voqea-hodisalar	Ta'sir kuchi
Umr yo'ldoshining o'limi	100
Ajrashish	73
Qamoqqa tushish	63
Turmush qurish	50
Ishini yo'qotish	47
Homiladorlik	40
Yaqin do'stining o'limi	37
10 000$ dan ortiq ssuda olish	31
Bo'lak chiqqan farzand	29
Maktabni o'zgartirish	20
10 000$ dank am ssuda olish	17
Yangi yil	12

a) Jadval funksiya bo‘ladimi yoki yo’qmi? Nima uchun?

b) Funksiyaning kirishi qanday bo‘ladi? Chiqish-natija qanday ko‘rinishda?

70 -71 misollarda keltirilgan diagrammalar funksiya bo‘ladimi?

70. Sendvichlar narxlari:

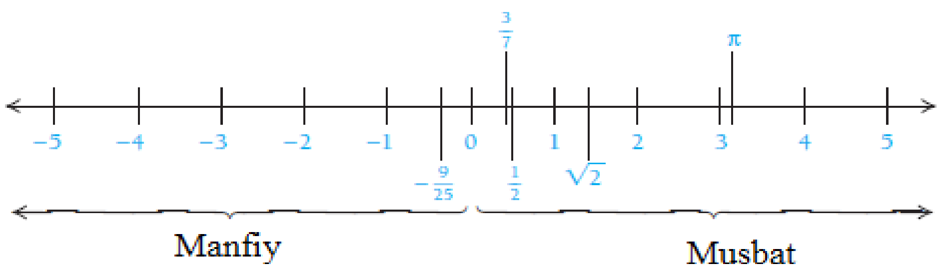
71. Sendvichlar kaloriyalari.

72. $f(x) = |x-2| + |x-1| - 5$ funksiya berilgan bo‘lsa, uning $f(-3)$, $f(-2)$, $f(0)$, $f(4)$ qiymatlarini toping.

73. $f(x) = |x+2| + |x+1| - 3$ funksiya grafigini chizing va uning $f(-3)$, $f(-2)$, $f(0)$, $f(4)$ qiymatlarini toping.

1.3. Funksiyaning aniqlanish va o‘zgarish sohalari

To‘plam – ob’yektlar uyumidir. Biz hisob fanida ko‘pincha haqiqiy sonlar to‘plamini qaraymiz. Sonlar o‘qining har bir nuqtasiga faqat bitta haqiqiy son mos qo‘yiladi.

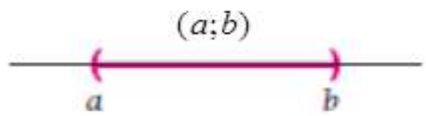


$-\dfrac{9}{25}$, 0, $\sqrt{2}$ sonlaridan iborat to‘plam $\left\{-\dfrac{9}{25}; 0; \sqrt{2}\right\}$ ko‘rinishda yoziladi. Bu usulga to‘plamni **ro‘yxat ko‘rinishi** deyiladi. Ro‘yxatda to‘plamning barcha elementi keltiriladi. To‘plamni berishning yana bir usuli **xarakteristik predikat** deb nomlanadi. **Xarakteristik predikat** usulida to‘plamga kirgan element qanday xossalarni qanoatlantirishi kerakligi beriladi. Masalan, 4 dan kichik bo‘lgan barcha haqiqiy sonlar to‘plami quyidagicha yoziladi: $\{x \,|\, x \in R,\ x < 4\}$ yoki qisqacha $\{x \,|\ x < 4\}$.

1.3.1. To‘plamning oraliq shaklida berilishi

To‘plamni ro‘yxat, xarakteristik predikat shaklida berish bilan birga **oraliq** shaklda ham tasvirlash mumkin. Agar a va b lar haqiqiy sonlar va

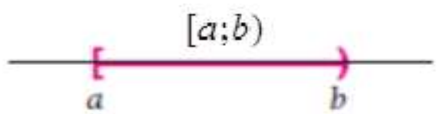
$a<b$ bo‘lsa, u holda a va b haqiqiy sonlar orasidagi barcha sonlar to‘plamini $(a;b)$ deb belgilaymiz. Bu oraliqning chetlaridagi a va b sonlar to‘plamga kirmaydi, ya‘ni to‘plamning elementlari shunday x haqiqiy sonlarki, ular a va b sonlar orasida yotadi:



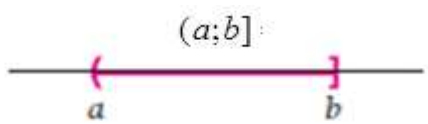
$$(a;b)=\{x\mid x\in R,\ a<x<b\} \text{ yoki } (a;b)=\{x\mid a<x<b\}.$$

Oddiy qavslar oraliq chetidagi elementlar to‘plamga kirmasligini bildiradi. Oddiy qavs bilan yozilgan oraliq **ochiq oraliq** deyiladi.

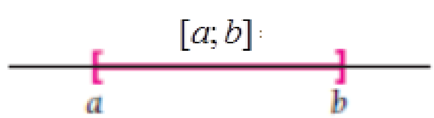
$$[a;b)=\{x\mid x\in R,\ a\le x<b\}$$



ko‘rinishdagi oraliqqa **yarim ochiq oraliq** deyiladi. Unda oraliqning chap chetidagi a element to‘plamga tegishli, lekin o‘ng tomonidagi b element to‘plamga tegishli emas.

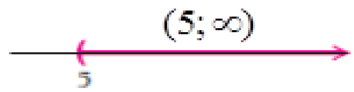


$(a;b]=\{x\mid x\in R,\ a<x\le b\}$ - **yarim ochiq oraliq**.

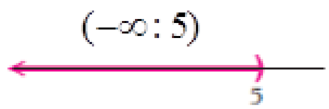


$[a;b]=\{x\mid x\in R,\ a\le x\le b\}$ - **yopiq oraliq yoki kesma** deyiladi.

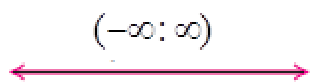
Oraliqning bir tomoni yoki ikkala chegarasi ham cheksizlikdan iborat bo‘lishi mumkin. Aytaylik, $(5;\infty)$ to‘plamning elementlari 5 dan katta haqiqiy sonlardan iborat:



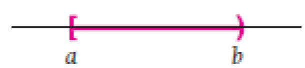
Yoki 5 dan kichik haqiqiy sonlar to‘plamini $(-\infty:5)$ yozmoqchi bo‘lsak, u holda quyidagi chizma hosil bo‘ladi:



Barcha haqiqiy sonlar $(-\infty:\infty)$ to‘plamini tasvirlaylik:



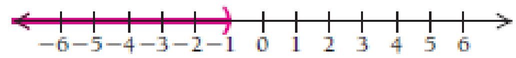
To’plamlar va grafiklar

Oraliq ko’rinishi	To’plam ko’rinishi	Grafiklar
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \le x \le b\}$	
$[a, b)$	$\{x \mid a \le x < b\}$	
$(a, b]$	$\{x \mid a < x \le b\}$	
(a, ∞)	$\{x \mid x > a\}$	
$[a, \infty)$	$\{x \mid x \ge a\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \le b\}$	
$(-\infty, \infty)$		

1-misol. Quyidagi oraliqlarni grafik yoki to‘plam shaklida yozing:

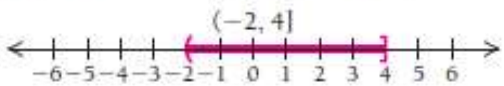
a) $\{x| -4 < x < 5\}$ b) $\{x|x \geq -2\}$

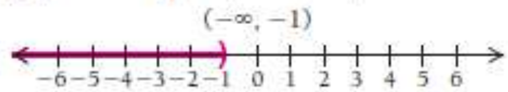
c) 

d) 

Yechilishi: ►

a) $\{x| -4 < x < 5\} = (-4, 5)$ b) $\{x|x \geq -2\} = [-2, \infty)$

c) 

d) 

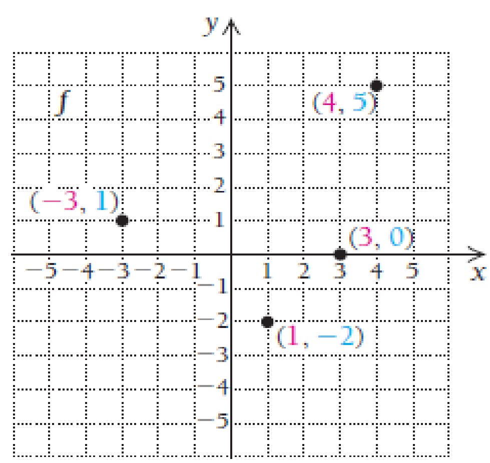
◄

1-vazifa. Quyidagi oraliqlarni mustaqil ravishda grafik yoki to‘plam shaklida yozing: **a)** $\{x| -3 < x < 4\}$; **b)** $\{x| -3 \leq x < 4\}$
c) $\{x| -3 < x \leq 4\}$; **d)** $\{x| -3 \leq x \leq 4\}$

1.3.2. Funksiyaning aniqlanish va o‘zgarish sohalarini topish

Bilamizki, funksiyaning biror nuqtasini ifodalovchi (a,b) sonlar juftligida 1-son aniqlanish sohasiga, 2-son o‘zgarish sohasiga tegishli.

2-misol. Quyidagi chizmadan nuqtalarning aniqlanish va o‘zgarish sohalarini toping:



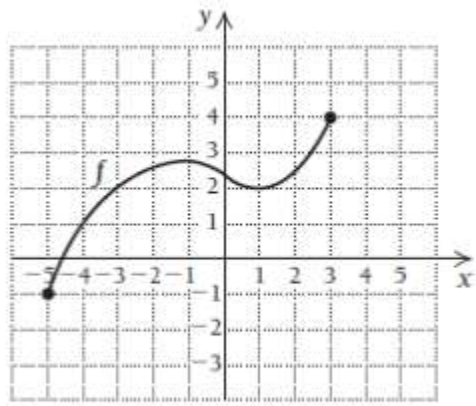
Yechilishi:

► Diagrammadan nuqtalarning joylashuviga qarab, chapdan o‘ngga tomon quyidagi to‘plamni yozamiz: $\{(-3,1);(1,-2);(3,0);(4,5)\}$.

Aniqlanish sohasi: $\{-3;\ 1;3;4\}$.

O‘zgarish sohasi: $\{1;\ -2;\ 0;5\}$. ◄

3-misol. Diagrammaga qarab, quyidagilarni toping:



a) Aniqlanish sohasidagi 1 soniga mos keluvchi o‘zgarish sohasi elementini toping $f(1)=?$

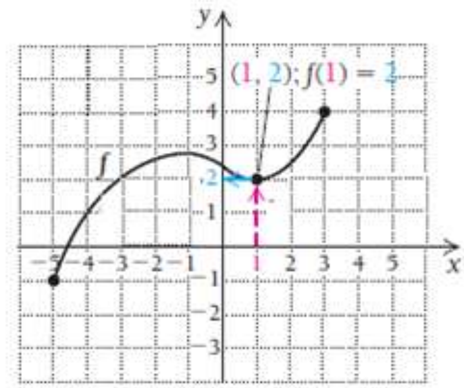
b) f ning aniqlanish sohasini toping.

c) $f(x)=1$ tenglikni qanoatlantiradigan barcha x larni toping.

d) f ning o‘zgarish sohasini toping.

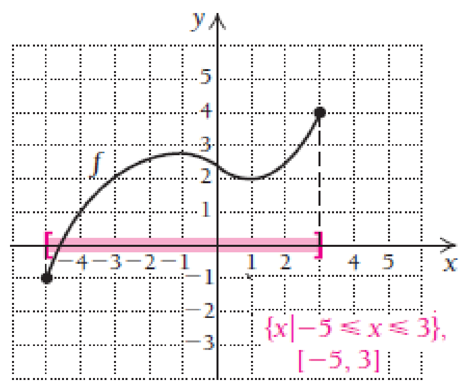
Yechilishi: ►

a) Funksiyaning aniqlanish sohasi gorizontal o‘qda yotuvchi sonlardan iborat. O‘zgarish sohasi elementlari esa vertikal o‘qda yotadi. Shuning uchun gorizontal o‘qdagi 1 soniga mos keluvchi o‘zgarish sohasi elementini topamiz: $f(1)=2$.

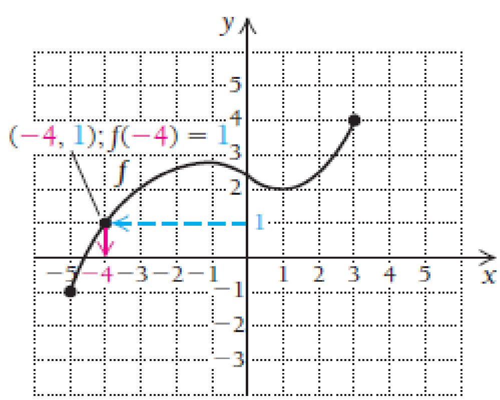


b) Funksiyaning aniqlanish sohasi X o‘qida bo‘lishini e’tiborga olsak, funksiya -5 dan 3 gacha aniqlanganini ko‘rish mumkin:

$$D(f)=\{x \mid -5 \le x \le 3\} \quad \text{yoki} \quad D(f)=[-5;\ 3]$$



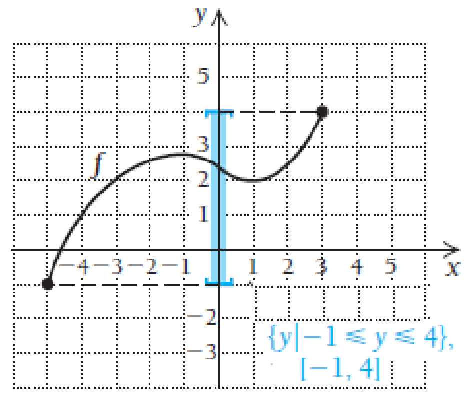
c) $f(x)=1$ tenglikni qanoatlantiradigan barcha x larni topamiz.



Buning uchun vertikal o‘qdan 1 ni topib olib, unga mos funksiya qiymati -4 ekanini aniqlaymiz: $f(-4)=1$.

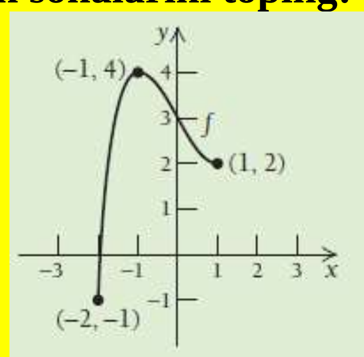
d) f ning o‘zgarish sohasini topamiz. O‘zgarish sohasi funksiyaga mos keladigan vertikal o‘qdagi elementlar bo‘ladi:

$$E(f)=\{y \mid -1 \le x \le 4\} \text{ yoki } E(f)=[-1;\ 4]$$



◄

2-vazifa. Funksiyaning aniqlanish va o‘zgarish sohalarini toping:



$f(1)=?$ $f(-1)=?$

Agar funksiya tenglama yoki formula bilan berilsa, uning aniqlanish sohasi cheksiz sonlar to‘pamidan, o‘zgarish sohasi esa chekli sonlar to‘plamidan iborat bo‘lishi ham mumkin.

4-misol. $f(x)=|x|$ funksiyaning aniqlanish va o‘zgarish sohalarini toping.

Yechilishi: ► x ning o‘rniga qanday sonlarni qo‘yish mumkin? Degan savolga javob izlaymiz. $f(x)=|x|$ tenglikda x ning o‘rniga har qanday haqiqiy sonni qo‘ysak ham tenglik ma‘noga ega bo‘ladi:

$$f(-3)=|-3|=3; \qquad f(0)=|0|=0; \qquad f(5)=|5|=5.$$

Shuning uchun funksiyaning aniqlanish sohasi $D(f)=R=(-\infty;\infty)$ dan iborat. O‘zgarish sohasi esa faqat nomanfiy sonlardan iborat $E(f)=[0;\ \infty]$. ◄

5-misol. $y=\dfrac{3}{x}$ funksiyaning aniqlanish va qiymatlar sohalarini toping.

Yechilishi: ► Ma’lumki, kasr son ma’noga ega bo‘lishi uchun uning maxraji noldan farqli bo‘lishi kerak. Shuning uchun, $x\neq 0$ deb olamiz, ya‘ni funksiyaning aniqlanish sohasi 0 dan boshqa barcha sonlarni o‘z ichiga oladi:
$$D(f)=(-\infty;0)\cup(0;\infty).$$

Qiymatlar sohasi esa $E(f)=(-\infty;0)\cup(0;\infty)$ ekanligiga funksiya grafigini chizib ishonch hosil qilish mumkin.◄

6-misol. $y=\dfrac{1}{2}(x-1)^{-1}$ funksiyaning aniqlanish sohasini toping.

Yechilishi: ► Ushbu funksiyani boshqacha ko‘rinishda yozish mumkin: $y=\dfrac{1}{2(x-1)}$. Endi xuddi oldingi misoldagidek muhokama yuritamiz, $2(x-1)\neq 0$ yoki $x\neq 1$. Demak, aniqlanish sohasi $D(f)=(-\infty;1)\cup(1;\infty)$ dan iborat. Qiymatlar sohasini esa grafik chizib ko‘rish mumkin.◄

7-misol. $y=\sqrt{3x+2}$ funksiyaning aniqlanish va qiymatlar sohalarini toping.

Yechilishi: ► Kvadrat ildiz ma’noga ega bo‘lishi uchun ildiz ostidagi ifoda manfiy bo‘lmasligi kerak, ya‘ni $3x+2\geq 0$, bunda $x\geq -\dfrac{2}{3}$. Demak, aniqlanish sohasi $[-\frac{2}{3}, +\infty)$ dan iborat. Aniqlanish sohasining qiymatlarini berilgan funksiya tenglamasiga qo‘yib, uning qiymatlar sohasini topish mumkin: $y(-\frac{2}{3}) = \sqrt{3\cdot\left(-\frac{2}{3}\right)+2} = 0$. Argument x ning o‘rniga cheksiz katta sonlar qo‘yilganda esa funksiya ham cheksizlikka intiladi. Bundan $E(f)=[0;\infty)$ deb qabul qilamiz. ◄

8-misol. $y=\frac{1}{\sqrt{4x-5}}$ funksiyaning aniqlanish sohasini toping.

Yechilishi: ► Agar yuqoridagidek muhokama yuritsak, $4x-5\geq 0$ bo‘lishi kerak, lekin bu son kasrning maxrajida bo‘lganligi uchun $4x-5>0$ o‘rinli bo‘ladi. Bundan $x>\frac{5}{4}$. Demak, funksiyaning aniqlanish sohasi $D(f)=(\frac{5}{4};\infty)$ dan iborat ekan. ◄

9-misol. $y=\lg(2x-1)$ funksiyaning aniqlanish sohasini toping.

Yechilishi: ► Logarifmik funksiya faqat musbat sonlarda aniqlangan. Shuning uchun, (2*x*-1)>0 bo‘lishi kerak. Bundan $x>\frac{1}{2}$ kelib chiqadi. Demak, aniqlanish sohasi $D(f)=(\frac{1}{2};\infty)$ dan iborat. ◄

10-misol. $y=\frac{1}{\lg(2x-1)}$ funksiyaning aniqlanish sohasini toping.

Yechilishi: ► Bu misolda ham logarifmik funksiyalar faqat musbat sonlar uchun aniqlanganini hisobga olsak,

$\begin{cases} 2x-1>0 \\ 2x-1\neq 1 \end{cases}$ bo‘ladi. Bundan $\begin{cases} x>\frac{1}{2} \\ x\neq 1 \end{cases}$ kelib chiqadi. Demak, funksiyaning aniqlanish sohasi $D(f)=(\frac{1}{2};1)\cup(1;\infty)$ dan iborat. ◄

1.3.3. Aniqlanish va o‘zgarish sohalarining tatbiqlari

11-misol. **Tadbirkorlik: murakkab foiz.** Faraz qilaylik, 500 $ ni t yilga har kvartalda 6 % daromad beradigan ishga tikdik. 1.1- bo‘limdagi 2-teoremaga ko‘ra $$A=P\left(1+\frac{i}{n}\right)^{nt}.$$

P=500 $, $i=6\%=0.06$, $n=4$, t qiymatlarni formulaga qo‘yamiz.

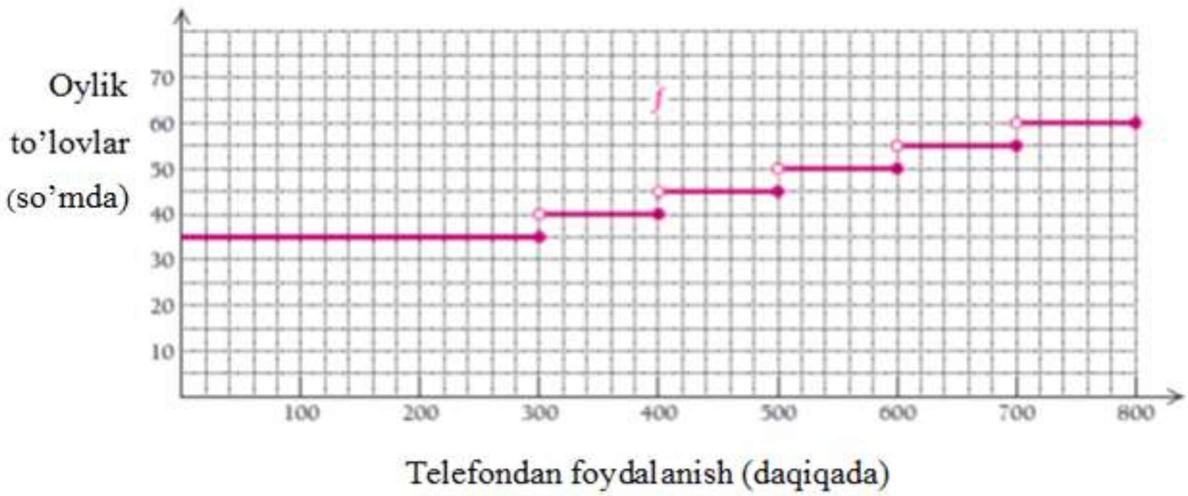
$$A=P\left(1+\frac{i}{n}\right)^{nt}=500\cdot\left(1+\frac{0.06}{4}\right)^{4t}=500\cdot(1.015)^{4t}$$

Funksiya A mablag‘ qo‘yilgan yillar davomidagi pullar miqdori. Funksiyaning aniqlanish sohasini toping.

Yechilishi: ► A funksiya t ga bog‘liq ravishda o‘zgaradi, ya‘ni t ning qabul qiladigan qiymatlari to‘plami funksiyaning aniqlanish sohasi bo‘ladi.

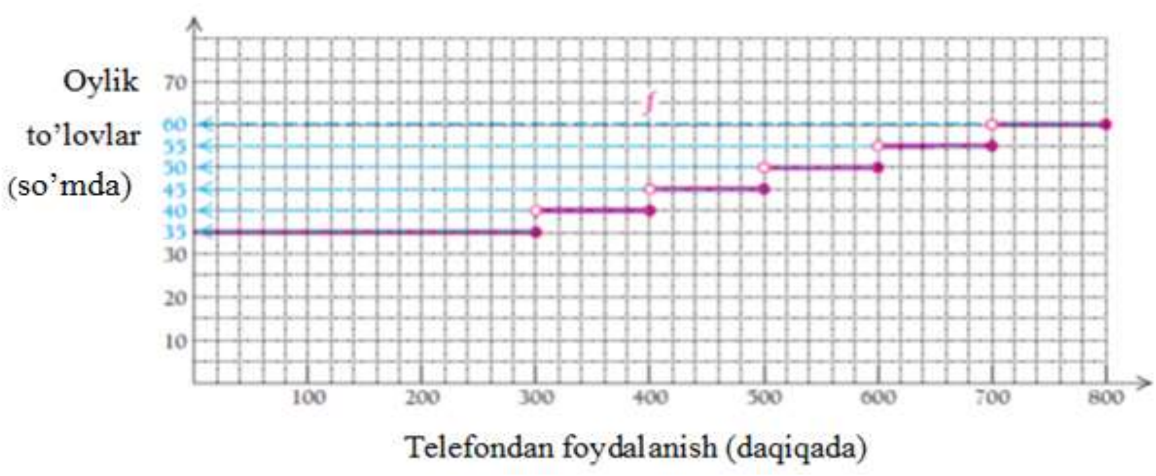
Formuladagi t kattalikni ixtiyoriy musbat haqiqiy son bilan almashtirish mumkin. Lekin t ning o‘rniga manfiy son qo‘yish mumkin emas, chunki yilning manfiy son bo‘lishi ma‘noga ega emas. Manfiy sonlarni chiqarib tashlasak, u holda funksiyaning aniqlanish sohasi barcha nomanfiy sonlar to‘plamidan iborat bo‘ladi: $D(A)=[0;\infty)$. ◄

12-misol. **Qo‘l telefonidan foydalanish ko‘rsatkichlari.** Yaqinda Ucell kompaniyasi mijozlarning talab rejasini, ya‘ni oylik hisobini quyidagi diagramma ko‘rinishida modellashtirishni taklif qildi. Keltirilgan funksiyaning o‘zgarish sohasini toping.



Yechilishi: ► Ushbu misolda funksiyaning o‘zgarish sohasi diagrammada keltirilgan turli oylik to‘lovlar hisoblanadi. Ko‘rib turibmizki, diagrammada ko‘k rang bilan berilgan 6 xil to‘lov summasi keltirilgan. Shunday qilib, funksiyaning o‘zgarish sohasini quyidagicha yozish mumkin:

$$\{35, 40, 45, 50, 55, 60\}$$



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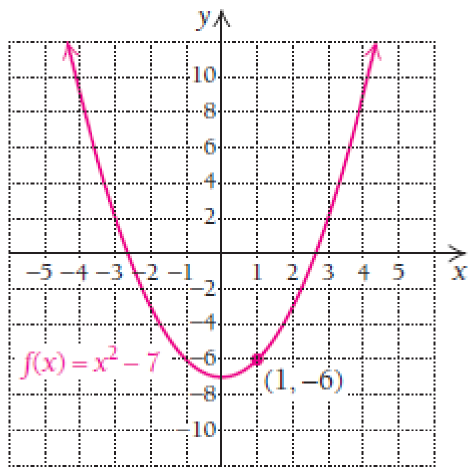
13-misol. $f(x)=x^2-7$ funksiya berilgan.

a) Kirishda 1 bo‘lganda, chiqishdagi qiymatni toping.

b) Funksiyaninf aniqlanish sohasi $D(f)=?$

c) Funksiyaning o‘zgarish sohasi $E(f)=?$

Yechilishi: ► Avvalo funksiya grafigini chizib olamiz.



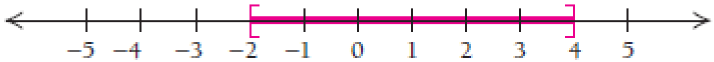
a) Grafikdan ko‘rish mumkinki, kirishda 1 bo‘lganda, chiqishdagi 1 ga mos qiymat -6 ga teng. Buni tenglamadan topish ham mumkin: $f(1)=1^2-7=-6$. Demak, $(1,\ -6)$ nuqta funksiya grafigiga tegishli.

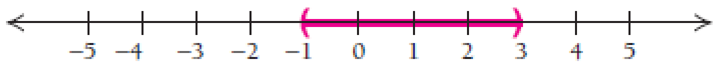
b) Funksiyaninf aniqlanish sohasi barcha kirishlardan iborat: $D(f)=(-\infty,\infty)$ yoki barcha haqiqiy sonlar to‘plamiga teng: $D(f)=R$;

c) Funksiyaning o‘zgarish sohasini grafikdan aniqlasak, $E(f)=[-7,\infty)$ bo‘ladi. ◄

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-10 misollarda grafiklarni oraliq va to‘plam ko‘rinishida ifodalang:

1.


2.


3.


4.


5.


6.


7.


8.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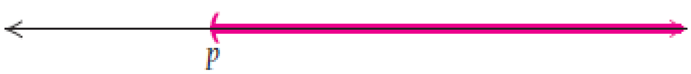


10.


11-20 misollarda oraliq va to‘plamlarni sonlar o‘qida grafik (diagramma) ko‘rinishida ifodalang:

11. $1 \le x \le 7$ oraliqdagi barcha sonlar;

12. $-7 < x \le 7$ oraliqqa tegishli barcha sonlar;

13. $\{x \mid x \in \mathrm{R},\ -2 < x \le 5\}$;

14. $\{x \mid x \in \mathrm{R},\ -5 < x < 6\}$;

15. $\{x \mid x \in \mathrm{R},\ x \le 5\}$;

16. $\{x \mid x \in \mathrm{R},\ x > -2\}$;

17. $\{x \mid x \in \mathrm{R},\ x \ge 7.5\}$;

18. $\{x \mid x \in \mathrm{R},\ x < 7.5\}$;

19. $\{x \mid x \in \mathrm{R},\ 3 \le x < 10\}$;

20. $\{x \mid x \in \mathrm{R},\ -6 \le x \le 6\}$

21-32 misollarda funksiyalar berilgan. Quyidagilarni toping:

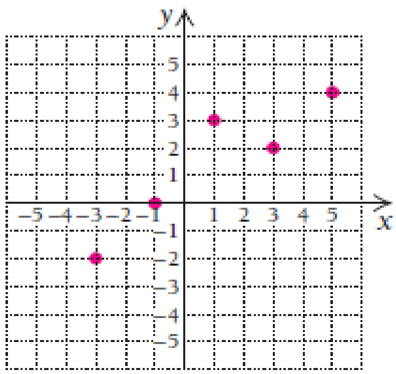
a) Kirishda 1 bo‘lganda, chiqishdagi qiymatni toping: $f(1) = ?$

b) Funksiyaninf aniqlanish sohasi $D(f) = ?$

c) Funksiyaning o‘zgarish sohasi $E(f) = ?$

d) $f(x) = 2$ bo‘lgan barcha qiymatlarni toping.

21.



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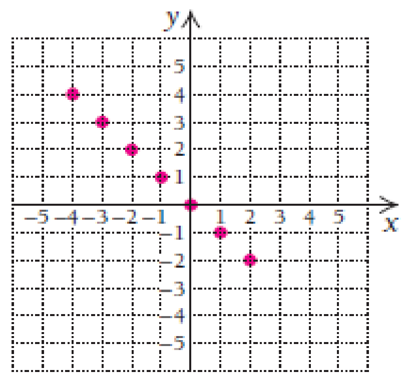
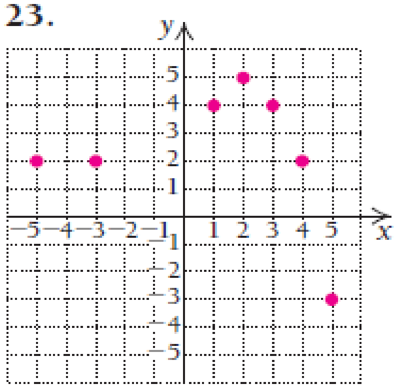


23.



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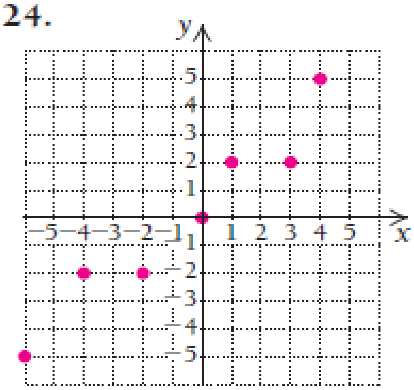

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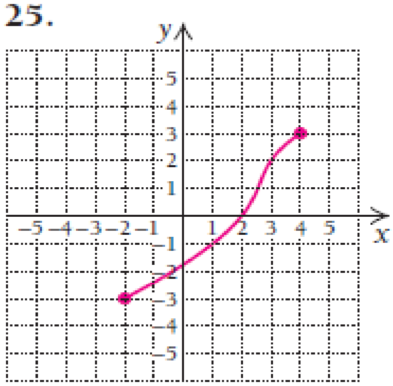

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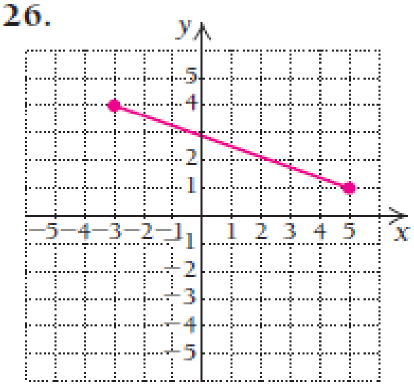

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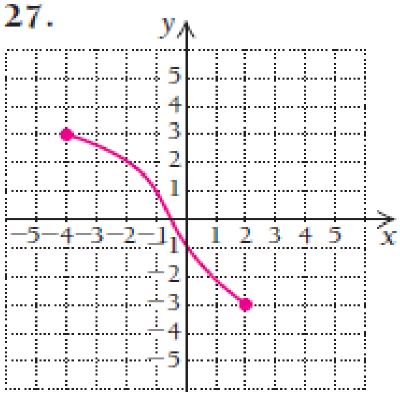

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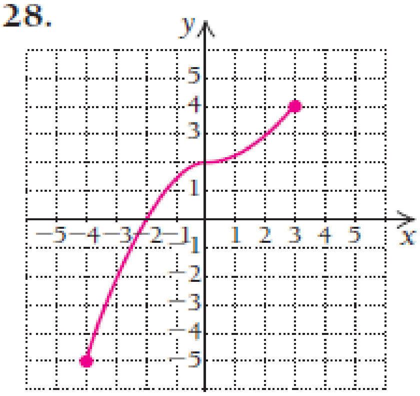

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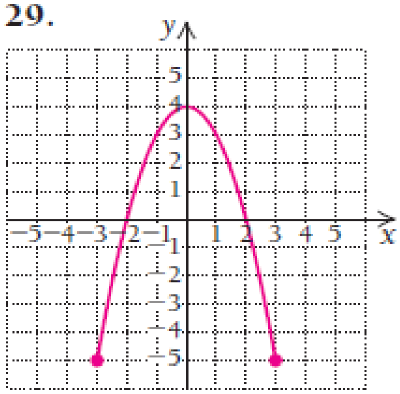

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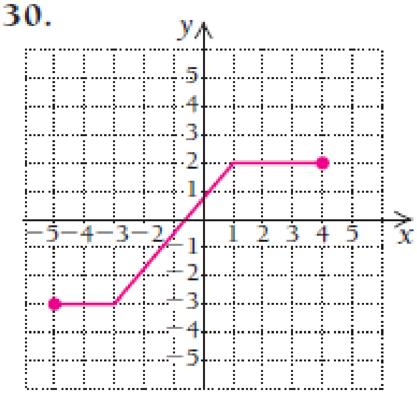

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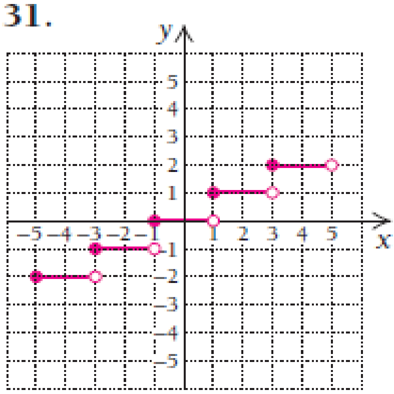

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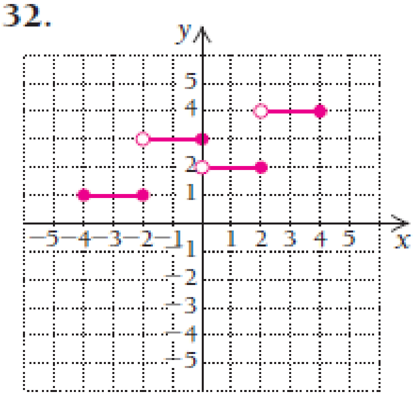


33-56 misollarda funksiyalarning aniqlanish sohasini toping.

33. $f(x)=\dfrac{2}{3-x}$; **34.** $f(x)=\dfrac{6}{2+x}$;

35. $f(x)=\sqrt{3x}$; **36.** $f(x)=\sqrt{3-x}$;

37. $f(x)=x^2-5x-6$; **38.** $f(x)=x^2-6$;

39. $f(x)=\frac{x-2}{6x-12}$;

40. $f(x)=\frac{2}{3x-1}$;

41. $f(x)=|x-5|$;

42. $f(x)=|x|-5$;

43. $f(x)=\frac{3x-2}{7-6x}$;

44. $f(x)=\frac{2x-1}{9-2x}$;

45. $f(x)=\sqrt{3x+5}$;

46. $f(x)=\sqrt{2-3x}$;

47. $h(x)=x^2-2x+1$;

48. $g(x)=4x^3+3x^2-x+6$;

49. $f(x)=\frac{2}{(3-x)^2}$;

50. $f(x)=\frac{2}{9-x^2}$;

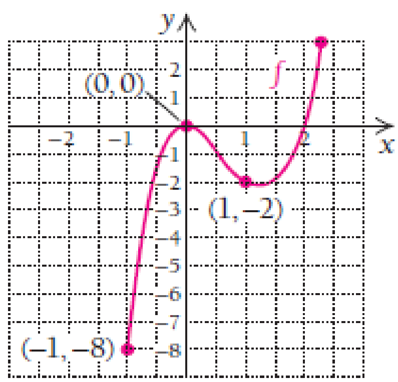
51. $f(x)=|x+7|$;

52. $f(x)=|x|+1$;

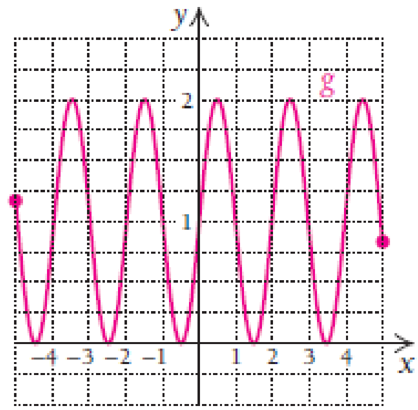
53. $f(x)=\frac{2x+1}{x^2-6x+5}$;

54. $f(x)=\frac{2x+1}{x^2-4x-5}$.

55. Grafigi berilgan funksiyaning barcha $f(x)\le 0$ tengsizlikni qanoatlantiruvchi qiymatlarini aniqlang:



56. Grafigi berilgan funksiyaning barcha $g(x)=1$ tenglikni qanoatlantiruvchi qiymatlarini aniqlang:



57- 58 misollar aniqlanish va o‘zgarish sohalarining tatbiqlariga doir:

57. Iqtisodiyot: murakkab foiz. Faraz qiling, 5000 $ ni yarim yilda 8 % daromad beradigan ishga t yilga sarfladingiz.

a) Agar A miqdor t ning funksiyasi bo‘lsa, A funksiya uchun formula yozing;

b) Funksiyaning aniqlanish sohasini toping.

58. Iqtisodiyot: murakkab foiz. Faraz qiling kollej uchun t yilda har kuni to‘lab borish sharti bilan 5 % foizga 3000 $ ssuda olindi.

a) Agar A miqdor t ning funksiyasi bo‘lsa, A funksiya uchun formula yozing;

b) Funksiyaning aniqlanish sohasini toping.

1.4. Chiziqli funksiyalar

1.4.1. Gorizontal va vertikal chiziqlar

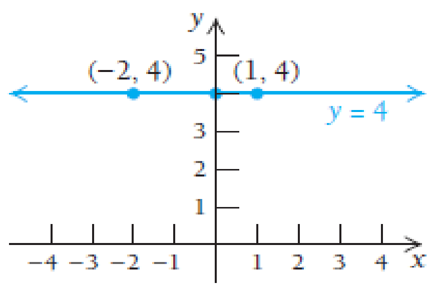
$x = a$ hamda $y = c$ tenglamalar grafiklarini qaraylik, bunda $a, c -$ haqiqiy sonlar.

1-misol. a) $y = 4$ tenglama grafigini chizing.

b) $y = 4$ grafigidan uning funksiya bo‘la olishini tekshiring.

Yechilishi: ► a) Grafik shunday sonlar juftligidan iboratki, ularning barchasida 2-koordinata 4 ga teng bo‘ladi. Aytaylik, (-2, 4) va (1, 4). Chunki $y = 4$ tenglama biz oldin tanishgan $y = 0x + 4$ tenglama bilan teng kuchli.

$y(-2) = 0 \cdot (-2) + 4 = 4$ va $y(1) = 0 \cdot 1 + 4 = 4$ munosabatlar o‘rinli.



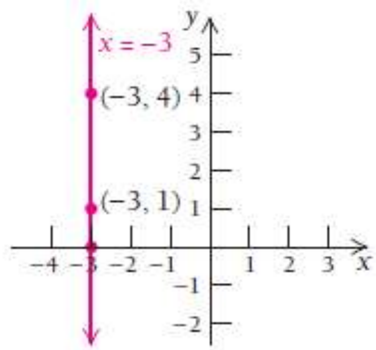
b) Vertilkal to‘g‘ri chiziq o‘tkazib, grafikni tekshirib ko‘ramiz va $y = 4$ tenglik funksiya bo‘ladi, deb xulosa qilamiz. ◄

2-misol. a) $x = -3$ tenglama grafigini chizing.

b) $x = -3$ grafigidan foydalanib, uning funksiya bo‘la olishini tekshiring.

Yechilishi: ► a) Grafik shunday sonlar juftligidan iboratki, ularning barchasida 1-koordinata -3 ga teng bo‘ladi. Aytaylik, (-3, 4) va (-3, 1).

Chunki $x=-3$ tenglama biz oldin tanishgan $x+0\cdot y=-3$ tenglama bilan teng kuchli. (-3, 4) nuqta grafikka tegishli bo‘lishini tekshiramiz: $-3+0\cdot 4=-3$ va (-3, 1) nuqtaning grafikka tegishli bo‘lishini tekshiramiz: $-3+0\cdot 1=-3$.



b) Vertilkal to‘g‘ri chiziq o‘tkazib, grafikni tekshirib ko‘ramiz va $x=-3$ tenglik **funksiya bo‘la olmaydi**, deb xulosa qilamiz. Chunki $x=-3$ vertical to‘g‘ri chiziqni o‘tkazsak, u berilgan tenglama grafigi bilan cheksiz ko‘p nuqtada kesishadi. ◄

1-vazifa. Tenglamalarning grafiklarini chizing va ular funksiya bo‘la oladimi? Tekshiring: **a)** $x=4$ **;**
b) $y=-3$ **.**

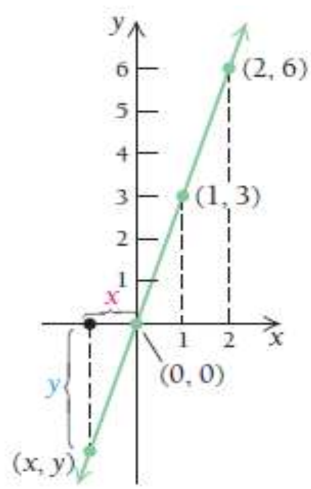
3-teorema. $y=c$ yoki $f(x)=c$ ning grafigi gorizontal chiziq bo‘lib, bu grafik funksiya bo‘ladi. Unga **o‘zgarmas funksiya** deyiladi. Grafigi vertikal chiziqdan iborat bo‘lgan $x=a$ tenglama esa funksiya bo‘la olmaydi.

1.4.2. $y = mx$ ko‘rinishidagi tenglamalar

Quyidagi jadvalni o‘rganib, unga mos tenglamalarni topishga harakat qilaylik:

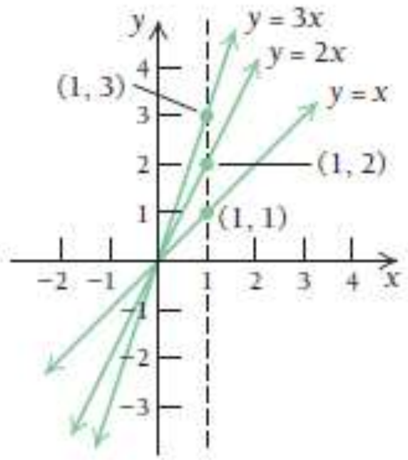
x	1	-1	$-\frac{1}{2}$	2	-2	3	-7	5
y	3	-3	$-\frac{3}{2}$	6	-6	9	-21	15

Ustunlarni qarab chiqib, y ning x ga nisbati 3 ga teng ekanini ko‘rish mumkin. Shunga ko‘ra, $\frac{y}{x} = 3$ deyish mumkin. Bundan $y = 3x$ kelib chiqadi. Jadvaldagi sonlar juftliklaridan $y = 3x$ tenglama grafigini chizish mumkin. Grafikka qarab, $y = 3x$ ning funksiya ekanini ko‘rish qiyin emas.

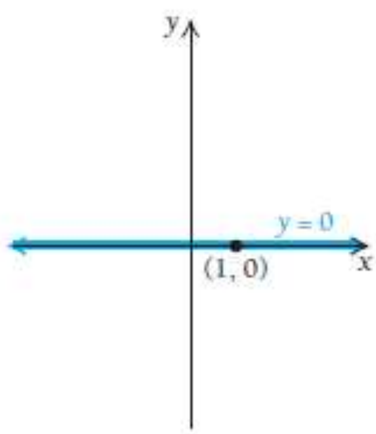


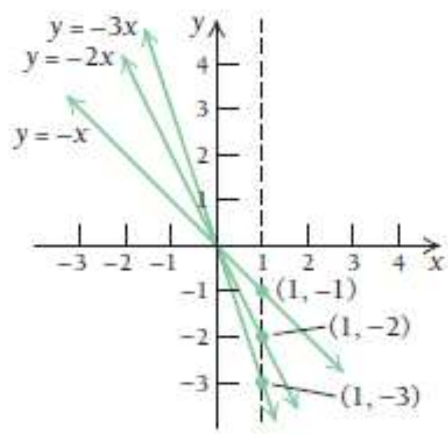
4-teorema. $y = mx$ yoki $f(x) = mx$ ning grafigi (0, 0) va $(1, m)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq bo‘lib, bu grafik funksiya bo‘ladi. Bunda m o‘zgarmas songa **to‘g‘ri chiziqning burchak koeffitsiyenti** deyiladi.

$y = mx$ funksiya grafigi m ning musbat qiymatlarida I va III choraklarda joylashadi. Burchak koeffitsiyenti katta bo‘lgan funksiya grafigi burchak koeffitsiyenti kichik bo‘lgan funksiya grafigidan tezroq o‘sadi.



Agar $m = 0$ bo‘lsa, $y = 0x$ yoki $y = 0$ funksiya gorizontal chiziqdan iborat bo‘ladi va bu chiziq X abssissalar o‘qi bilan ustma-ust tushadi.





$y = mx$ funksiya grafigi m ning manfiy qiymatlarida II va IV choraklarda joylashadi. Bunda burchak koeffitsiyenti kichik bo‘lgan funksiya grafigi burchak koeffitsiyenti katta funksiya grafigidan tezroq kamayadi.

Gullarning qatorlari bir xil burchak koeffitsiyentiga ega

2-vazifa. a) $y=\frac{1}{2}x$; **b)** $y=-\frac{1}{2}x$ **tenglamalar grafiklarini chizing.**

1.4.3. Og‘ma

$y=mx$ to‘g‘ri chiziqdan ko‘pgina masalalarning tatbiqlarida foydalanamiz, bunda m biror musbat son. Uni biz **og‘ma** deb nomlaymiz, m ni esa **proporsionallik koeffitsiyenti** deymiz.

Ta‘rif. Agar $y=mx$ tenglamada m biror musbat son bo‘lsa, y o‘zgaruvchi x ga **to‘g‘ri proporsional** ravishda o‘zgaradi deyiladi.

3-misol. Fizika. Yer va Oydagi vaznlar. Oydagi M jismning massasi bilan Yerdagi E jism massasi proporsional. Astronaft Yerda tortib ko‘rganida 180 kg bo‘lgan jism Oyda tortib ko‘rilganda 28.8 kg chiqdi.

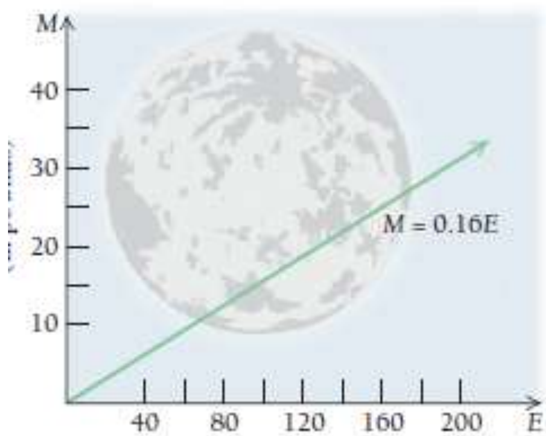
a) Ushbu moslik tenglamasini tuzing.

b) Astronaft Yerda tortib ko‘rganida 120 kg bo‘lgan jism Oyda tortib ko‘rilganda qancha bo‘ladi?

Yechilishi: ► a) Proporsionallik $M = mE$ tenglama bilan aniqlanadi.

Bundan m ni topamiz: $28.8 = m \cdot 180$, $\frac{28.8}{180} = m$, $0.16 = m$.

Demak, proporsionallik koeffitsiyenti 0.16 ga teng ekan: $M = 0.16E$.

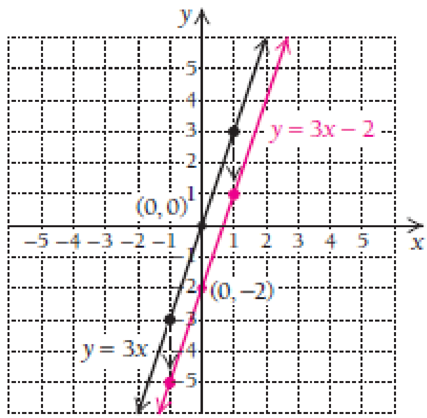


b) Astronaft Yerda tortib ko‘rganida 120 kg bo‘lgan jism Oyda qancha vaznga ega bo‘lishini aniqlaymiz: $M = 0.16E = 0.16 \cdot 120 = 19.2$

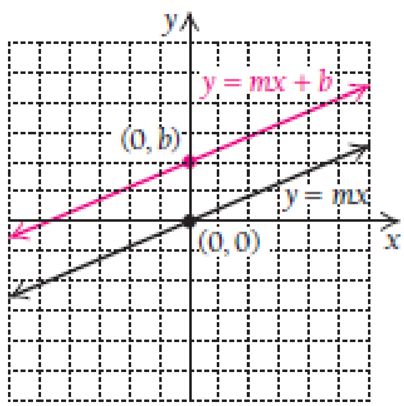
Yerda 120 kg vaznga ega bo‘lgan jism Oyda 19.2 kg bo‘lar ekan.◄

1.4.4. $y = mx + b$ ko‘rinishidagi tenglamalar

$y = 3x$ va $y = 3x - 2$ tenglamalarni qaraylik. $y = 3x - 2$ ning grafigi $y = 3x$ ning grafigidan 2 birlik pastdan, ya‘ni (0, -2) nuqtadan o‘tadi. Ikkala grafik ham funksiya bo‘ladi.



Ta‘rif. Chiziqli funksiya deb, grafigi $y = mx$ funksiyaga parallel va $(0, b)$ nuqtadan o‘tuvchi $y = mx + b$ yoki $f(x) = mx + b$ funksiyaga aytiladi.



Agar $m=0$ bo‘lsa, $y=0\cdot x+b=b$ o‘zgarmas funksiya hosil bo‘ladi (3-teorema). Bunday funksiyaning grafigi gorizontal chiziqdan iborat.

1.4.5. Og‘ma tenglamasi

Har bir vertikal bo‘lmagan l to‘g‘ri chiziq uning m burchak koeffitsiyenti va Y o‘qini kesib o‘tadigan $(0,b)$ nuqtasi orqali aniqlanadi.

Ta‘rif. Y o‘qini kesib o‘tuvchi $y=mx+b$ to‘g‘ri chiziqqa **o‘gma** deyiladi.

4-misol. $3x-5y=8$ tenglamadan og‘ma tenglamasini keltirib chiqaring va Y o‘qini kesib o‘tuvchi nuqtasini aniqlang.

Yechilishi: ► Tenglamani y ga nisbatan yechamiz: $3x-5y=8$

$5y=3x-8$ tenglamaning har ikki tomoniga $5y$ ni qo‘shamiz.

$y=\frac{3}{5}x-\frac{8}{5}$ tenglamaning har ikki tomonini 5 ga bo‘lamiz.

$\frac{3}{5}$ - og‘ish koeffitsiyenti; $\left(0,-\frac{8}{5}\right)$ nuqtada Y o‘qini kesadi. ◄

5-misol. (-1,-5) nuqtadan o‘tuvchi va o‘g‘ish koeffitsiyenti 3 bo‘lgan og‘ma tenglamasini tuzing.

Yechilishi: ► Shartga ko‘ra, $y = mx + b$ og‘ma tenglamasidagi b ozod had noma‘lum, og‘ish koeffitsiyenti $m = 3$ va to‘g‘ri chiziq (-1, -5) nuqtadan o‘tadi, ya‘ni $x = -1$, $y = -5$. Shularni hisobga olib,

$y = mx + b$ dan $-5 = 3 \cdot (-1) + b$ tenglikni hosil qilamiz.

$-5 = -3 + b$

$b = -2$. Demak, og‘ma tenglamasi $y = 3x - 2$ hosil bo‘ladi.◄

Umuman olganda, agar (x_1, y_1) nuqta

$$y = mx + b \qquad (1)$$

to‘g‘ri chiziqqa tegishli bo‘lsa, u holda

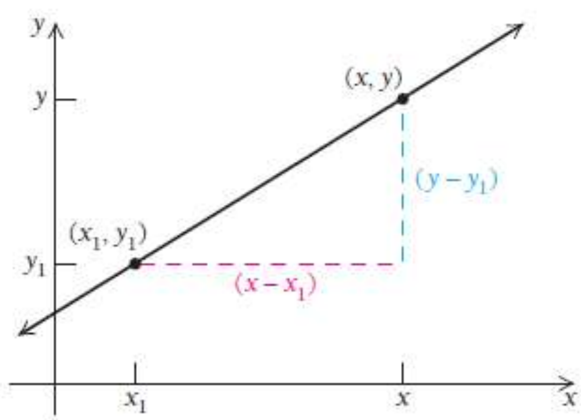
$$y_1 = mx_1 + b \qquad (2)$$

tenglik o‘rinli bo‘ladi. (2) tenglamaning chap va o‘ng tomonlarini mos ravishda (1) tenglikdan ayiramiz:

$y - y_1 = (mx + b) - (mx_1 + b)$;

$y - y_1 = mx + b - mx_1 - b$;

$y - y_1 = m(x - x_1)$ tenglikni hosil qilamiz.



Ta‘rif. $y - y_1 = m(x - x_1)$ ga to‘g‘ri chiziqning **nuqtadagi og‘ma tenglamasi** deyiladi. Bunda nuqta (x_1, y_1) va m og‘ish koeffitsiyenti.

6-misol. (-1,-5) nuqtadan o‘tuvchi va o‘g‘ish koeffitsiyenti $\frac{2}{3}$ bo‘lgan og‘ma tenglamasini tuzing.

Yechilishi: ► $y - y_1 = m(x - x_1)$ tenglamadan foydalanamiz.

$x_1 = -1, y_1 = -5$, $m = \frac{2}{3}$ qiymatlarni o‘rniga qo‘yamiz.

$$y - (-5) = \frac{2}{3}(x - (-1))$$

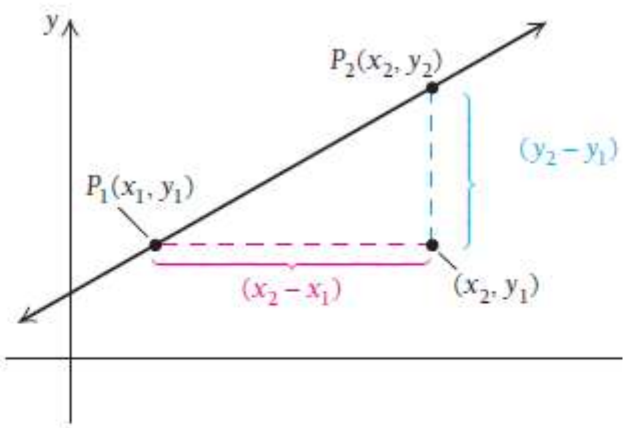
$$y + 5 = \frac{2}{3}(x + 1)$$

$$y = \frac{2}{3}x + \frac{2}{3} - 5$$

Og‘ma tenglamasi $y = \frac{2}{3}x - \frac{13}{3}$ hosil bo‘ladi. ◄

3-vazifa. (-3, 6) nuqtadan o‘tuvchi va o‘g‘ish koeffitsiyenti $-\frac{2}{3}$ bo‘lgan og‘ma tenglamasini mustaqil tuzing.

1.4.6. Og‘ish koeffitsiyentini hisoblash



Agar og‘maning ikkita nuqtasining koordinatasi ma‘lum bo‘lsa, og‘ma chiziqni hisoblash usulini o‘rganamiz. Faraz qilaylik, (x_1, y_1) va (x_2, y_2) nuqtalar vertikal bo‘lmagan to‘g‘ri chiziqning ikki turli P_1 va P_2 nuqtalari bo‘lsin.

E’tibor bering, y dagi $y_2 - y_1$ va x dagi $x_2 - x_1$ ayirmalar o‘zgarsa, og‘maning og‘ish burchagi ham o‘zgaradi. Buni ko‘rish uchun og‘maning nuqtasidan foydalanamiz.

(x_2, y_2) nuqta $y - y_1 = m(x - x_1)$ og‘mada yotganligi uchun, uning tenglamasini qanoatlantiradi: $y_2 - y_1 = m(x_2 - x_1)$.

To‘g‘ri chiziq vertikal bo‘lmaganligi sababli $x_2 - x_1 \neq 0$, ya‘ni x_1 va x_2 lar bir-biridan farq qiladi. m ni hisoblash uchun tenglikni $x_2 - x_1 \neq 0$ ga bo‘lish mumkin.

5-teorema. (x_1, y_1) va (x_2, y_2) nuqtalardan o‘tuvchi og‘maning og‘ish koeffitsiyenti $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ ga teng.

7-misol. (-1,5) va (-2, 9) nuqtalardan o‘tuvchi to‘g‘ri chiziqning og‘ish koeffitsiyentini hisoblang.

Yechilishi: ► $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-5}{-2-(-1)} = \frac{4}{-1} = -4.$$

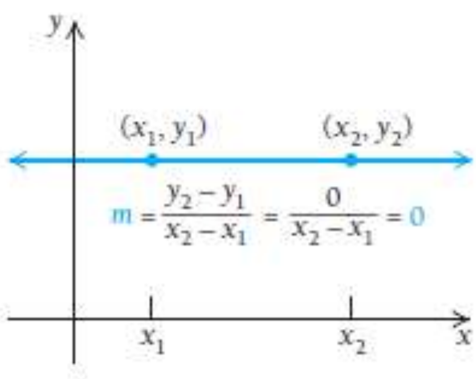
Biz bu yerda (-1,5) ni P_1 va (-2, 9) ni P_2 deb oldik.

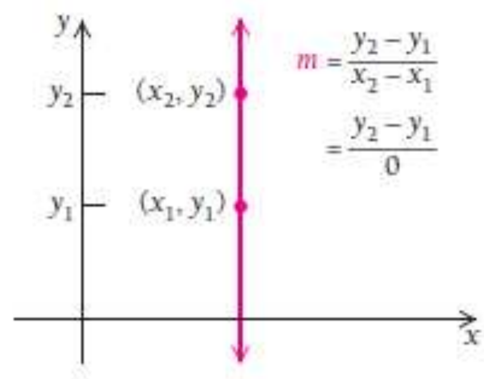
Agar (-1,5) nuqtani P_2 va (-2, 9) nuqtani P_1 deb olsak, natija o‘zgaradimi?

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-9}{-1-(-2)} = \frac{-4}{1} = -4.$$

Ko‘rib turibmizki, og‘ish koeffitsiyenti nuqtalarni olish tartibiga bog‘liq emas ekan.◄

Agar to‘g‘ri chiziq gorizontal bo‘lsa, u holda y dagi $y_2 - y_1 = 0$ bo‘ladi, shuning uchun **gorizontal to‘g‘ri chiziqning og‘ish koeffitsiyenti** $m = 0$ **ga teng.**



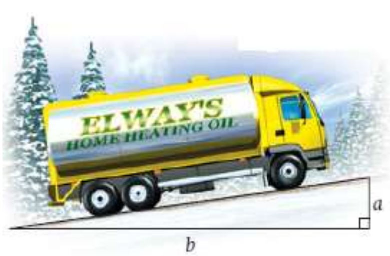


Agar to‘g‘ri chiziq vertikal bo‘lsa, u holda x dagi $x_2 - x_1$ ayirma 0 ga teng boʻla olmaydi, shuning uchun **vertikal toʻgʻri chiziqning ogʻish koeffitsiyenti aniqlanmagan.**

4-vazifa. (2, 3) va (1, -4) nuqtalardan oʻtuvchi toʻgʻri chiziqning ogʻish koeffitsiyentini mustaqil hisoblang.

1.4.7. Ogʻmaning tatbiqlari

Turmushda ogʻmaning juda koʻp sohalarda tadbiq qilinishini koʻrish mumkin. Masalan, **yoʻl qurilishida** tepalikka koʻtarilish balandligi boʻyicha yoʻlning tikkalik darajasini koʻrsatish uchun 2%, 3% va 6% koʻrsatkichlar ishlatiladi. 3% li daraja $\left(3\% = \frac{3}{100}\right)$ yoʻlning har 100 metr oraliqda 3 metrga koʻtarilishini bildiradi.



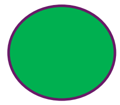
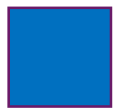
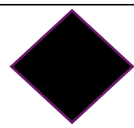
Arxitekturada tomning ogʻmalik koʻrsatkichi juda muhim. Tom qancha tikka boʻlsa, unda qor shuncha kam toʻplanadi.

Nogironlik aravachalarini yasashda ham ogʻmalik darajasini e’tiborga olishadi. Aravachaning tepalikka chiqishida ogʻmalik darajasi $\frac{1}{12}$ dan oshmasligi kerak.

Tog‘ chang‘isida qiyinchiliklarni baholash yoki og‘maning yana bir tatbig‘i – gradiyent kattaligi mavjud.

Tog‘ hududlarida juda murakkab chang‘i yo‘llari barpo qilingan. Yo‘llarning qiyinlik darajasi chang‘i kurorti operatorlari tomonidan belgilanadi. iPod , iPhone va Samsung telefonlarida qiyinlik darajalari to‘g‘ri yoki noto‘g‘ri qo‘yilganligini baholaydigan ko‘pgina ilovalar mavjud. Gradiyentni baholash uchun qo‘lingizni yerga parallel tuting, bu 0^0 li gradiyent bo‘ladi. Agar qo‘lingiz 45^0 burchakka ochilsa, 100% li gradiyent, 22.5^0 burchak bo‘lsa, 41% li gradiyent bo‘ladi. 3.5^0 burchak esa bor-yo‘g‘i 6% gradiyentga tengdir.

Chimyon tog‘ chag‘i yo‘llari.

Baholanayotgan daraja nomlanishi	Belgilanishi	Tavsiflanishi	Qiyinlik darajasi
Yashil doira		sodda	Bu yo‘llar keng va tekislangan bo‘ladi. Og‘ish gradiyenti 6-25 % (100% og‘ish 45^0 ga teng).
Ko‘k kvadrat		o‘rtacha	Bu chang‘i yo‘llarining og‘ish gradiyenti 25-40 % ga teng. Ular ishlov berilgan va ko‘p foydalanuvchilarga ega.
Qora romb		murakkab	40 % dan yuqori og‘ ish gradiyentiga ega. Ishlov berilmagan va murakkab chang‘i yo‘llari hisoblanadi.

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-9 misollarda funksiya grafiklarini chizing:

1. $x=5$ **2.** $x=-3$ **3.** $x=1.5$

4. $x=-4.5$ **5.** $y=2$ **6.** $y=-4$

7. $y=2.75$ **8.** $y=-3.25$ **9.** $x=-1.25$

10-21 misollarda og‘ish koeffitsiyentini va Y o‘qini kesib o‘tish nuqtasini aniqlang:

10. $y=2.75x$ **11.** $y=-3x$ **12.** $y=\frac{1}{2}x$

13. $f(x)=0.5x+1$ **14.** $f(x)=-0.5x$ **15.** $f(x)=5x-2$

16. $g(x)=x-3.25$ **17.** $g(x)=-x+4$ **18.** $g(x)=\frac{1}{4}x-5$

19. $y=2$ **20.** $y=-7$ **21.** $y=0.25$

22-30 misollarda og‘ish koeffitsiyentini va Y o‘qini kesib o‘tish nuqtasini aniqlang:

22. $y-2x=5$ **23.** $y-4x=1$ **24.** $3x-y=-6$

25. $2x+y-3=0$ **26.** $2x-y+3=0$ **27.** $2x+2y+12=0$

28. $x=y-3$ **29.** $x=-3y+4$ **30.** $x=5-2y$

31-40 misollarda to‘g‘ri chiziqning og‘ish koeffitsiyenti va Y o‘qini kesib o‘tish nuqtasi ma‘lum bo‘lsa, tenglamasini tuzing:

31. Og‘ish koeffitsiyenti $m=-5$ va Y o‘qini kesib o‘tish nuqtasi $(-2,-3)$;

32. Og‘ish koeffitsiyenti $m=-2$ va Y o‘qini kesib o‘tish nuqtasi $(2,3)$;

33. Og‘ish koeffitsiyenti $m=7$ va Y o‘qini kesib o‘tish nuqtasi $(1,\ 7)$;

34. Og‘ish koeffitsiyenti $m=-3$ va Y o‘qini kesib o‘tish nuqtasi $(-2,\ 5)$;

35. Og‘ish koeffitsiyenti 2 va Y o‘qini kesib o‘tish nuqtasi $(3,\ 0)$;

36. Og‘ish koeffitsiyenti 5 va Y o‘qini kesib o‘tish nuqtasi $(5,\ 0)$;

37. Og‘ish koeffitsiyenti 0.5 va Y o‘qini kesib o‘tish nuqtasi $(0,-6)$;

38. Y o‘qini kesib o‘tish nuqtasi $(0,6)$ va og‘ish koeffitsiyenti 3.5;

39. Y o‘qini kesib o‘tish nuqtasi $(4,6)$ va og‘ish koeffitsiyenti 0;

40. Y o‘qini kesib o‘tish nuqtasi $(-2,-3)$ va og‘ish koeffitsiyenti 0;

41-52 misollarda (x_1,y_1) va (x_2,y_2) nuqtalardan o‘tuvchi to‘g‘ri chiziqning og‘ish koeffitsiyentini toping. Bular orasida og‘ish koeffitsiyenti aniqlanmagan to‘g‘ri chiziqlar bormi?:

41. $(5,-3)$ va $(-2,\ 1)$ nuqtalar; **42.** $(-2,\ 1)$ va $(6,\ 3)$ nuqtalar;

43. $(2,-3)$ va $(-1,-4)$ nuqtalar; **44.** $(-3,-5)$ va $(1,-6)$ nuqtalar;

45. $(3,-7)$ va $(3,-9)$ nuqtalar;

46. $(-4,2)$ va $(-4,10)$ nuqtalar;

47. $(\frac{4}{5},-3)$ va $(\frac{1}{2},\frac{2}{5})$ nuqtalar;

48. $(-\frac{3}{16},-\frac{1}{2})$ va $(\frac{5}{8},-\frac{3}{4})$ nuqtalar;

49. $(x,3x)$ va $(x+h,3(x+h))$ nuqtalar;

50. $(x,4x)$ va $(x+h,4x+4h)$ nuqtalar;

51. $(x,2x+3)$ va $(x+h,2(x+h)+3)$ nuqtalar;

52. $(x,3x-1)$ va $(x+h,3(x+h)-1)$ nuqtalar;

53-64 misollarda ikki nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasini tuzing.

53. $(5,-3)$ va $(-2,1)$ nuqtalar;

54. $(-2,1)$ va $(6,3)$ nuqtalar;

55. $(2,-3)$ va $(-1,-4)$ nuqtalar;

56. $(-3,-5)$ va $(1,-6)$ nuqtalar;

57. $(3,-7)$ va $(3,-9)$ nuqtalar;

58. $(-4,2)$ va $(-4,10)$ nuqtalar;

59. $(\frac{4}{5},-3)$ va $(\frac{1}{2},\frac{2}{5})$ nuqtalar;

60. $(-\frac{3}{16},-\frac{1}{2})$ va $(\frac{5}{8},-\frac{3}{4})$ nuqtalar;

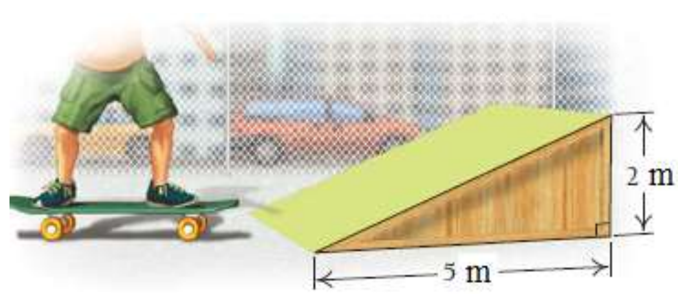
61. $(x,3x)$ va $(x+h,3(x+h))$ nuqtalar;

62. $(x,4x)$ va $(x+h,4x+4h)$ nuqtalar;

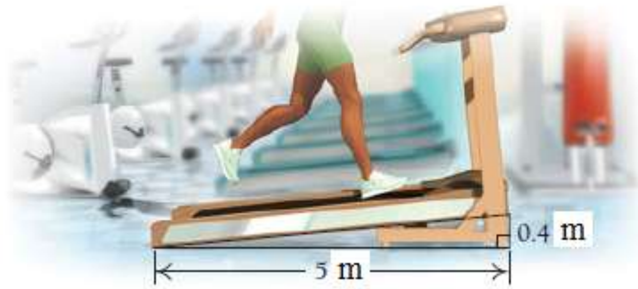
63. $(x,2x+3)$ va $(x+h,2(x+h)+3)$ nuqtalar;

64. $(x,3x-1)$ va $(x+h,3(x+h)-1)$ nuqtalar;

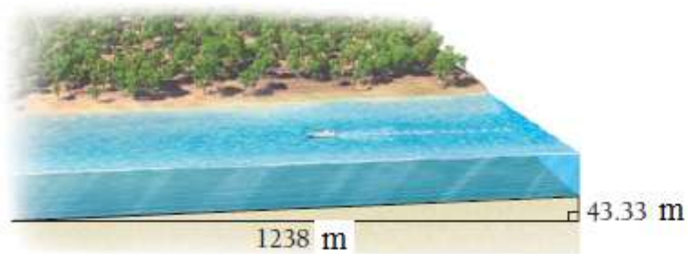
65. Skeytbord trampi og‘ish gradiyentini toping.



66. Yugurish qurilmasidagi bir martalik og‘ish darajasini aniqlang.



67. Daryoning og‘ish burchagini toping va natijani foizda ifodalang.



1.4-bo‘limning amaliy tatbiqlariga doir

1. Iqtisod va tadbirkorlik

68. Pullik avtomobil yo‘li. Pullik yo‘l talabiga ko‘ra transport vositalari qanchalik og‘ir bo‘lsa, shosse uchun shunchalik javobgarlikka ega. Shuning uchun haydovchi transport vositasining og‘irligiga proportsoinal ravishda to‘lov qilishga majburdir. Faraz qiling, “Toyota Campy” avtotransport vositasi 3350 kg og‘irlikka ega bo‘lib, 80 km yo‘ldan o‘tganligi uchun 2.70 $ to‘ladi.

a) Yo‘l uchun to‘lov summasini tarasport vositasining og‘irligi funksiyasi sihatida tenglamasini tuzing.

b) Agar 3700 kg.lik “Jeep Cherokee” shu yo‘ldan o‘tsa, qancha to‘lov qilishi kerak?

69. Printer katridji. Talabalar bo‘limidagi nusxa ko‘chirish va chop qilish qurilmasida katridj siyohlari har yili ro‘yxatdan o‘tuvchi talabalar soniga proportsional ravishda sarflanadi.

a) Agar bo‘limda 5800 talaba ro‘yxatdan o‘tganda 16 marta katridj qurilmasi almashtirilgan bo‘lsa, buni funksiya sifatida ifodalang.

b) Agar 73100 talaba ro‘yxatdan o‘tsa, necha marta katridj qurilmasi almashtirilar edi?

70. Foyda va zararni hisoblash. Anvar o‘t o‘rish qurilmasida gazon kesib, pul topishga qaror qildi. U bir m^2 yerdagi gazonni kesish uchun dastlabki narxni 25 000 so‘m deb baholadi. Benzin va xizmat ko‘rsatishga ketadigan harajatlar $1m^2$ uchun 4 000 so‘m.

a) x m^2 yerdagi gazonni kesish uchun $C(x)$ umumiy narx funksiyasini yozing.

b) Anvarni hisobiga ko‘ra, narx funksiyasi $C(x) = 9000x - 25000$ ga teng bo‘lsa, kvadrat metriga qancha so‘ramoqda?

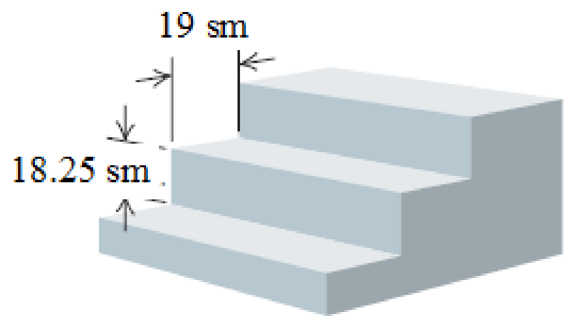
71. Odam bosh miyasining og‘irligi. Odamning bosh miya og‘irligi uning tana og‘irligiga to‘g‘ri proportsional.

a) Ma‘lumki, 60 kg og‘irlikka ega odamning miyasi 1300 g bo‘ladi. Bu bog‘liqlikni funksiya sifatida tasvirlang.

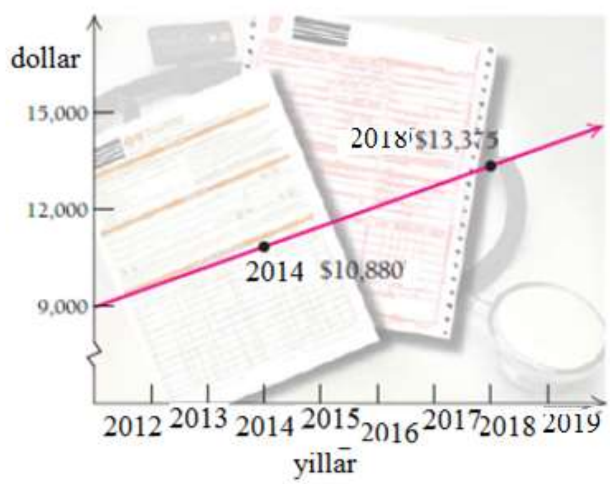
b) Bog‘lanishdagi o‘zgarmas sonni foiz orqali ifodalang.

c) 80 kg vaznli odamning miyasi qancha og‘irlikda bo‘ladi?

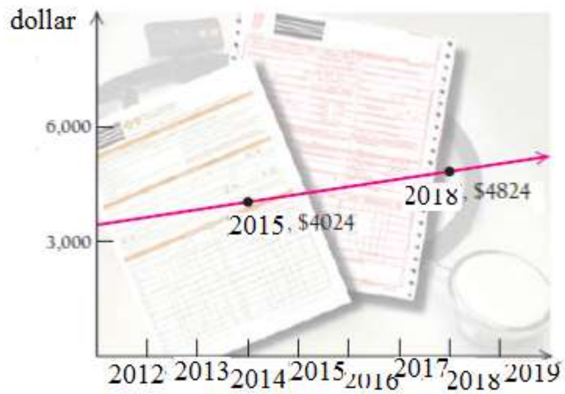
72. Zina pog‘onalariga talab. Davlat qurilish standartlariga ko‘ra universitet binosining qavat oralig‘i zinapoyalarida bitta pog‘ona balandligi 18.25 sm va kengligi 19 sm bo‘lishi kerak (19 okt. 2017 yil DQSt). Rasmga qarang. Standartga asosan zina pog‘onasining maksimal ko‘rsatkichi qanday bo‘lishi kerak?



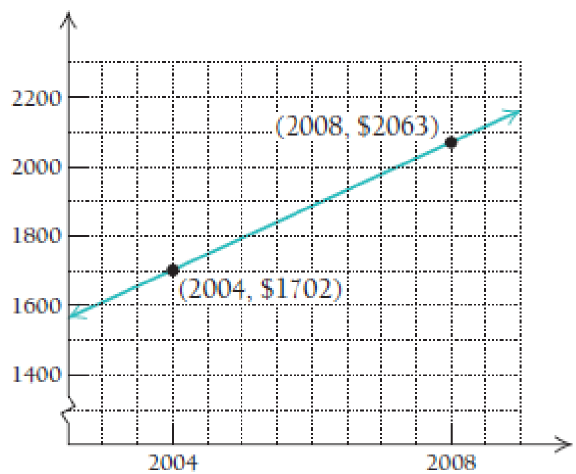
73. Tibbiy sug‘urta to‘lovi. Sog‘lom oila sug‘urtasidagi har yilgi to‘lov miqdorining o‘rtacha o‘sishini toping.



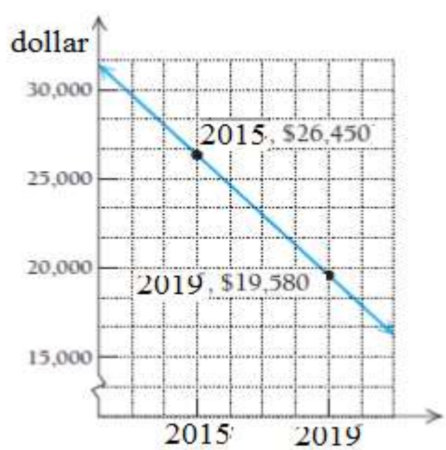
74. Tibbiy sug‘urta to‘lovi. Sog‘lom oila sug‘urtasidagi har yilgi to‘lov miqdorining 1 kishi uchun to‘lanadigan o‘rtacha qiymatini toping.



75. Magistratura ta‘limi. Diagrammadan ikki yillik magistratura ta‘limida kontrakt pulining o‘rtacha o‘sishini toping.



76. Nikoh to‘yi sarf-harajatlari. Diagrammadan foydalanib, to‘yga ketadigan pul miqdorining o‘rtacha o‘zgarishini toping.



77. Elektrtejamkorlik. Elektr simlarining R qarshiligi bevosita ularning qalinligiga T bog‘liq.

a) agar $T=3$ bo‘lganda $R=12.51$ bo‘lsa, ularning bog‘liqlik tenglamasini tuzing.

b) $T=6$ bo‘lganda R nimaga teng bo‘ladi?

78. Nerv impulsining tezligi. Nerv tolalarida impuls harakatining tezligi 118 m/s ga teng. Oyoqdagi impuls t sekund ichida $D=118t$ masofaga yetib boradi. Bo‘yi 1.85 m bo‘lgan kishining miyasidagi impuls oyoq barmoqlariga qancha vaqtda yetib boradi?

79. Mushaklar kuchi. Odamning M mushaklarining kuchi uning W tana og‘irligiga to‘g‘ri proporsional.

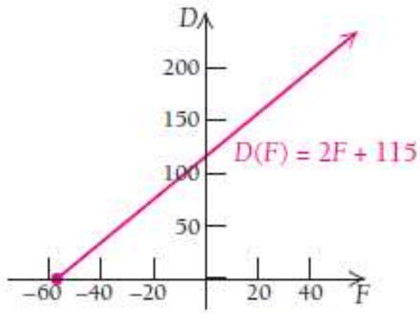
a) Og‘irligi 80 kg bo‘lgan odamning muskullarining kuchi 32 kg bo‘ladi. M ni W ning funksiyasi sifatida tasvirlang.

b) Tenglamadagi o‘zgarmas koeffitsiyentni foizda ifodalang va uning tenglamasini tuzing.

c) 48 kg massali odamning mushak kuchini toping.

80. Muzda to‘xtash yo‘lining uzunligi. Agar havo harorati t^0 F bo‘lsa, muzda konkida to‘xtash yo‘lining uzunligi havo haroratining funksiyasi sifatida quyidagicha yoziladi: $L(t) = 2t + 115$,

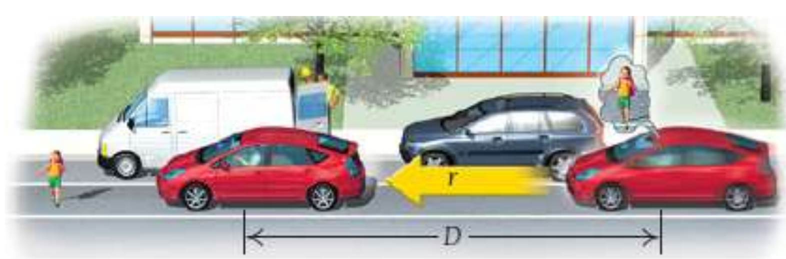
bunda harorat Farengeyt o‘lchamida berilgan. Farengeytdan Seltsiyga o‘tish formulasi: $(32^0 F - 32)\cdot\frac{5}{9} = 0^0 C$, ya‘ni 32^0 F=0^0 C ga teng.



a) $L(0^0)$, $L(-20^0)$, $L(10^0)$ va $L(32^0)$ Farengeyt o‘lchamida hisoblang.

b)Nima uchun aniqlanish sohasi $[-57.5^0 F,\ 32^0 F]$ oraliqda bo‘lishi kerak?

81. Reaksiya vaqti. Avtomobil boshqarayorganingizda birdaniga yo‘lga bola yugurib chiqqanini ko‘rdingiz. Bu vaqtda miya favqulotda holatni ro‘yxatga oladi va tormozni bosish uchun oyoqlaringizga signal jo‘natadi. U holda reaksiya masofasini D deb belgilaymiz. D masofa r ning funksiyasi (rasmga qarang) bo‘lib, $D(r) = \frac{11r + 5}{10}$ chiziqli funksiyadan iborat.

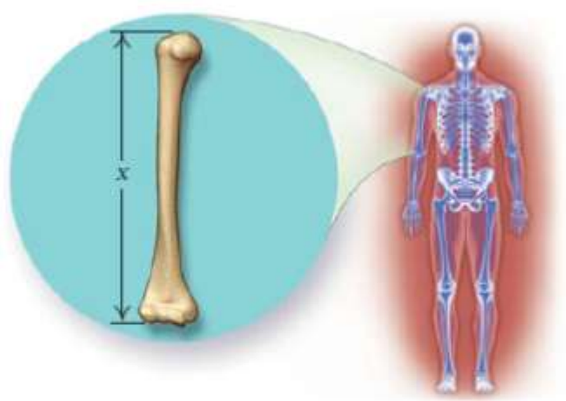


a) $D(5)$, $D(10)$, $D(20)$, $D(50)$ va $D(65)$ ni hisoblang.

b) $D(r)=\dfrac{11r+5}{10}$ ni grafigini chizing.

c) $D(r)=\dfrac{11r+5}{10}$ funksiyaning aniqlanish sohasini toping.

82. Odam bo‘yini baholash. Antropolog ma‘lum suyaklar uzunligini hisobga olgan holda erkak yoki ayolning bo‘yini aniqlashda chiziqli funksiyadan foydalanadi. Yelka suyagi – bu tirsakdan yelkagacha bo‘lgan suyak.

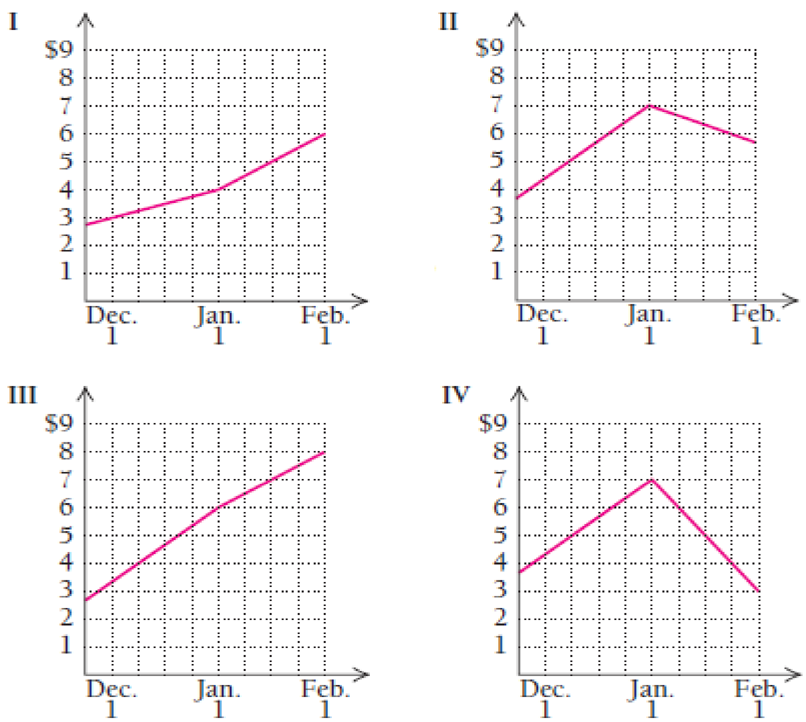


Agar erkak yelka suyagi uzunligini x sm desak, u holda erkakning bo‘yi $M(x)=2.89x+70.64$ sm ga teng bo‘ladi. Agar ayol yelka suyagi uzunligini x sm desak, u holda ayolning bo‘yi $F(x)=2.75x+71.48$ sm ga teng bo‘ladi. Ba‘zi qazishma ishlarida yelka suyagi 26 sm bo‘lgan murdalar topildi.

a) Agar biz bu erkak deb faraz qilsak, uning bo‘yi qancha bo‘lgan?

b) Agar biz bu ayol deb faraz qilsak, uning bo‘yi qancha bo‘lgan?

83. Tadbirkorlik. Quyidagi diagrammalarga mos javoblarni ayting:\
a) 1 yanvardan keyin kunlik savdo o‘sdi, lekin juda sekinlik bilan o‘sdi;
b) 1 yanvardan keyin kunlik savdo oldingi davrlarga nisbatan tezroq kamaydi;
c) 1 yanvardan keyin kunlik savdo oldingiga nisbatan 2 marta tez o‘sdi;
d) 1 yanvardan keyin kunlik savdo dekabrdagiga nisbatan 2 marta kamaydi;



83. 2000 yilda Internetdan foydalanadigan 18 dan 29 yoshgacha bo‘lgan yoshlar 72 % ni tashkil qilar edi. 2019 yilga kelib bu ko‘rsatkich 92 % ga yetdi.
a) Yilni x koordinata, foizni y koordinata deb belgilab, koordinata sistemasida chiziq grafigini chizing.
b) Berilgan ma‘lumotlar asosida to‘g‘ri chiziq tenglamasini tuzing.

c) Qaysi yilda foydalanuvchilar 100 % ga yetadi.

d) Nima uchun bu funksiyadan bir necha yil keyingi ma‘lumotni olish uchun foydalanib bo‘lmaydi.

1.5. Chiziqli bo'lmagan funksiyalar va modellar

Shunday funksiyalar borki, ularning grafiklari to'g'ri chiziq bo'lmaydi. Ularni nochiziqli funksiyalar deb yuritiladi. Fanni o'rganish jarayonida biz ko'p marta uchratadigan ana shunday nochiziqli funksiyalarning ba'zilarini o'rganamiz.

1.5.1. Kvadratik funksiyalar

Ta'rif. $y = ax^2 + bx + c$, bunda $a \neq 0$ ko'rinishidagi funksiyaga **kvadratik funksiya** deyiladi.

Biz kvadratik funksiyalarni oldingi mavzularda grafiklar chizishni o'rgangan paytimizda $f(x) = x^2$ va $g(x) = x^2 - 1$ ko'rinishlarda ishlatgan edik.

$y = ax^2 + bx + c$ kvadratik funksiya grafigiga **parabola** deyiladi.

a) Agar $a > 0$ bo'lsa, parabola tarmoqlari tepaga,
agar $a < 0$ bo'lsa, parabola tarmoqlari pastga qaragan bo'ladi.

b) Parabola uchining birinchi koordinatasi quyidagi formuladan topiladi:

$$x = -\frac{b}{2a}.$$

c) $x = -\frac{b}{2a}$ vertikal chiziq parabolaning **simmetriya chizig'i** deyiladi.

1-misol. $f(x) = x^2 - 2x - 3$ funksiya grafigini chizing.

Yechilishi:

►Berilgan funksiya kvadratik funksiya, uni to‘liq shaklda yozamiz:

$$f(x) = 1x^2 - 2x - 3.$$

Shunga ko‘ra, $a = 1$, $b = -2$, $c = -3$. Shuningdek, $a > 0$ bo‘lgani uchun parabola tarmoqlari tepaga qaragan bo‘ladi. Parabola uchining x koordinatasini topamiz:

$$x = -\frac{b}{2a} = -\frac{-2}{2} = 1.$$

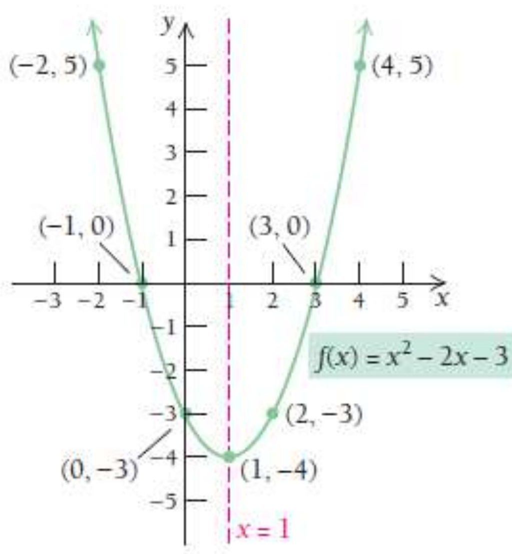
Endi parabola uchining y koordinatasini hisoblaymiz:

$$f(1) = 1^2 - 2 \cdot 1 - 3 = -4.$$

Demak, parabola uchi $(1, -4)$ nuqtada ekan.

$x = 1$ vertikal chiziq esa parabolaning simmetriya chizig‘i bo‘ladi.

x	$f(x)$
1	−4
0	−3
2	−3
3	0
4	5
−1	0
−2	5



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2-misol. $f(x) = -2x^2 + 10x - 7$ funksiya grafigini chizing.

Yechilishi:

► Berilgan funksiya $f(x) = -2x^2 + 10x - 7$ da $a = -2$, $b = 10$, $c = -7$. Shuningdek, $a < 0$ bo‘lgani uchun parabola tarmoqlari pastga qaragan bo‘ladi. Parabola uchining x koordinatasini topamiz:

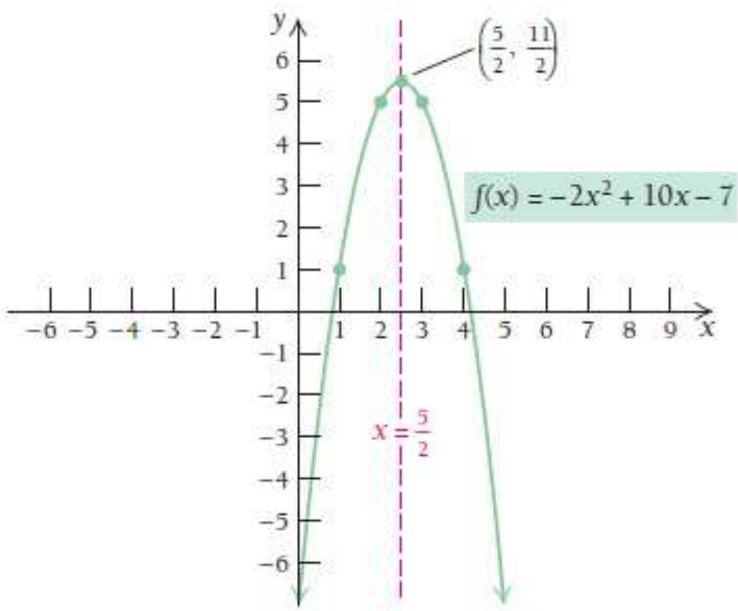
$$x = -\frac{b}{2a} = -\frac{10}{2 \cdot (-2)} = 2.5,$$

Endi x ning qiymatini berilgan tenglamaga qo‘yib, parabola uchining y koordinatasini hisoblaymiz: $f(2.5) = -2 \cdot 2.5^2 + 10 \cdot 2.5 - 7 = -12.5 + 25 - 7 = 5.5$.

Demak, parabola uchi $(2.5, 5.5)$ nuqtada ekan.

$x = 2.5$ vertikal chiziq esa parabolaning simmetriya chizig‘i bo‘ladi.

x	$f(x)$
$\frac{5}{2}$	$\frac{11}{2}$
0	−7
1	1
2	5
3	5
4	1
5	−7



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1-vazifa. Quyidagi funksiyalar grafiklarini mustaqil chizing:

a) $f(x) = 2x^2 + 3x - 7$

b) $f(x) = -x^2 - 10x - 7$

Agar $a \neq 0$ bo‘lsa, $ax^2 + bx + c = 0$ tenglamaning yechimi $y = ax^2 + bx + c$ funksiya grafigining X o‘qiga urinish yoki uni kesib o‘tish (agar ular mavjud bo‘lsa) nuqtalarining birinchi koordinatasi bo‘ladi.

6-teorema: Har qanday $ax^2 + bx + c = 0$, $a \neq 0$ kvadrat tenglamaning yechimi $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ga teng bo‘ladi.

1.5.2. $n-$darajali algebraik ko‘phad ko‘rinishidagi funksiyalar

Ta‘rif. $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$ funksiyaga $n-$**darajali algebraik ko‘phad ko‘rinishidagi funksiya** deyiladi, bunda $n \neq 0$ butun son, $a_n,\ a_{n-1}, ..., a_2, a_1, a_0$ haqiqiy sonlar bo‘lib, ularga **koeffitsiyentlar** deyiladi.

Chiziqli va kvadratik funksiyalar $n-$darajali algebraik ko‘phad ko‘rinishidagi funksiyalarning xususiy xoli hisoblanadi.

Misol uchun, $f(x) = -7$ o‘zgarmas funksiya;

$f(x) = 3x - 7$ chiziqli funksiya;

$f(x) = 2x^2 + 3x - 7$ kvadratik funksiya;

$f(x) = x^3 + 2x^2 + 3x - 7$ kubik yoki 3-darajali funksiya.

$n-$darajali algebraik ko‘phad ko‘rinishidagi funksiyalarni $f(x) = a_n x^n$ va $f(x) = a_n x^n + a_0$ dan tashqari daraja ko‘rsatkichi oshgan sari ularning grafigini chizish murakkablashib boradi. Shuning uchun grafik chizishga mo‘ljallangan dasturlardan foydalaniladi.

2-vazifa. Quyidagi funksiyalar grafiklarini mustaqil chizing:

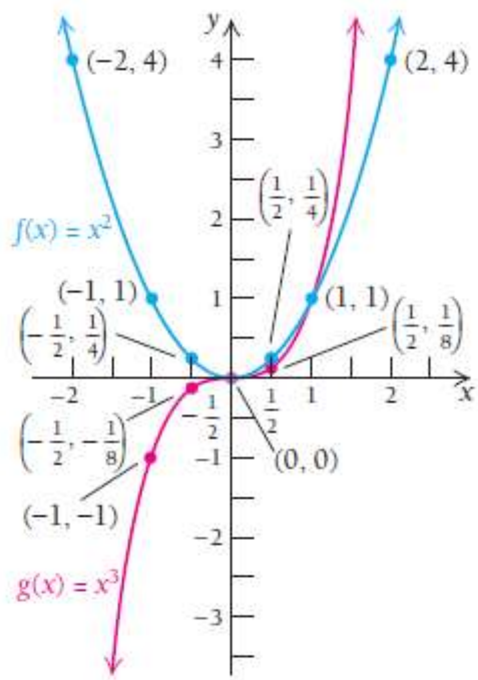
a) $f(x) = 3 - x^2$;

b) $g(x) = x^3 - 1$

3-misol. $f(x) = x^2$ va $g(x) = x^3$ funksiyalar grafiklarini chizing.

Yechilishi: ►

x	x^2	x^3
-2	4	-8
-1	1	-1
$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{8}$
0	0	0
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
1	1	1
2	4	8



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1.5.3. Ratsional funksiyalar

Ta‘rif. Ikkita algebraik ko‘phadning bo‘linmasidan iborat

$$R(x)=\frac{f(x)}{g(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+...+b_2x^2+b_1x+b_0}$$

funksiyaga **ratsional funksiya** deyiladi. Bunda $n\neq 0$ butun son, $a_n,\ a_{n-1},...,a_2,a_1,a_0$ va $b_m,\ b_{m-1},...,b_2,b_1,b_0$ haqiqiy sonlar bo‘lib, ularga **koeffitsiyentlar** deyiladi.

$$f(x)=\frac{x^2-16}{x+4},\qquad g(x)=\frac{x-3}{x^2-2x-1},$$

$$k(x)=\frac{2x^2+5x}{x-3}\qquad h(x)=\frac{x^2+7x-6}{1}=x^2+7x-6$$

ko‘rinishdagi funksiyalar ratsional funksiyalardir.

Shuni esda tutingki, $h(x)$ ko‘rinishdagi ko‘phad shaklidagi funksiya ham ratsional funksiya hisoblanadi.

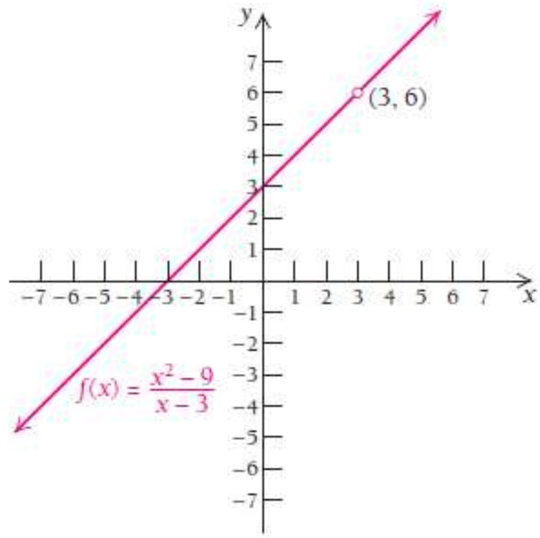
Ratsional funksiyaning aniqlanish sohasini topish uchun uning maxrajini 0 ga tenglaymiz. Hosil bo‘lgan tenglamaning yechimlari funksiyaning aniqlanish sohasiga kirmaydi. **Nima uchun shunday?**

4-misol. $f(x)=\dfrac{x^2-9}{x-3}$ funksiyaning aniqlanish sohasini toping.

Yechilishi: ► $x-3\neq 0,\qquad x\neq 3$

Aniqlanish sohasi: $D(f)=(-\infty;\ 3)\cup(3;\ \infty)$.

x	$f(x)$
−3	0
−2	1
−1	2
0	3
1	4
2	5
4	7



Biroq bu funksiya grafigini chizishdan oldin uni soddalashtiramiz:

$$f(x)=\frac{x^2-9}{x-3}=\frac{(x-3)(x+3)}{x-3}=x+3.$$

Shuning uchun $f(x)=x+3$, $x\neq 3$ deb qabul qilamiz. ◄

5-misol. $g(x)=\dfrac{x-3}{x^2-2x-1}$ funksiyaning aniqlanish sohasini toping.

Yechilishi: ► Kasrning maxrajini nolga tenglab olamiz:

$$x^2-2x-1\neq 0$$

Kvadrat tenglamaning yechimini topish uchun formulani yozamiz:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{4-4\cdot(-1)}}{2}=\frac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$$

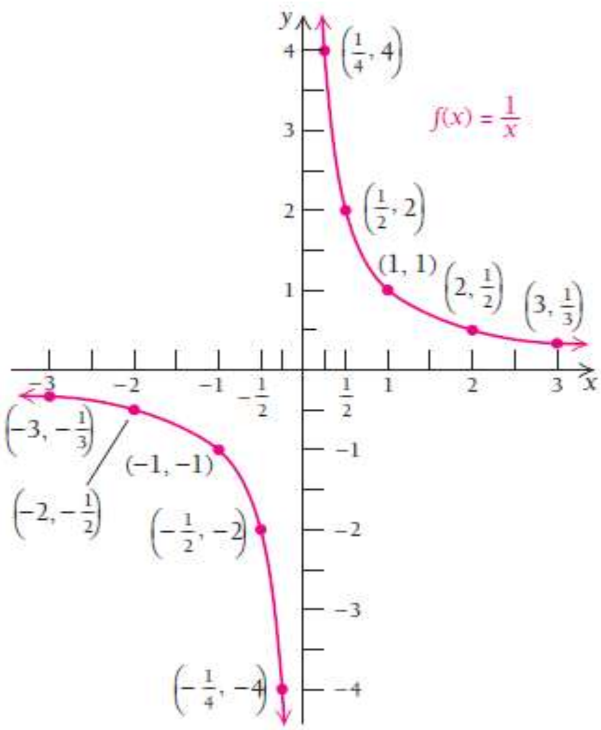
Bundan funksiyaning aniqlanish sohasi

$D(g)=(-\infty;\ 1-\sqrt{2})\cup(1-\sqrt{2};\ 1+\sqrt{2})\cup(1+\sqrt{2};\ \infty)$ ekani kelib chiqadi.◄

6-misol. $f(x)=\dfrac{1}{x}$ funksiya grafigini chizing.

Yechilishi: ► $f(x)=\dfrac{1}{x}$ funksiya uchun $x\neq 0$ va bu funksiya $(-\infty;\ 0)$ oraliqda ham, $(0;\ \infty)$ oraliqda ham kamaymoqda, shunga ko‘ra $D(f)=(-\infty;\ 0)\cup(0;\ \infty)$, ya‘ni funksiya o‘zining aniqlanish sohasida kamayuvchi deb gapiriladi.

x	$f(x)$
-3	$-\frac{1}{3}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$-\frac{1}{4}$	-4
$\frac{1}{4}$	4
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$



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3-vazifa. Quyidagi funksiyalar grafiklarini mustaqil chizing:

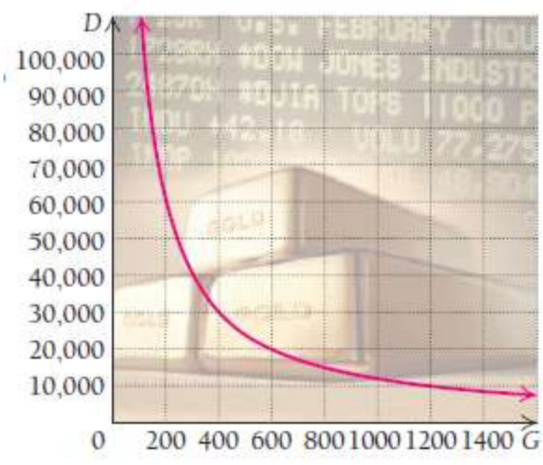
a) $f(x)=\dfrac{x^2-9}{x+3}$

b) $f(x)=-\dfrac{1}{x}$

Ta‘rif. Agar biror k musbat son mavjud bo‘lib, $y=\frac{k}{x}$ tenglik o‘rinli bo‘lsa, u holda y o‘zgaruvchi x ga **teskari proporsional** deyiladi.

7-misol. Tadbirkorlik: Aksiya va tilla. Ba‘zi iqtisodchilar nazariy jihatdan aksiya narxi tilla narxiga teskari proporsional deb bilishadi. Agar shunday bo‘lsa, tilla narxi oshganda aksiya narxi kamayishi yoki aksincha, tilla narxi pasayganda aksiya narxi ko‘tarilishi kerak. Keling faraz qilaylik, Dou Jons sanoat indeksi, ya‘ni D - barcha aksiyalar bahosining indeksi, G - tilla bahosiga teskari proporsional (misqoliga 1$). Bir kuni Dou Jons sanoat indeksi 10 619.70 $ bo‘lib, o‘sha kuni tilla narxi 1129.60 $ bo‘lgan. Agar tilla narxi 1400$ ga ko‘tarilgan bo‘lsa, Dou Jons sanoat indeksi nimaga teng bo‘ladi?

Yechilishi: ▶ Bilamizki, $D=\frac{k}{G}$, bundan $10\,619.7=\frac{k}{1129.6}$, $k=11\ 996\ 013.12$ kelib chiqadi. Demak, $D=\frac{11\ 996\ 013.12}{G}$, endi $G=1400$ ni hisoblaymiz: $D=\frac{11\ 996\ 013.12}{1400}\approx 8\ 568.6$.



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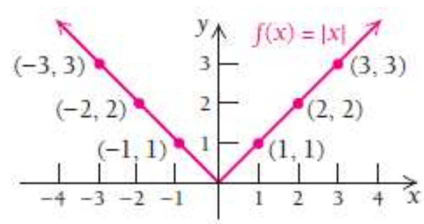
1.5.4. Modulli funksiyalar

Sonning absolyut qiymati – bu sonlar o‘qida o‘sha son bilan 0 soni orasidagi masofaga teng. x ning absolyut qiymatini $|x|$ deb belgilaymiz va $f(x)=|x|$ funksiyani modulli funksiya deymiz. Modulli funksiyalar hisob fanida katta ahamiyatga ega, uning grafigi o‘ziga xos V shaklida bo‘ladi.

7-misol. $f(x)=|x|$ funksiyaning grafigini chizing.

Yechilishi: ► Dastlab jadval to‘ldiramiz, so‘ngra jadvaldagi nuqtalarni koordinata sistemasida belgilab, shu nuqtalarni tutashtirib chiqamiz:

x	$f(x)$
−3	3
−2	2
−1	1
0	0
1	1
2	2
3	3



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Modulli funksiyani bo‘lakli aniqlangan funksiya sifatida tasvirlash mumkin:

$$f(x)=|x|=\begin{cases} x, & \text{agar } x \ge 0 \\ -x, & \text{agar } x < 0 \end{cases}$$

4-vazifa. Quyidagi funksiyalar grafiklarini mustaqil chizing:

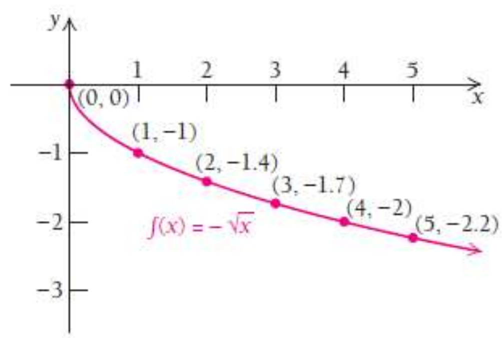
a) $f(x) = \left|\frac{x}{2}\right|$; b) $f(x) = \left|x^2 - 4\right|$

1.5.5. Darajali funksiyalar

8-misol. $f(x) = -\sqrt{x}$ kvadrat ildizli funksiyaning grafigini chizing.

Yechilishi: ► Ushbu funksiyaning aniqlanish sohasi barcha nomanfiy sonlardan iborat: $D(f) = [0;\ \infty)$.

x	0	1	2	3	4	5
$f(x) = -\sqrt{x}$	0	−1	−1.4	−1.7	−2	−2.2



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5-vazifa. Quyidagi funksiyalar grafiklarini mustaqil chizing:

a) $f(x) = |x+3|$; b) $f(x) = \sqrt{x+4}$

9-misol. $f(x) = \sqrt[4]{3x-15}$ funksiyaning aniqlanish sohasini toping.

Yechilishi: ► Ushbu funksiyaning aniqlanish sohasi ildiz ko‘rsatkichi juft bo‘lganligi uchun barcha nomanfiy sonlardan iborat:

$\sqrt[4]{3x-15} \geq 0$, $3x - 15 \geq 0$, $3x \geq 15$, $x \geq 5$,

$$D(f)=\{x \mid x \ge 5\}=[5;\ \infty)\,.\quad ◀$$

k – daraja ko‘rsatkichli $f(x)=ax^k$ ko‘rinishdagi **darajali funksiyalar** ham amaliyotda keng qo‘llaniladi.

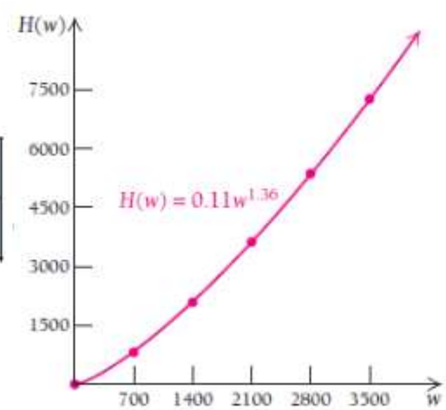
10-misol. Zoologiya. Shaxsiy hudud[3]. Hayvonlarning shaxsiy hududlari – bu ular harakatlanadigan, ov qiladigan maydondir. Yirtqich hayvonlarning shaxsiy hududlarini funksiya sifatida approksimatsiyalash mumkin: $H(w)=0.11w^{1.36}$, bunda w – yirtqichning massasi, gr. $H(w)$ – shaxsiy hududi yuzasi, gektar.

Yechilishi: ► Bu funksiya qiymatlarini hisoblashda uni

$$w^{1.36}=w^{\frac{136}{100}}=\sqrt[100]{w^{136}}=\sqrt[25]{w^{34}}$$

deb o‘zgartirib, kalkulyatorning y^x funksiyasidan foydalanishimiz mumkin.

w	0	700	1400	2100	2800	3500
$H(w)$	0	814.2	2089.9	3627.5	5364.5	7266.5



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[3]. М. Молюков, И. Песков «Красная книга мира», 2019 г.

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-12 misollarda funksiyalar grafiklarini bitta koordinata sistemasida tasvirlang:

1. $y=\frac{1}{4}x^2$ va $y=-\frac{1}{4}x^2$

2. $y=\frac{1}{3}x^2$ va $y=-\frac{1}{3}x^2$

3. $y=x^2$ va $y=x^2-2$

4. $y=x^2$ va $y=x^2+3$

5. $y=-2x^2$ va $y=-2x^2+1$

6. $y=-3x^2$ va $y=-3x^2+2$

7. $y=|x|$ va $y=|x-3|$

8. $y=|x|$ va $y=|x+1|$

9. $y=x^3$ va $y=x^3+2$

10 $y=x^3$ va $y=x^3+1$

11. $y=\sqrt{x}$ va $y=\sqrt{x+2}$

12. $y=\sqrt{x}$ va $y=\sqrt{x-1}$

13-32 misollarda funksiyaning aniqlanish sohasini toping va grafigini chizing. Funksiyalarning qaysilari parabola bo‘ladi? Agar parabola bo‘lsa, uning simmetriya chizig‘ini aniqlang.

13. $f(x)=x^2+4x-8$

14. $f(x)=x^3+2x-7$

15. $f(x)=2x^4-4x^2-3$

16. $g(x)=3x^2-8x$

17. $g(x)=x^2-4x+3$

18. $g(x)=x^2-6x+5$

19. $y=-x^2+2x-1$

20. $y=-x^2-x+6$

21. $h(x)=2x^2-6x+1$

22. $h(x)=3x^2-6x+5$

23. $g(x)=\frac{4}{x}$

24. $g(x)=-\frac{4}{x}$

25. $y=\frac{3}{x}$

26. $y=-\frac{3}{x}$

27. $g(x)=\frac{1}{x^2}$

28. $g(x)=\frac{1}{1-x}$

29. $g(x)=\frac{x^2+5x+6}{x+3}$

30. $y=\frac{x^2+7x+10}{x+2}$

31. $y=\frac{x^2-1}{x+1}$

32. $y=\frac{x^2-25}{x-5}$

33. $y=\frac{3}{|x|}$

34. $y=\frac{1}{|x|}$

35-44 misollarda kvadrat tenglamani yeching:

35. $x^2-2x=2$

36. $x^2-2x+1=6$

37. $x^2+6x=1$

38. $x^2+4x=3$

39. $4x^2=4x-1$

40. $-4x^2=4x-1$

41. $3x^2+8x+2=0$

42. $2y^2-5y=1$

43. $p+7+\frac{9}{p}=0$

44. $1-\frac{1}{z}=\frac{1}{z^2}$

45-54 misollarda funksiyaning aniqlanish sohasini toping:

45. $y=\sqrt{5x+7}$

46. $f(x)=\sqrt{6-15x}$

47. $g(x)=\frac{x^2-81}{x-9}$

48. $y=\frac{x^4+17}{x^2+6x+5}$

49. $g(x)=\frac{x^3+2x-81}{x^2-5x+6}$

50. $y=\sqrt[6]{8-x}$

51. $y=\frac{x^3}{x^2-5x+6}$

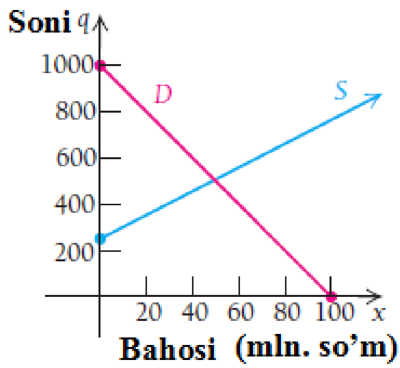
52. $y=\sqrt[5]{8-x}$

53. $y=\frac{1}{x^2-5x+6}$

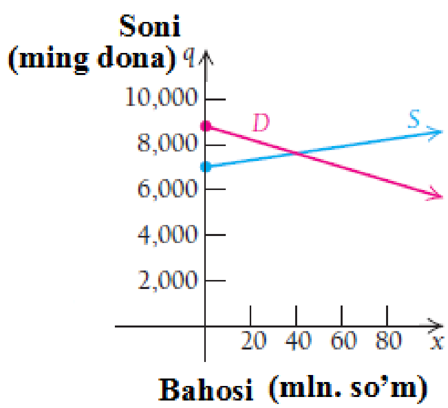
54. $y=\sqrt[3]{x+5}$

55-62 misollarda talab va taklif funksiyalarining o'zaro muvozanatlashadigan nuqtasini toping:

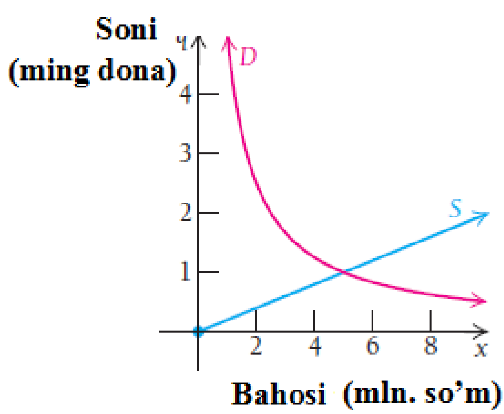
55. Talab: $D(x)=1000-10x$; Taklif: $S(x)=250+5x$



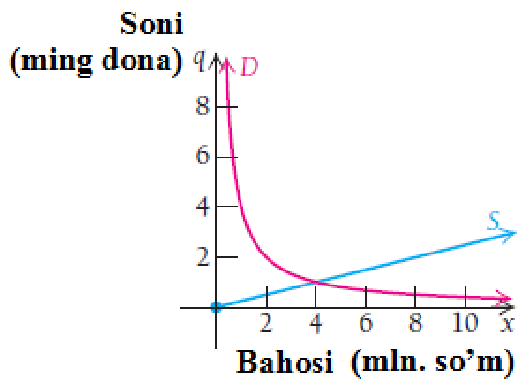
56. Talab: $D(x)=8800-30x$; Taklif: $S(x)=7000+15x$



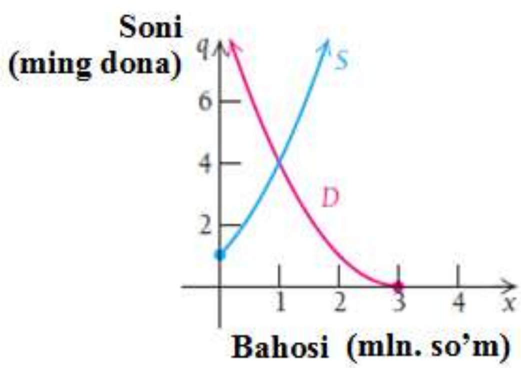
57. Talab: $D(x)=\dfrac{5}{x}$; Taklif: $S(x)=\dfrac{x}{5}$



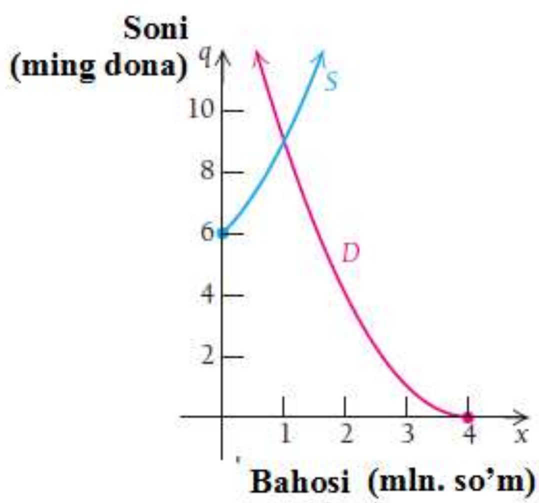
58. Talab: $D(x)=\dfrac{4}{x}$; Taklif: $S(x)=\dfrac{x}{4}$



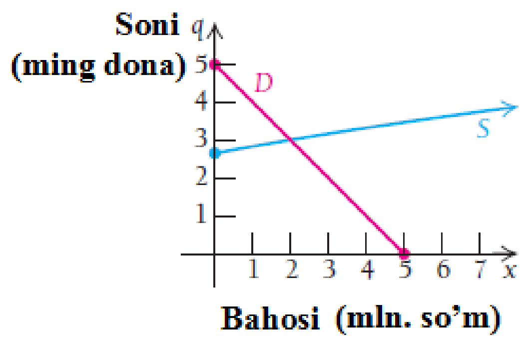
59. Talab: $D(x)=(x-3)^2$; Taklif: $S(x)=x^2+2x+1$



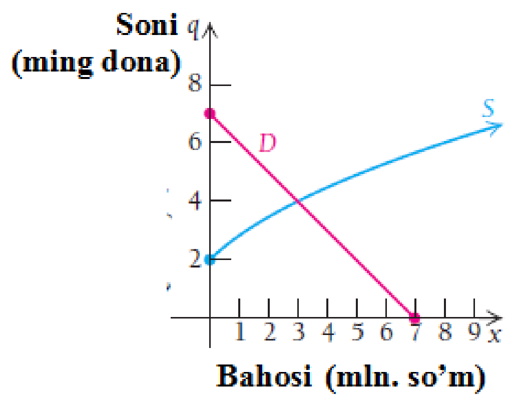
60. Talab: $D(x)=(x-4)^2$; Taklif: $S(x)=x^2+2x+6$



61. Talab: $D(x)=5-x$; Taklif: $S(x)=\sqrt{x+7}$



62. Talab: $D(x)=7-x$; Taklif: $S(x)=2\sqrt{x+1}$



63. Talab. Sotilgan plazmali televizorlarning narxi uning soniga teskari proporsional. Agar 85 000 dona televizorning har biri 2 900 000 so‘mdan sotilgan bo‘lsa, ular 850 000 so‘m deb kelishilganda nechta televizor sotilgan bo‘lar edi?

64. Fizika: Radarning o‘lchash masofasi. x Watt quvvatga ega bo‘lgan ARSR-3 markali radarning maksimal o‘lchash masofasi (mill hisobida) $R(x)=11.74x^{0.25}$ funksiya bilan approksimatsiyalanadi.

a) Agar radarning quvvati 40 000 Wt, 50 000 Wt, 60 000 Wt bo‘lsa, uning maksimal o‘lchash masofasini aniqlang.

b) Funksiyaning grafigini chizing.

65. Shaxsiy hudud. Hayvonlarning shaxsiy hududlari – bu ularning ov qiladigan maydonidir. Yirtqich hayvonlarning shaxsiy hududlarini funksiya sifatida approksimatsiyalash mumkin: $H(w)=0.059w^{0.92}$,

bunda $w-$yirtqichning massasi, gr; $H(w)-$shaxsiy hududi yuzasi, gektar.

Keltirilgan jadvalni to‘ldiring va funksiya grafigini yasang:

w	**0**	**1 000**	**2 000**	**3 000**	**4 000**	**5 000**	**6 000**	**7 000**
$H(w)$	0	34.0						

66. Ekologiya: Muhitning ifloslanishini nazorat qilish. Muhitning ifloslanish darajasini nazorat qilish vazifasi barcha davlatlarni qiynab kelayotgan masaladir. Agar muhit ifloslanishi nazorat qilinmasa, juda ayanchli holat yuz berishi mumkin. 1980 yilni $t=0$ va 2015 yilni $t=35$ deb hisoblansa, ko‘pchilik shaharlarda $P(t)=1000t^{1.25}+14000$ funksiya t yillardagi 1sm^3 muhitning o‘rtacha ifloslanish darajasini ifodalaydi.

a) Muhitning 2000, 2005, 2014 yillardagi ifloslanish ko‘rsatkichini aniqlang.

b) [0; 50] oraliqdagi funksiya grafigini chizing.

67. Tibbiyot: Tana yuzasi va vazn. Vazni 75 kg bo‘lgan bemorning nurlantiriladigan tana yuzasi $f(h)=0.144h^{0.5}$ funksiya orqali approksimatsiyalanadi. Bunda $f(h)$ yuza m^2 da, bo‘yining h balandligi sm da hisoblangan.

a) Og‘irligi 75 kg va bo‘yining balandligi 180 sm bo‘lgan kishining nurlantiriladigan taxminiy tana yuzasini hisoblang.

b) Og‘irligi 75 kg va bo‘yining balandligi 170 sm bo‘lgan kishining nurlantiriladigan taxminiy tana yuzasini hisoblang.

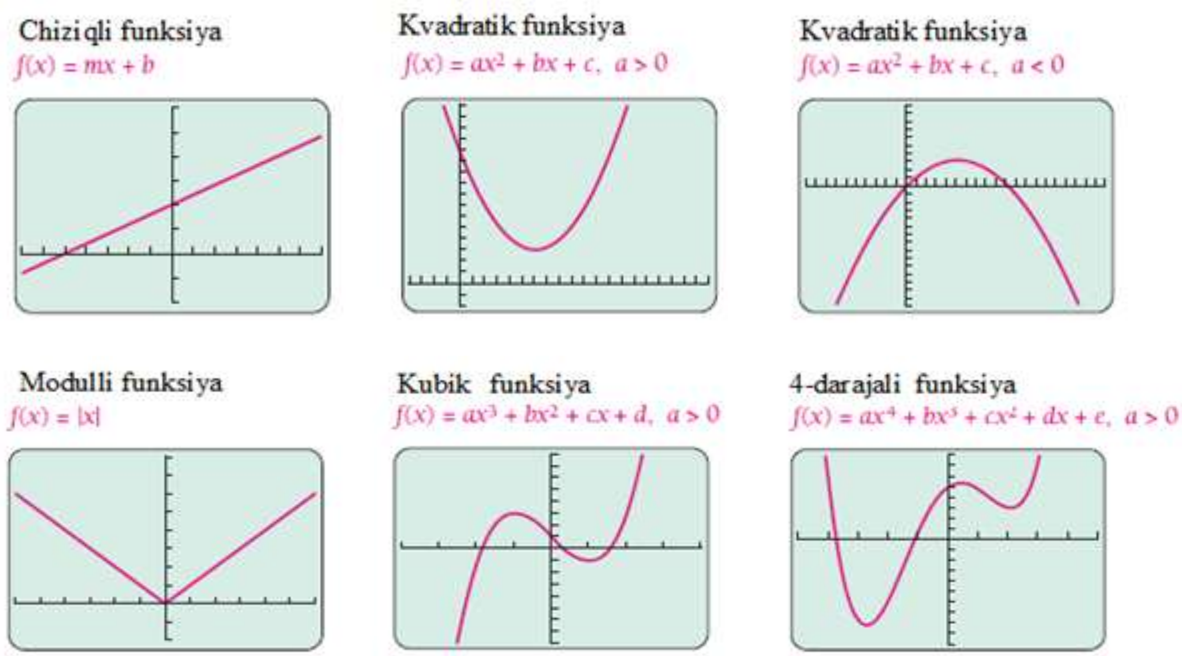
c) $0 \leq h \leq 200$ oraliq uchun funksiya grafigini chizing.

68. Zip qonuni. Zip qonuniga ko‘ra, odamlar S soni shaharlar N soniga teskari proporsional ekan. 2008 yilda Amerikaning 52 ta shaharlarining har birida 350 000 nafardan ortiq kishi istiqomat qilgan. 350 000 dan 500 000 gacha, 300 000 dan 600 000 gacha kishi yashaydigan Amerika shaharlari sonini baholang.

1.6. Funksiyalarni approksimatsiyalash

1.6.1. Ma‘lumotlar bazasi asosida funksiyani aniqlash

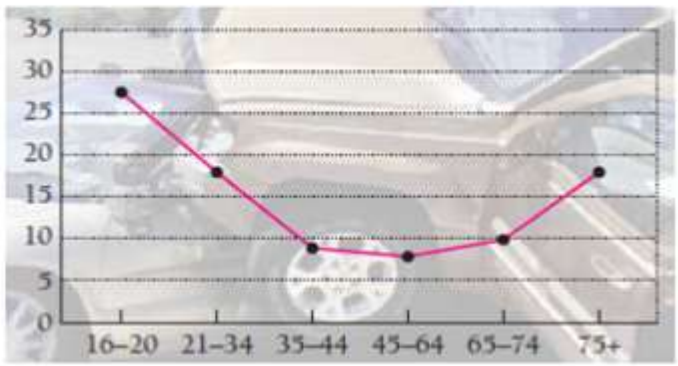
Oldingi mavzularda siz bilan chiziqli, kvadratik, kubik, modulli, 4-darajali funksiyalarni ko‘rib chiqdik. Bu funksiyalarning grafiklarini yana yodga olaylik:



Endi real hayotdagi ba‘zi ma‘lumotlarga asoslangan masalalarni ko‘rib chiqaylik. Agar bizga biror ma‘lumotlar bazasi berilgan bo‘lsa, ular qaysi funksiyaga mos keladi? Buni aniqlashning oddiy usuli **nuqtalarning geometrik o‘rni** deb ataluvchi ma‘lumotlar diagrammasidan foydalanishdir. Shunga qarab, funksiyalarning qaysi birining grafigiga o‘xshashini aniqlaymiz.

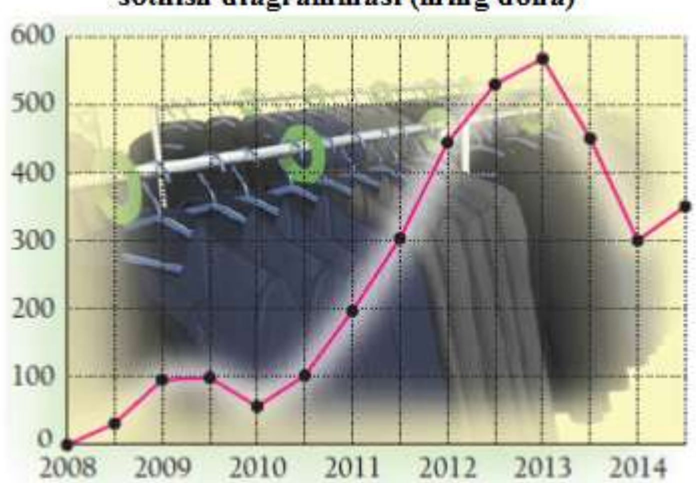
Masalan, ma‘lumotlarni grafigini chizsak, to‘g‘ri chiziq hosil bo‘lsa, chiziqli funksiya deb xulosa qilamiz. Agar diagramma parabolaga o‘xshab, oldin o‘sib borib, keyin kamayib ketsa, kvadrat funksiya deyish mumkin.

1-misol. 2019 yilda avtohalokatda olamdan o'tgan haydovchilar soni

► 

Ma‘lumotlar asosida chizilgan diagrammada grafik parabolaga o‘xshab, oldin kamayib, keyin o‘sgan $f(x)=ax^2+bx+c,\ a>0$. ◄

2-misol. D'Maretti firmasida kostyum-shimlarning yillar bo'yicha sotilish diagrammasi (ming dona)

► 

Diagrammadagi ko‘tarilish va tushishlar bu funksiya kvadratik ham, chiziqli ham bo‘lmasligini ko‘rsatadi. Lekin bu funksiya ko‘phad ko‘rinishidagi funksiya bo‘lishi mumkin. ◄

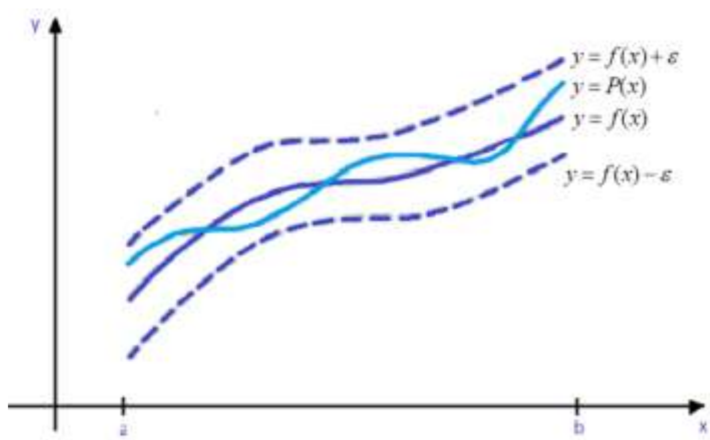
Ba‘zi diagrammalar asosida, ularning alohida qiymatlarini olgan holda funksiyaning analitik ifodasini tuzish mumkin.

Approksimatsiyalash – **yaqinlashtirish** degan ma’noni bildiradi.

Ko‘pincha amaliy masalalarni yechishda qandaydir $y = f(x)$ funksional bog‘lanishlar qiymatlarini hisoblashga to‘g‘ri keladi. Bunday masalalarda ikkita holat bo‘lishi mumkin:

1. $[a;b]$ oraliqda x va y lar orasidagi oshkor bog‘lanish ma’lum emas, faqat $\{x_i, y_i\}$, $i = \overline{1,n}$ tajriba ma’lumotlari jadvali ma’lum. Bu jadvaldan $[x_i, x_{i/2}] \in [a, b]$ oraliqda $y = f(x)$ bog‘lanishni aniqlash talab qilinadi.

2. $y = f(x)$ bog‘lanish ma’lum va uzluksiz, biroq u amaliy hisoblashlar uchun murakkablik qiladi. Bunday holda $y = f(x)$ funksiyani va uning (hosilasi, maksimum va minimum qiymatlari, funksiya integrali kabi) xarakteristikalarini hisoblash ishlarini soddalashtirish kerak bo‘ladi.



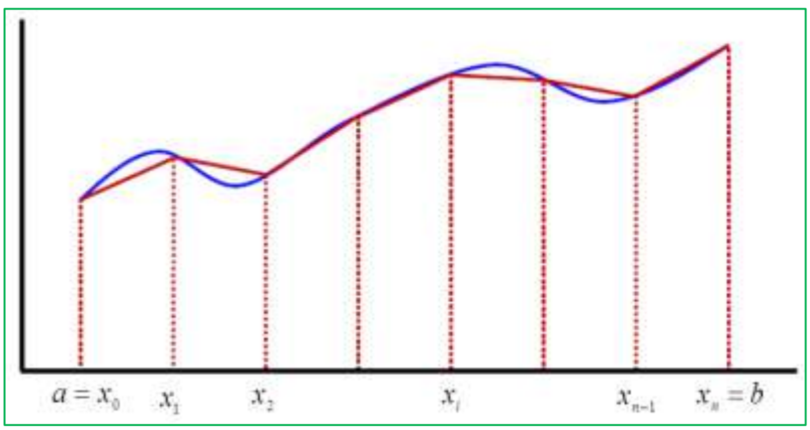
Shuning uchun moddiy resurslarni va vaqtni iqtisod qilish maqsadida qandaydir boshqa $y = P(x)$ funksional bog‘lanish tuziladi. Bu tuzilgan bog‘lanish $y = f(x)$ ga uning asosiy parametrlari bo‘yicha yaqin bo‘lishi, hisoblash oson va qulay bo‘lishi kerak, ya’ni $y = f(x)$ funksiyani aniqlanish sohasida **approksimatsiyalash** kerak.

$y = P(x)$ funksiyaga **approksimatsiyalovchi funksiya** deyiladi.

Agar yaqinlashishni biror $\{x_i\}$, $i = \overline{1,n}$ diskret to‘plamda bajarsak, u holda approksimatsiyaga **nuqtaviy approksimatsiya** deyiladi.

$[a;b]$ oraliqda yotuvchi, tajriba asosida olingan x_i nuqtalarni o‘sish tartibida raqamlab chiqamiz va ularni **tugunlar** deb ataymiz:

$$a \le x_0 \le x_1 \le x_2 \le ... \le x_n \le b$$



Nuqtaviy approksimatsiyalash turlaridan biri - interpolyatsiyalashdir.

Biz ushbu bo‘limda 1-holatni, ya’ni ma’lumotlar jadval ko‘rinishida berilganda, funksiya ko‘rinishini tiklashni o‘rganamiz.

Interpolyatsiyalash tarixini o‘rganadigan bo‘lsak, uning turli xil ko‘rinishlariga duch kelamiz, biroq ularning aksariyati quyosh sistemasi haqidagi ma’lumotlardan iborat.

Bizgacha yetib kelgan manbaalarga ko‘ra, **er. avv. III asrda** qadimgi Vavilonda olimlar **quyosh, oy va planetalarning joylashuvini** bashorat qilish uchun chiziqli interpolyatsiyadan foydalanishgan. Katta yer egalari bu ma’lumotlar asosida qachon yerga don sochish mumkinligini bilishgan.

Er. avv. 150 yillarda Gretsiyada olim Gipparx **fazoviy jismlar o‘rnini** hisoblashda sinusoidal chiziqli interpolyatsiyadan foydalangan.

Eramizning 600 yillarida Xitoyda Li Dzu **“Imperiya taqvimi”**ni tuzishda Grigoriy-Nyuton interpolyatsiyasiga o‘xshash kvadratik interpolyatsiya formulasidan foydalangan.

Eramizning 625 yilida Hindistonda hind astronomi Braxmagupta tengmas oraliqlar uchun **sinusoidal kvadratik interpolyatsiya** metodini qo‘llagan...

Yer ustida amaliy maqsadlarda interpolyatsiya formulalari keng qo‘llanib kelinmoqda. Lekin **okean uchun topilgan interpolyatsion yangilik**[4] juda muhim ahamiyatga ega bo‘ldi. 100 yillar davomida dengizchilar kenglik va uzunlikni aniqlash uchun tuzilgan maxsus jadvaldan foydalanishadi. Bu jadval biz o‘rganishni maqsad qilgan interpolyatsion formulalar asosida hisoblangan. Ilk davrlardayoq Fransiyada dengizchilar uchun jadvallar ko‘plab nusxada qonuniy va noqonuniy tarzda chop qilina boshlandi. Biroq, hisoblash ishlarida hisobchilar xatoliklarga yo‘l qo‘yishar edi. Natijada kemalarning adashib ketish holatlari kuzatildi. Hukumat bu jadvallarni yaratgan kishilarni “gilyatina”da boshini ola boshladi va shundan so‘ng jadvallar tuzish to‘xtab qoldi.

Charlz Bebbeg hisoblashlardagi xatolarni oldini olish maqsadida perfokarta yordamida dasturlashtirilgan mexanik kompyuter yaratdi. Hozirda uni zamonaviy hisoblashning otasi deyishadi.

II jahon urushidan oldin Fransiyada to‘xtatib qo‘yilgan jadval tuzish musobaqasi yana avj oldi. Endi barcha davlatlar juda aniqlik bilan tuzilgan “uzunlik va kenglikni aniqlab beradigan jadval”ga ehtiyoj seza boshlashdi.

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4. Dengizchi lug‘ati (Словарь моряка)

Suv osti kemalarining yo‘nalishni aniqlashi, suvda yoki quruqlikda dushman joylashgan nuqtani nishonga olish juda aniqlikni taqozo qiladi…

Interpolyatsiyalashdan maqsad shuki, (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) ma’lumotlar (nuqtalar) ma’lum. Biroq bizga $y = f(x)$ funksiyaning boshqa x nuqtadagi qiymati kerak bo‘lsin. Ma’lumotlar asosida (x, y) ni aniqlash mumkinmi? Misol uchun, raketa tezligining 10, 15, 20-sekundlardagi tezliklarini o‘lchay olganmiz. Lekin 16-sekunddagi tezligi nimaga teng?

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

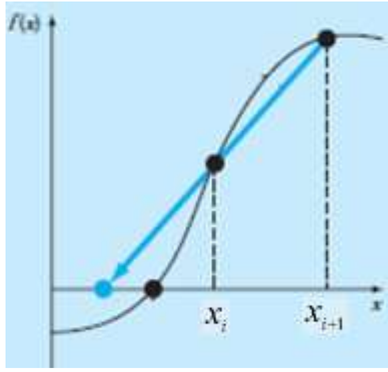
Ana shunday holatlarda interpolyatsiyalash formulalari kerak bo‘ladi.

Hozirda ko‘pgina amaliy masalalar interpolyatsiyalash yordamida yechiladi. Eng oddiy interpolyatsilash – bu chiziqli interpolyatsiyalashdir.

1.6.2. Chiziqli interpolyatsiya

Agar $[a;b]$ oraliqda berilgan jadval yoki diagrammadan foydalanib, algebraik ikkihad tuzsak, unga **chiziqli interpolyatsiya** deyiladi:

$$f(x) = a_0 + a_1 x.$$



Berilgan oraliqqa tegishli ikkita $x_i,\ x_{i+1}$ tugunni olamiz va 2 noma'lumli chiziqli tenglamalar sistemasini tuzamiz:

$$\begin{cases} f(x_i) = a_0 + a_1 x_i \\ f(x_{i+1}) = a_0 + a_1 x_{i+1} \end{cases}$$

Bu formuladan $a_0,\ a_1$ ni topib, chiziqli interpolyatsiya formulasiga qo'yamiz.

Umuman olganda $[x_{i-1}, x_i] \in [a,b]$ siniq chiziq kesmasi uchun (x_{i-1}, y_{i-1}) va (x_i, y_i) nuqtalardan o'tuvchi to'g'ri chiziq tenglamasini tuzish mumkin. Kompyuterda hisoblash uchun chiziqli interpolyatsiyaning ishchi formulasini tuzamiz: $$\frac{y - y_{i-1}}{y_i - y_{i-1}} = \frac{x - x_{i-1}}{x_i - x_{i-1}}$$

Bu tenglamadan y ni topib olamiz: $y - y_{i-1} = \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot (y_i - y_{i-1})$

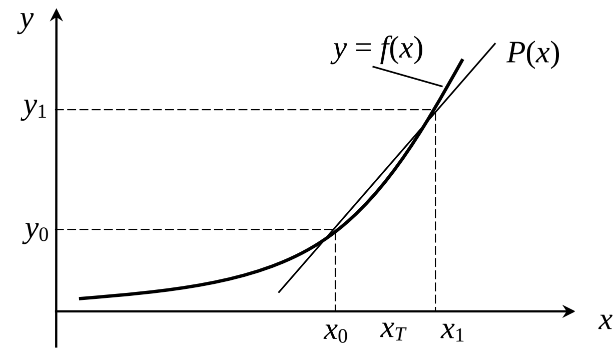
$$y = \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot (y_i - y_{i-1}) + y_{i-1}$$

$$y = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} x + y_{i-1} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}} x_{i-1}$$

Belgilash kiritamiz: $a_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}, \quad b_i = y_{i-1} - a_i x_{i-1}, \quad i = \overline{1,n}$.

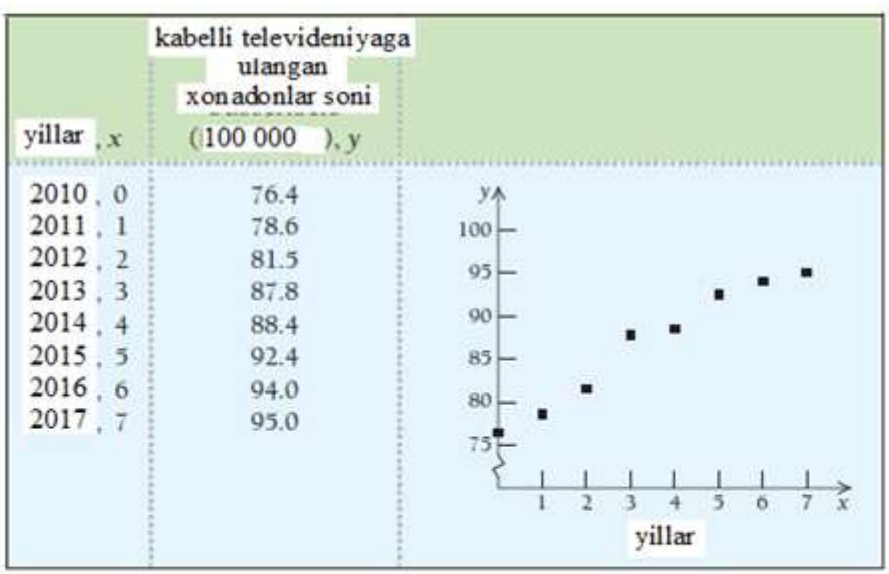
U holda **ishchi formula:** $y = a_i x_T + b_i,\ x_{i-1} \le x_T \le x_i$ hosil bo'ladi.

Chizmadan ko‘rish mumkinki, ishchi formulani amalga oshirish uchun oldin x_T qiymat tushadigan oraliqni aniqlash kerak, so‘ngra bu oraliq chegaralaridan foydalanish mumkin.



Biz oldingi mavzularda o‘rgangan $y - y_1 = m(x - x_1)$ tenglamadan ham foydalanishimiz mumkin.

3-misol. Kabelli televideniya ulanuvchilari. Quyidagi ma‘lumot O‘zbekistonda nechta xonadon kabelli televideniyaga ulanganligini ko‘rsatadi.



yillar, x	kabelli televideniyaga ulangan xonadonlar soni (100 000), y
2010, 0	76.4
2011, 1	78.6
2012, 2	81.5
2013, 3	87.8
2014, 4	88.4
2015, 5	92.4
2016, 6	94.0
2017, 7	95.0

a) Ma‘lumotlarga mos keladigan chiziqli funksiyani toping;

b) Modeldan foydalanib, 2025 yilda nechta xonadon kabelli televideniyaga ulanishini bashorat qiling.

Yechilishi: ►

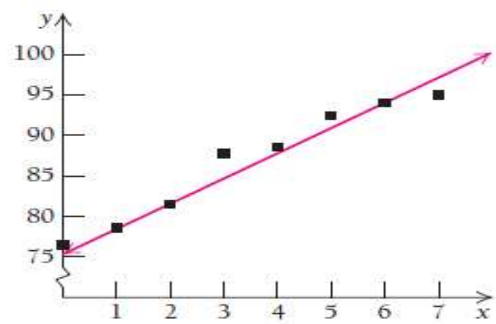
a) Funksiya tenglamasini tuzish uchun ixtiyoriy 2 ta qiymatni olamiz.

1) Kabelli televideniyaga ulangan xonadonlar soni 78.6 va 92.4 qiymatlarni tanladik. Bular 1 va 5 ga to‘g‘ri keladi.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{92.4-78.6}{5-1}=\frac{13.8}{4}=3.45$$

$y-y_1=m(x-x_1)$ ga ko‘ra, $y-92.4=3.45(x-5)$

$y=3.45x+75.15$ chiziqli funksiya tenglamasini hosil qildik.



2) Agar kabelli televideniyaga ulangan xonadonlar soni 81.5 va 94.0 qiymatlarni tanlasak-chi. Bular 2 va 6 ga to‘g‘ri keladi.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{94.0-81.5}{6-2}=\frac{12.5}{4}=3.125$$

$y-y_1=m(x-x_1)$ ga ko‘ra,

$y-94.0=3.125(x-6)$, ya’ni

$y=3.125x+75.25$ tenglama hosil bo‘ladi.

$y=3.125x+75.25$ tenglama $y=3.45x+75.15$ ga yaqin. Ular ikkalasi ham taxminiy tenglamalar hisoblanadi.

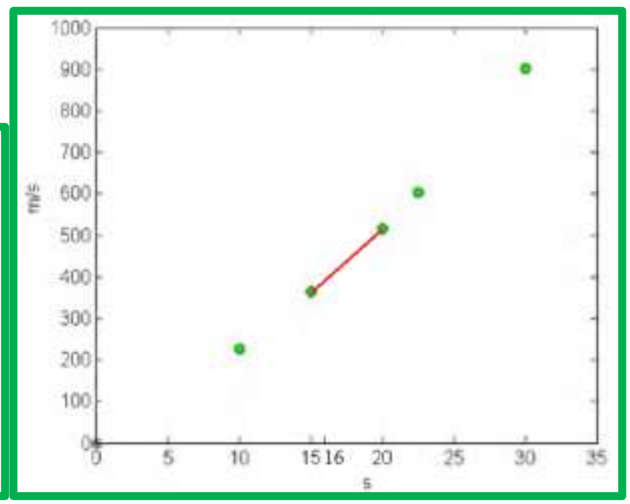
b) Modeldan foydalanib, 2025 yilda nechta xonadon kabelli televideniyaga ulanishini bashorat qilamiz. 2025-2010=15, u holda

$$y = 3.125x + 75.25 = 3.125 \cdot 15 + 75.25 = 122.125$$

Ya'ni 2025 yilda 12 212 500 xonadon kabelli televideniyaga o'tishar ekan. ◄

4-misol. Raketaning ko'tarilish tezligi vaqtning funksiyasi sifatida jadvalda keltirilgan. $t = 16$ sekunddagi raketa tezligini hisoblang.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Yechilishi: ► Chiziqli interpolyatsiyani tuzib, tezlikni aniqlaymiz:

$$v(t) = a_0 + a_1 t.$$

$a_0,\ a_1$ ni topish uchun $t = 16$ sekundga eng yaqin nuqtalarni aniqlaymiz, ular quyidagilar:

$$t_0 = 15; \quad v(t_0) = 362.78$$
$$t_1 = 20; \quad v(t_1) = 517.35$$

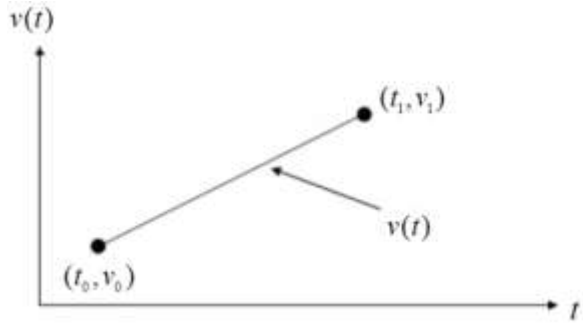
Bu nuqtalarni formulaga qo'yib, tenglamalar sistemasini tuzamiz:

$$\begin{cases} v(15) = a_0 + 15a_1 = 362.78 \\ v(20) = a_0 + 20a_1 = 517.35 \end{cases} \qquad \begin{matrix} a_0 = -100.93 \\ a_1 = 30.914 \end{matrix}$$

$$v(t) = a_0 + a_1 t = -100.93 + 30.914t$$

Ko‘tarilish tezligining chiziqli interpolyatsiyasi:

$$v(t) = 30.914t - 100.93, \quad 15 \le \text{t} \le 20$$



Shunday qilib, raketaning $t = 16$ sekunddagi tezligini topamiz:

$$v(16) = 30.914 \cdot 16 - 100.93 = 393.7 \quad \text{m/s}$$ ◄

5-misol. Jadval bilan berilgan $y = f(x)$ funksiya qiymatini x = 0,4 nuqtada chiziqli interpolyatsion formuladan foydalanib hisoblang:

i	0	1	2	3
x_i	0	0,1	0,3	0,5
y_i	–0,5	0	0,2	1

Yechilishi: ► Dastlab ishchi formulani yozib olamiz:

$$y = a_i x + b_i \qquad x_{i-1} \le x_T \le x_i,$$

bunda $a_i = \dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}$, $b_i = y_{i-1} - a_i x_{i-1}$, $i = \overline{1,n}$, $x_t = 0.4$; $0.3 \le x_t \le 0.5$;

Jadvaldagi $x_{i-1} = 0.3$; $x_i = 0.5$; $y_{i-1} = 0.2$; $y_i = 1$ qiymatlar yordamida koeffitsiyentlarni hisoblaymiz: $a_i = \dfrac{y_i - y_{i-1}}{x_i - x_{i-1}} = \dfrac{1 - 0.2}{0.5 - 0.3} = \dfrac{0.8}{0.2} = 4;$

$$b_i = y_{i-1} - a_i x_{i-1} = 0.2 - 4 \cdot 0.3 = -1;$$

Natijada funksiyamizning ko‘rinishi $y = 4x–1$ ekanligini aniqladik. Endi $x=0.4$ qiymatda shu chiziqli funksiyaning son qiymatini aniqlaymiz:

$$y = 4 \cdot 0.4 - 1 = 0.6 \quad ◀$$

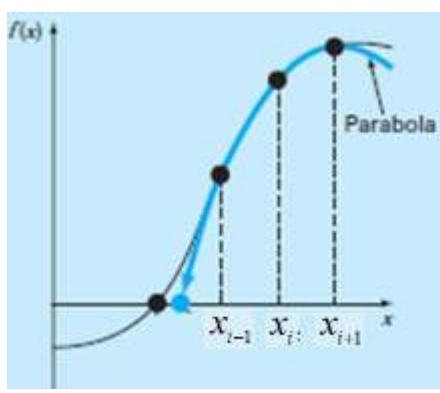
1.6.3. Kvadratik interpolyatsiya

Agar $[a;b]$ oraliqda berilgan jadval yoki diagrammadan foydalanib, 3 ta haddan iborat ko‘phad tuzilsa, unga **kvadratik (parabolik) interpolyatsiya** deyiladi: $f(x) = a_0 + a_1 x + a_2 x^2$.

Bunda oraliqqa tegishli uchta x_{i-1}, x_i, x_{i+1} tugunni olamiz va 3 noma’lumli chiziqli tenglamalar sistemasini tuzamiz:

$$\begin{cases} f(x_{i-1}) = a_0 + a_1 x_{i-1} + a_2 x_{i-1}^2 \\ f(x_i) = a_0 + a_1 x_i + a_2 x_i^2 \\ f(x_{i+1}) = a_0 + a_1 x_{i+1} + a_2 x_{i+1}^2 \end{cases}$$

Bu formuladan a_0, a_1, a_2 koeffitsiyentlarni topib, kvadratik interpolyatsiya formulasiga qo‘yamiz.



6-misol. Fiziologiya. Uyqu vaqtlari va o‘lim darajasi. Olim G. Morovichning to‘plagan ma‘lumotlariga ko‘ra, erkaklarning o‘limi bilan ularning kuniga o‘rtacha necha soat uxlashi o‘zaro bog‘liq ekan.

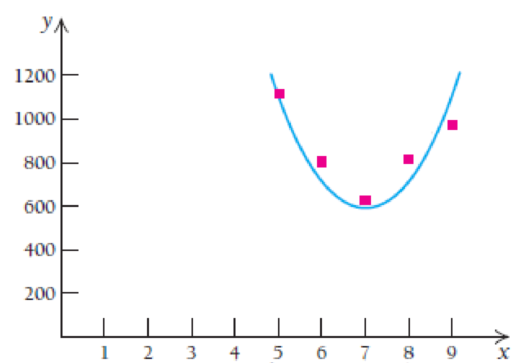
a) Ma‘lumotlar asosida grafik chizing. Grafik kvadrat funksiya bo‘ladimi?

b) Kvadrat funksiya tenglamasini tuzing.

c) Modeldan foydalanib, 2 soat, 8 soat va 10 soat uxlaydigan kishilarning o‘limi ko‘rsatkichini aniqlang.

O’rtacha uyqu soati, x	O’lim ko’rsatkichi, y (100 000 kishi hisobida)
5	1121
6	805
7	626
8	813
9	967

Yechilishi: ► a) Jadval asosida grafik chizamiz:



b) Model kvadrat funksiyaga o‘xshaydi. Shuning uchun

$$f(x) = ax^2 + bx + c,\ a > 0$$

$a,\ b,\ c$ koeffitsiyentlarni topamiz:

$$\begin{cases} 1121 = a \cdot 5^2 + b \cdot 5 + c \\ 626 = a \cdot 7^2 + b \cdot 7 + c \\ 967 = a \cdot 9^2 + b \cdot 9 + c \end{cases} \qquad \begin{cases} 1121 = 25a + 5b + c \\ 626 = 49a + 7b + c \\ 967 = 81a + 9b + c \end{cases}$$

Tenglamalar sistemasini hosil qilamiz va uni yechib, $a = 104.5,\ b = -1501.5,\ c = 6016$ ni hosil qilamiz. Bu sistemani yechish uchun MathLab, Mathcad dasturlaridan foydalanish mumkin. Natijada $y = 104.5x^2 - 1501.5x + 6016$ kvadrat funksiyani hosil qilamiz.

c) 2 soat uyquda $y = 104.5 \cdot 2^2 - 1501.5 \cdot 2 + 6016 = 3431$

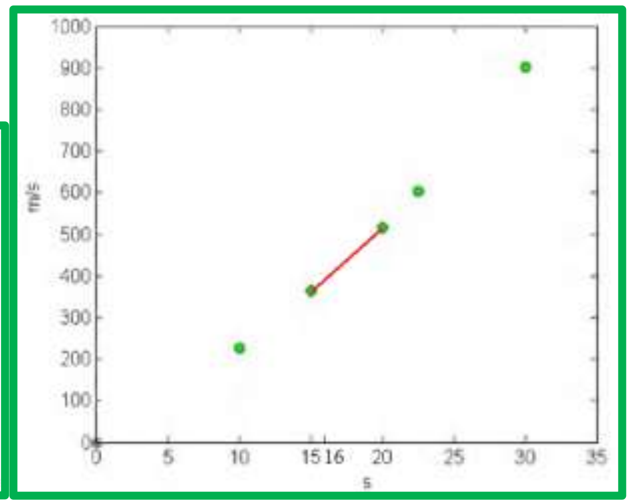
8 soatda $y = 104.5 \cdot 8^2 - 1501.5 \cdot 8 + 6016 = 692$

10 soatda $y = 104.5 \cdot 10^2 - 1501.5 \cdot 10 + 6016 = 1451$ ga teng. ◄

7-misol. Raketaning ko‘tarilish tezligi vaqtning funksiyasi sifatida jadvalda keltirilgan. $t = 16$ sekunddagi raketa tezligini hisoblang.

(4-misolda buni chiziqli interpolyatsiya bilan hisoblagan edik.)

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Yechilishi: ► Kvadratik interpolyatsiyani tuzamiz, buning uchun tezlikni aniqlaymiz: $v(t) = a_0 + a_1 t + a_2 t^2$.

$a_0,\ a_1,\ a_2$ larni topish uchun $t = 16$ sekundga eng yaqin bo‘lgan 3 ta nuqtani aniqlaymiz, ular quyidagilar:

$t_0 = 10, \quad v(t_0) = 227.04$
$t_1 = 15, \quad v(t_1) = 362.78$
$t_2 = 20, \quad v(t_2) = 517.35$

$$\begin{cases} v(10) = a_0 + 10a_1 + 10^2 a_2 = 227.04 \\ v(15) = a_0 + 15a_1 + 15^2 a_2 = 362.78 \\ v(20) = a_0 + 20a_1 + 20^2 a_2 = 517.35 \end{cases}$$

Hosil qilingan tenglamalar sistemasini yechamiz:

$$\begin{cases} a_0 + 10a_1 + 100a_2 = 227.04 \\ a_0 + 15a_1 + 225a_2 = 362.78 \\ a_0 + 20a_1 + 400a_2 = 517.35 \end{cases} \qquad \begin{aligned} a_0 &= 12.05 \\ a_1 &= 17.733 \\ a_2 &= 0.3766 \end{aligned}$$

Raketa tezligi uchun kvadratik interpolyatsiyani tuzamiz:

$$v(t) = 0.3766t^2 + 17.733t + 12.05, \quad 10 \le \mathrm{t} \le 20$$

Shunda raketaning $t = 16$ sekundgagi ko‘tarilish tezligi quyidagiga teng bo‘ladi:

$$v(t) = 0.3766 \cdot 16^2 + 17.733 \cdot 16 + 12.05 = 392.19 \quad \text{m/s}$$

Chiziqli interpolyatsiya formulasi yordamida topilgan tezlik qiymati:

$$v(16) = 393.7 \quad \text{m/s}$$

Kvadratik interpolyatsiya formulasi yordamida topilgan tezlik qiymati esa:

$$v(16) = 392.19 \quad \text{m/s}$$

Haqiqiy qiymatga yaqinlashish xatosini aniqlaymiz:

$$|\in_a| = \left| \frac{392.19 - 393.7}{392.19} \right| \cdot 100\,\% = 0.3841\,\%$$

◀

8-misol. 5-misoldagi chiziqli interpolyatsiya usulida yechilgan misolni yanada aniqroq approksimatsiyalamoqchi bo‘lsak, uni 3 ta tugun nuqtasini olib hisoblash kerak. Jadval bilan berilgan $y = f(x)$ funksiya qiymatini x=0.4 nuqtada kvadratik interpolyatsiya formulasidan foydalanib hisoblaymiz:

i	0	1	2	3
x_i	0	0,1	0,3	0,5
y_i	–0,5	0	0,2	1

Yechilishi: ► Ishchi formulani yozib olamiz:

$$y=a_i x^2+b_i x+c_i \qquad x_{i-1} \le x_T \le x_{i+1}.$$

a_i, b_i, c_i koeffitsiyentlarni aniqlash uchun tenglamalar sistemasini tuzamiz. Buning uchun x_t = 0,4 nuqtaga eng yaqin bo‘lgan 3 ta nuqtani tanlaymiz: x_{i-1} = 0,1; x_i = 0,3; x_{i+1} = 0,5.

y_{i-1} = 0; y_i = 0,2; y_{i+1} = 1.

va mos tenglamalarni hosil qilamiz:

$$\left.\begin{array}{c} a_i x_{i-1}^2 + b_i x_{i-1} + c_i = y_{i-1} \\ a_i x_i^2 + b_i x_i + c_i = y_i \\ a_i x_{i+1}^2 + b_i x_{i+1} + c_i = y_{i+1} \end{array}\right\} \Rightarrow \begin{cases} 0{,}01a_i + 0{,}1b_i + c_i = 0; \\ 0{,}09a_i + 0{,}3b_i + c_i = 0{,}2; \\ 0{,}25a_i + 0{,}5b_i + c_i = 1; \end{cases}$$

Tenglamalar sistemasini yechib, a, b, c koeffitsiyentlarni aniqlaymiz:

$$a = 0 - \frac{75}{3} \cdot \frac{1}{5} + \frac{25}{2} = 7.5; \quad b = -2; \quad c = 0{,}125;$$

Natijada izlangan funksiya ko‘rinishini olamiz: $y = 7{,}5x^2 - 2x + 0{,}125$.

Endi x = 0,4 qiymat uchun hosil bo‘lgan kvadratik funksiyaning son qiymatini aniqlaymiz. Natija y = 0,525 ga teng. ◄

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5. Autar Kaw, E.Erik Kalu “Numerical Methods with Applications: Abridged”, 2-nd edition, 2011. 756 p. (8-misolga ko‘rsatma).

1.6.4. Lagranj interpolyatsion ko'phadi

Tugunlar soni ortib borishi bilan hisoblash ishlari ham murakkablashib ketadi. Chunki *n* noma'lumli tenglamalar sistemasini yechish amaliyoti ko'p vaqt va xotira (EHM) talab qiladi.

Shuning uchun tenglamalar sistemasi tuzishni talab qilmaydigan va faqat arifmetik hisoblashlar bajariladigan ancha sodda, (n+1) ta qo'shiluvchidan iborat Lagranj interpolyatsion formulasidan foydalaniladi:

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

Bunda $L_i(x)$ ga Lagranj interpolyatsion ko'phadi deyiladi:

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

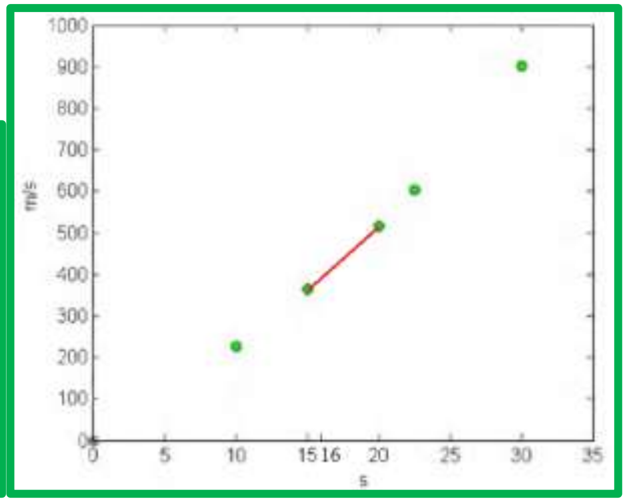
9-misol. Raketaning ko'tarilish tezligi vaqtning funksiyasi sifatida jadvalda keltirilgan. $t = 16$ sekunddagi raketa tezligini hisoblang.

Yechilishi: ► 1-tartibli Lagranj interpolyatsion formulasini tuzamiz. Bunda 2 ta tugunni olamiz:

$$v(t) = \sum_{i=0}^{1} L_i(t) v(t_i) = L_0(t) v(t_0) + L_1(t) v(t_1)$$

$$t_0 = 15; \quad v(t_0) = 362.78$$
$$t_1 = 20; \quad v(t_1) = 517.35$$

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



$$L_0(t)=\prod_{\substack{j=0\\ j\neq 0}}^{1}\frac{t-t_j}{t_0-t_j}=\frac{t-t_1}{t_0-t_1} \qquad L_1(t)=\prod_{\substack{j=0\\ j\neq 1}}^{1}\frac{t-t_j}{t_1-t_j}=\frac{t-t_0}{t_1-t_0}$$

$$v(t)=\frac{t-t_1}{t_0-t_1}v(t_0)+\frac{t-t_0}{t_1-t_0}v(t_1)=\frac{t-20}{15-20}\cdot 362.78+\frac{t-15}{20-15}\cdot 517.35$$

$$v(16)=\frac{16-20}{15-20}\cdot 362.78+\frac{16-15}{20-15}\cdot 517.35=0.8\cdot 362.78+0.2\cdot 517.35=393.69 \text{ m/s}$$

2-tartibli Lagranj interpolyatsion formulasini tuzamiz. Bunda 3 ta tugunni olamiz:

$$v(t)=\sum_{i=0}^{2}L_i(t)v(t_i)=L_0(t)v(t_0)+L_1(t)v(t_1)+L_2(t)v(t_2)$$

$$L_0(t)=\prod_{\substack{j=0\\ j\neq 0}}^{2}\frac{t-t_j}{t_0-t_j}=\frac{t-t_1}{t_0-t_1}\cdot\frac{t-t_2}{t_0-t_2},$$

$$L_1(t)=\prod_{\substack{j=0\\ j\neq 1}}^{2}\frac{t-t_j}{t_1-t_j}=\frac{t-t_0}{t_1-t_0}\cdot\frac{t-t_2}{t_1-t_2}$$

$$L_2(t)=\prod_{\substack{j=0\\ j\neq 2}}^{2}\frac{t-t_j}{t_2-t_j}=\frac{t-t_0}{t_2-t_0}\cdot\frac{t-t_1}{t_2-t_1}$$

$$v(t)=\frac{t-t_1}{t_0-t_1}\cdot\frac{t-t_2}{t_0-t_2}v(t_0)+\frac{t-t_0}{t_1-t_0}\cdot\frac{t-t_2}{t_1-t_2}v(t_1)+\frac{t-t_0}{t_2-t_0}\cdot\frac{t-t_1}{t_2-t_1}v(t_2)$$

$$t_0=10,\quad v(t_0)=227.04$$
$$t_1=15,\quad v(t_1)=362.78$$
$$t_2=20,\quad v(t_2)=517.35$$

$$v(16)=\frac{16-15}{10-15}\cdot\frac{16-20}{10-20}\cdot 227.04+\frac{16-10}{15-10}\cdot\frac{16-20}{15-20}\cdot 362.78+$$

$$+\frac{16-10}{20-10}\cdot\frac{16-15}{20-15}\cdot 517.35=392.19\ \text{m/s}$$

Approksimatsiyalash xatoligi:

$$|\in_a|=\left|\frac{392.19-393.69}{392.19}\right|\cdot 100\,\%=0.3841\,\%$$

3-tartibli Lagranj interpolyatsion formulasini tuzamiz. Bunda 4 ta tugunni olamiz:

$$v(t)=\sum_{i=0}^{3}L_i(t)v(t_i)=L_0(t)v(t_0)+L_1(t)v(t_1)+L_2(t)v(t_2)+L_3(t)v(t_3)$$

$$t_0=10,\quad v(t_0)=227.04$$
$$t_1=15,\quad v(t_1)=362.78$$
$$t_2=20,\quad v(t_2)=517.35$$
$$t_3=22.5,\quad v(t_3)=602.97$$

$$L_0(t)=\prod_{\substack{j=0\\ j\neq 0}}^{3}\frac{t-t_j}{t_0-t_j}=\frac{t-t_1}{t_0-t_1}\cdot\frac{t-t_2}{t_0-t_2}\cdot\frac{t-t_3}{t_0-t_3}$$

$$L_1(t)=\prod_{\substack{j=0\\ j\neq 1}}^{3}\frac{t-t_j}{t_1-t_j}=\frac{t-t_0}{t_1-t_0}\cdot\frac{t-t_2}{t_1-t_2}\cdot\frac{t-t_3}{t_1-t_3}$$

$$L_2(t)=\prod_{\substack{j=0\\ j\neq 2}}^{3}\frac{t-t_j}{t_2-t_j}=\frac{t-t_0}{t_2-t_0}\cdot\frac{t-t_1}{t_2-t_1}\cdot\frac{t-t_2}{t_2-t_3}$$

$$L_3(t)=\prod_{\substack{j=0\\j\neq 3}}^{3}\frac{t-t_j}{t_3-t_j}=\frac{t-t_0}{t_3-t_0}\cdot\frac{t-t_1}{t_3-t_1}\cdot\frac{t-t_2}{t_3-t_2}$$

$$v(t)=\frac{t-t_1}{t_0-t_1}\cdot\frac{t-t_2}{t_0-t_2}\cdot\frac{t-t_3}{t_0-t_3}v(t_0)+\frac{t-t_0}{t_1-t_0}\cdot\frac{t-t_2}{t_1-t_2}\cdot\frac{t-t_3}{t_1-t_3}\cdot v(t_1)+$$

$$+\frac{t-t_0}{t_2-t_0}\cdot\frac{t-t_1}{t_2-t_1}\cdot\frac{t-t_3}{t_2-t_3}v(t_2)+\frac{t-t_0}{t_3-t_0}\cdot\frac{t-t_1}{t_3-t_1}\cdot\frac{t-t_2}{t_3-t_2}\cdot v(t_3)$$

$$v(16)=\frac{16-15}{10-15}\cdot\frac{16-20}{10-20}\cdot\frac{16-22.5}{10-22.5}\cdot 227.04+\frac{16-10}{15-10}\cdot\frac{16-20}{15-20}\cdot\frac{16-22.5}{15-22.5}\cdot 362.78+$$

$$+\frac{16-10}{20-10}\cdot\frac{16-15}{20-15}\cdot\frac{16-22.5}{20-22.5}\cdot 517.35+\frac{16-10}{22.5-10}\cdot\frac{16-15}{22.5-15}\cdot\frac{16-20}{22.5-20}\cdot 602.97=$$

$$=392.06\ \text{ m/s}$$

Approksimatsiyalash xatoligi:

$$|\in_a|=\left|\frac{392.06-392.19}{392.06}\right|\cdot 100\,\%=0.033269\,\%$$

◄

10-misol. 5- va 8- misollarda chiziqli hamda kvadratik interpolyatsiya asosida yechilgan misolni Lagranj interpolyatsion formulasidan foydalanib hisoblang:

i	0	1	2	3
x_i	0	0,1	0,3	0,5
y_i	–0,5	0	0,2	1

Yechilishi: ► Dastlab ishchi formulani yozib olamiz:

$$L(x)=\sum_{i=0}^{n}y_i\prod_{\substack{j=0\\j\neq i}}^{n}\frac{x-x_j}{x_i-x_j};$$

Bizning holda n = 3 gacha, shu sababli: $L(x)=\sum_{i=0}^{3}y_i\prod_{\substack{j=0\\j\neq i}}^{3}\frac{x-x_j}{x_i-x_j}=$

$$= y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} +$$

$$+ y_2 \frac{(x-x_1)(x-x_0)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} =$$

$$= \frac{125}{3} x^3 - 30x^2 + \frac{91}{12} x - 0{,}5;$$

$x = 0{,}4$ bo‘lganda $y \approx L(x) = 0{,}3999$.

Berilgan jadval asosida n=1 va $x_T = 0{,}4$ bo‘lgan hol uchun Lagranj ko‘phadini tuzamiz:

$$L(x) = \sum_{i=0}^{1} y_i \prod_{\substack{j=0 \\ j \neq i}}^{1} \frac{x - x_j}{x_i - x_j} = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{(x - x_0)}{(x_1 - x_0)} =$$

$$= 0{,}2 \frac{x - 0{,}5}{0{,}3 - 0{,}5} + 1 \frac{(x - 0{,}3)}{(0{,}5 - 0{,}3)} = 5x - 1{,}5 - x + 0{,}5 = 4x - 1;$$

Bu esa chiziqli interpolyatsion formula bilan ustma-ust tushadi. Berilgan jadval asosida n=2 va $x_T = 0{,}4$ bo‘lgan hol uchun Lagranj ko‘phadini tuzamiz:

$$y \approx L(x) = \sum_{i=0}^{2} y_i \prod_{\substack{j=0 \\ i \neq j}}^{2} \frac{x - x_j}{x_i - x_j};$$

Qaralayotgan $[x_1, x_3]$ intervalda $x_0 = 0{,}1;$ $x_1 = 0{,}3;$ $x_2 = 0{,}5;$
$y_0 = 0;$ $y_1 = 0{,}2;$ $y_2 = 1$

qiymatlarni olamiz. U holda 2–tartibli Lagranj interpolyatsion ko‘phadi hosil bo‘ladi:

$$y \approx L(x) = 0{,}2 \cdot \frac{(x - 0{,}1)(x - 0{,}5)}{(0{,}3 - 0{,}1)(0{,}3 - 0{,}5)} + 1 \cdot \frac{(x - 0{,}1)(x - 0{,}3)}{(0{,}5 - 0{,}1)(0{,}5 - 0{,}3)} = 7{,}5x^2 - 2x + 0{,}125.$$

Bu tenglik kvadratik interpolyatsiya formulasi bilan bir xil. ◄

1.6.5. Nyuton interpolyatsion formulasi (bo‘lingan ayirmalar)

Agar x nuqta berilgan ma’lumotlar diapozoniga tushmasa, u holda bu interpolyatsiya emas, **ekstropolyatsiya** deyiladi. Bunda qaysi turdagi interpolyatsion formulani ishlatgan ma’qul?

Algebraik ko‘phad ancha qulay, chunki algebraik ko‘phadni trigonometrik va eksponensial ko‘phadlarga qaraganda

1) xatoligini aniqlash,

2) differensiallash,

3) integrallash oson.

Nyuton interpolyatsion formulasini bo‘lingan ayirmalar usuli deb ham yuritiladi. Keling, dastlab bo‘lingan ayirmalar usulida chiziqli va kvadratik interpolyatsion formulalarni keltirib chiqaramiz.

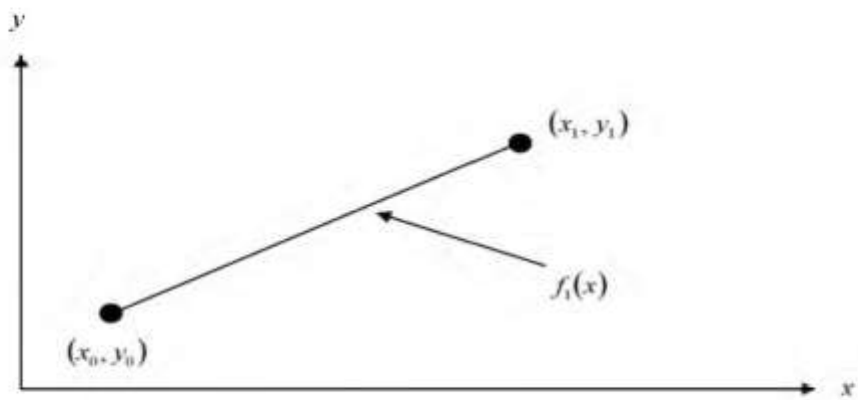
1-tartibli Nyuton interpolyatsion ko‘phadi

Bizga (x_0, y_0) va (x_1, y_1) nuqtalar ma’lum bo‘lsin. Ikki nuqtadan o‘tuvchi to‘g‘ri chiziq tenglamasini tuzish uchun

$$f(x) = a_0 + a_1 x \quad \text{yoki} \quad y - y_1 = m(x - x_1) \quad \text{yoki} \quad f(x_1) = b_0 + b_1(x - x_0)$$

ko‘rinishdagi ikkihadlardan foydalanish mumkin.

Biz $y = f(x)$ funksiyani tuzish uchun $f(x_1) = b_0 + b_1(x - x_0)$ dan foydalanamiz:



Agar $x = x_0$ bo‘lsa, u holda $f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$ ga;

agar $x = x_1$ bo‘lsa, $f_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) = f(x_0) + b_1(x_1 - x_0)$ ga ega

bo‘lamiz. Bundan $b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ ni topish mumkin. Topilgan

$$b_0 = f(x_0) \quad \text{va} \quad b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

qiymatlarni chiziqli interpolyatsiya formulasiga qo‘yamiz:

$f(x_1) = b_0 + b_1(x - x_0)$ ifoda

$$f(x_1) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \qquad (4)$$

ko‘rinishga keladi. (4)-ga bo‘lingan ayirmalar usulida topilgan **chiziqli interpolyatsiya formulasi** deyiladi.

11-misol. Raketaning $t = 16$ sekunddagi ko‘tarilish tezligini hisoblaymiz.

Yechilishi: ► Chiziqli interpolyatsiyadan tezlikni aniqlaymiz:

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$v(t) = b_0 + b_1(t - t_0)$$

$t = 16$ sekundga eng yaqin nuqtalarni aniqlaymiz, ular quyidagilar:

$$t_0 = 15; \quad v(t_0) = 362.78$$
$$t_1 = 20; \quad v(t_1) = 517.35$$

$b_0,\ b_1$ koeffitsiyentlarni aniqlab, tezlik funksiyasini tuzamiz:

$$b_0 = v(t_0) = 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{517.35 - 362.78}{20 - 15} = 30.914$$

Ko‘tarilish tezligining chiziqli interpolyatsiyasi:

$$v(t) = b_0 + b_1(t - t_0) = 362.78 + 30.914(t - 15), \quad 15 \le t \le 20$$

Shunday qilib, raketaning $t = 16$ sekunddagi tezligini topamiz:

$$v(16) = 362.78 + 30.914(16 - 15) = 393.69 \text{ m/s.}$$

Agar $v(t) = 362.78 + 30.914(t - 15)$ ifodani soddalashtirsak, to‘g‘ridan-to‘g‘ri topilgan $v(t) = 30.914t - 100.93$, $15 \le t \le 20$ formulaga kelamiz.◄

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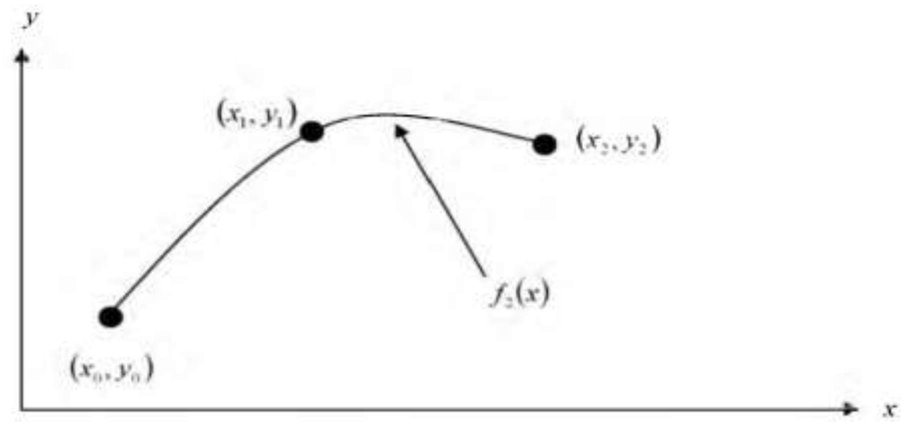
6. Autar Kaw, E.Erik Kalu “Numerical Methods with Applications: Abridged”, 2-nd edition, 2011. 756 p.

2-tartibli Nyuton interpolyatsion ko‘phadi

(x_0, y_0), (x_1, y_1) va (x_2, y_2) nuqtalarga mos $y = f(x)$ kvadratik interpolyatsiya formulasini tuzish uchun $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$ dan foydalanamiz.



Kvadratik formulani quyidagicha izlaymiz:

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1).$$

Agar $x = x_0$ bo‘lsa, $f_2(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1) = b_0$, ya’ni $b_0 = f(x_0)$ ga; agar $x = x_1$ bo‘lsa,

$$f_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1) = f(x_0) + b_1(x_1 - x_0),$$

ya’ni $f(x_1) = f(x_0) + b_1(x_1 - x_0)$ ga ega bo‘lamiz. Bundan $b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ ni topish mumkin.

Agar $x = x_2$ bo‘lsa, $f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$ ni hosil qilamiz, ya’ni

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1).$$

Bundan b_2 ni topib olamiz: $$b_2 = \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} - \dfrac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}.$$

Shunday qilib, **bo‘lingan ayirmalar usulida kvadratik interpolyatsiya formulasini** quyidagicha tuzishimiz mumkin:

$$f_2(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0) + \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} - \dfrac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}(x-x_0)(x-x_1)$$

12-misol. Raketaning ko‘tarilish tezligi vaqtning funksiyasi sifatida jadvalda keltirilgan. $t=16$ sekunddagi raketa tezligini hisoblang.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Yechilishi: ► Kvadratik interpolyatsiyani tuzamiz, buning uchun tezlikni aniqlaymiz: $v(t) = b_0 + b_1(t-t_0) + b_2(t-t_0)(t-t_1)$

$b_0,\ b_1,\ b_2$ koeffitsiyentlarni topish uchun $t=16$ sekundga eng yaqin bo‘lgan 3 ta nuqtani aniqlaymiz, ular quyidagilar:

$$t_0 = 10, \quad v(t_0) = 227.04$$
$$t_1 = 15, \quad v(t_1) = 362.78$$
$$t_2 = 20, \quad v(t_2) = 517.35$$

Shunda $b_0 = v(t_0) = 227.04$;

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10} = 27.148;$$

$$b_2 = \frac{\dfrac{v(t_2) - v(t_1)}{t_2 - t_1} - \dfrac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} = \frac{\dfrac{517.35 - 362.78}{20 - 15} - \dfrac{362.78 - 227.04}{15 - 10}}{20 - 10} =$$

$$= \frac{30.914 - 27.148}{10} = 0.3766$$

topilgan koeffitsiyentlarni formulaga qo‘yib, tezlik ifodasini topamiz:

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) =$$

$$= 227.04 + 27.148(t - 10) + 0.3766(t - 10)(t - 15), \quad 10 \le t \le 20.$$

Endi $t = 16$ sekunddagi raketa tezligini hisoblaymiz:

$$v(16) = 227.04 + 27.148(16 - 10) + 0.3766(16 - 10)(16 - 15) = 392.19 \text{ m/s}.$$

Bo‘lingan ayirmalar usulida raketa tezligi uchun topilgan kvadratik interpolyatsiya formulasi quyidagicha bo‘ladi:

$$v(t) = 227.04 + 27.148(t - 10) + 0.3766(t - 10)(t - 15), \quad 10 \le t \le 20$$

yoki $$v(t) = 0.3766t^2 + 17.733t + 12.05, \quad 10 \le t \le 20 \quad ◄$$

$n-$tartibli Nyuton interpolyatsion formulasi

Agar 2 ta yoki 3 ta emas, 4 ta nuqtani olib, ko‘phad tuzmoqchi bo‘lsak, unda bo‘lingan ayirmalar qanday tuziladi?

Keling kvadrat interpolyatsiyadan foydalanib, umumiy formula topishga urinib ko‘ramiz. Bizga quyidagilar ma’lum:

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1),$$

bunda $b_0 = f(x_0)$, $b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$, $b_2 = \dfrac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$ mos ravishda birinchi, ikkinchi va uchinchi bo‘lingan ayirmalarni ifodalaydi.

Birinchi bo‘lingan ayirmani $f[x_0] = f(x_0)$ deb belgilaymiz.

Ikkinchi bo‘lingan ayirma $f[x_1, x_0] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$;

Uchinchi bo‘lingan ayirma

$f[x_2, x_1, x_0] = \dfrac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \dfrac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$ ga teng, bu yerda $f[x_0]$, $f[x_1, x_0]$, $f[x_2, x_1, x_0]$ kvadrat qavslar ichida bo‘lingan ayirmada ishtirok qiladigan funksiyalarning argumentlari ko‘rsatilgan. Bulardan foydalanib, bo‘lingan ayirmalar formulasini soddaroq ko‘rinishda yozish mumkin:

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1).$$

Endi ushbu formulani $n+1$ ta (x_0, y_0), (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) tugun nuqta uchun umumlashtiramiz:

$$f_n(x) = b_0 + b_1(x - x_0) + ... + b_n(x - x_0)(x - x_1)...(x - x_{n-1}).$$

Ushbu formulaga **Nyutonning** $n+1$ **ta tugun nuqta uchun bo‘lingan ayirmalar usulida topilgan interpolyatsion formulasi** deyiladi.

Bunda $b_0 = f[x_0]$,

$b_1 = f[x_1, x_0],$

$b_2 = f[x_2, x_1, x_0],$

$\dots$

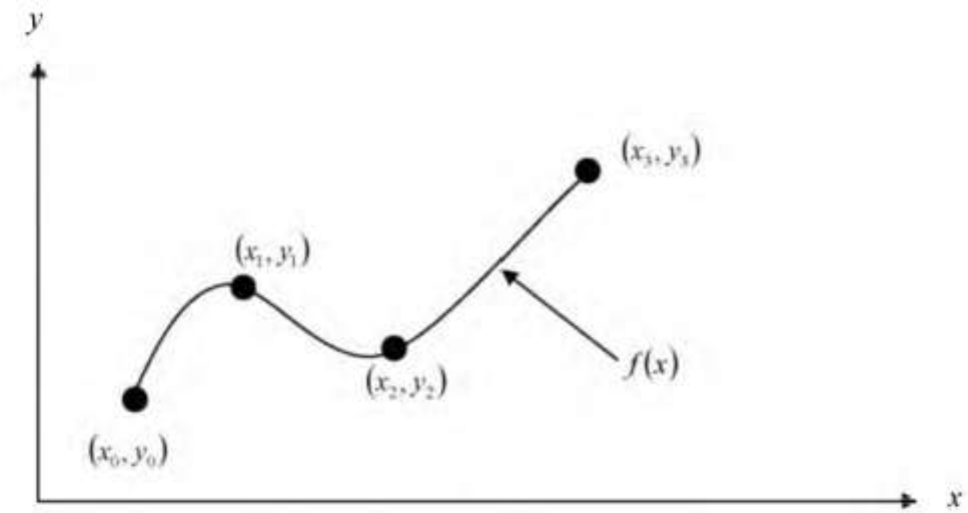
$b_{n-1} = f[x_{n-1}, x_{n-2}, ..., x_0],$

$b_n = f[x_n, x_{n-1}, ..., x_0].$

Har bir $m-$bo‘lingan ayirma $b_m = f[x_m, ..., x_0] = \dfrac{f[x_m, ...x_1] - f[x_{m-1}, ..., x_0]}{x_m - x_0}$

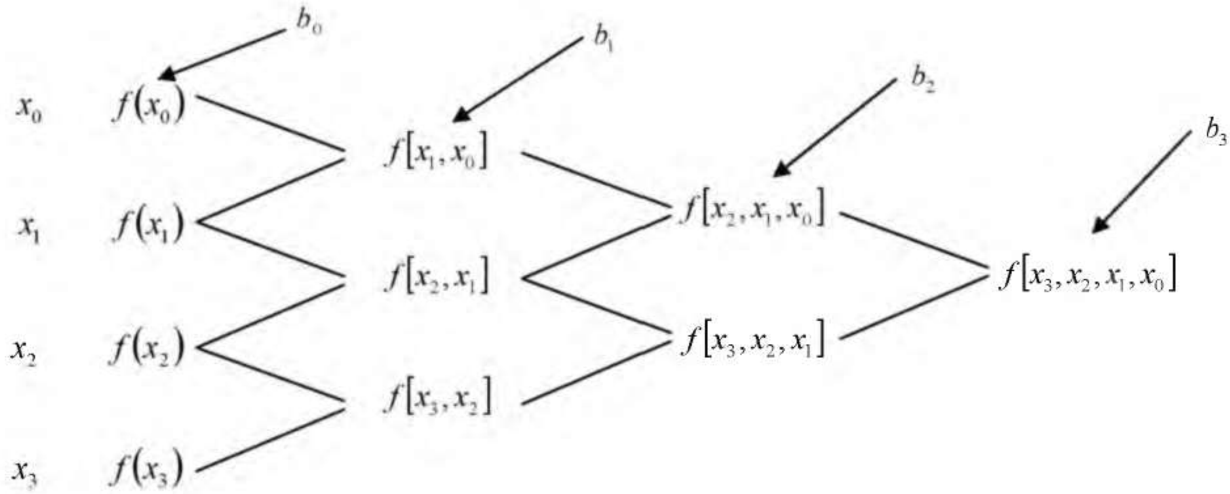
ifodadan topiladi.

Misol uchun, 3-tartibli ko‘phad tuzishda 4 ta (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) tugun nuqta qatnashadi, ularni quyidagicha bog‘laymiz:



$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) +$$
$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

Bo‘lingan ayirmalarning kubik interpolyatsion formulasini sxematik tasvirlaymiz:



13-misol.

a) Raketaning $t=16$ sekunddagi ko‘tarilish tezligini bo‘lingan ayirmalar usulidan foydalanib hisoblang;

b) Raketaning $t=11\text{s}$ va $t=16\text{s}$ vaqtlar oralig‘ida bosib o‘tgan masofasini toping;

c) Raketaning $t=16$ sekunddagi tezlanishini aniqlang.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Yechilishi: ►

a) Raketaning $t=16$ sekunddagi ko‘tarilish tezligini bo‘lingan ayirmalar usulidan foydalanib hisoblash uchun $t=16\text{s}$ ga eng yaqin bo‘lgan 4 ta nuqtani tanlaymiz:

$$t_0 = 10, \quad v(t_0) = 227.04$$
$$t_1 = 15, \quad v(t_1) = 362.78$$
$$t_2 = 20, \quad v(t_2) = 517.35$$
$$t_3 = 22.5, \quad v(t_3) = 602.97$$

Tezlik formulasini yozamiz:

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

$b_0,\ b_1,\ b_2,\ b_3$ koeffitsiyentlarni topamiz:

$$b_0 = v[t_0] = v(t_0) = 227.04;$$

$$b_1 = v[t_1, t_0] = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10} = 27.148;$$

$$b_2 = v[t_2, t_1, t_0] = \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0} = \frac{\dfrac{v(t_2) - v(t_1)}{t_2 - t_1} - \dfrac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} =$$

$$= \frac{\dfrac{517.35 - 362.78}{20 - 15} - \dfrac{362.78 - 227.04}{15 - 10}}{20 - 10} = \frac{30.914 - 27.148}{10} = 0.3766;$$

$$b_3 = v[t_3, t_2, t_1, t_0] = \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0}, \text{ bunda}$$

$$v[t_3, t_2, t_1] = \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1} = \frac{\dfrac{v(t_3) - v(t_2)}{t_3 - t_2} - \dfrac{v(t_2) - v(t_1)}{t_2 - t_1}}{t_3 - t_1} =$$

$$= \frac{\dfrac{602.97 - 517.35}{22.5 - 20} - \dfrac{517.35 - 362.78}{20 - 15}}{22.5 - 15} = \frac{34.248 - 30.914}{22.5 - 15} = 0.44453,$$

$v[t_2, t_1, t_0] = 0.3766$ ni oldin hisoblagan edik. Shunda

$$b_3 = v[t_3,t_2,t_1,t_0] = \frac{v[t_3,t_2,t_1]-v[t_2,t_1,t_0]}{t_3-t_0} = \frac{0.44453-0.3766}{22.5-10} = 0.0054347\,.$$

Endi tezlikning umumiy formulasini yozish mumkin:

$$v(t) = b_0 + b_1(t-t_0) + b_2(t-t_0)(t-t_1) + b_3(t-t_0)(t-t_1)(t-t_2)$$

$$v(t) = 227.04 + 27.148(t-10) + 0.3766(t-10)(t-15) +$$
$$+ 0.0054347(t-10)(t-15)(t-20)$$

$t = 16$ sekunddagi tezlik esa

$$v(t) = 227.04 + 27.148(16-10) + 0.3766(16-10)(16-15) +$$
$$+ 0.0054347(16-10)(16-15)(16-20) = 392.06 \quad \text{m/s.}$$

Tezlikning umumiy formulasini topish uchun qavslarni ochib, soddalashtiramiz:

$$v(t) = 227.04 + 27.148(t-10) + 0.3766(t-10)(t-15) +$$
$$+ 0.0054347(t-10)(t-15)(t-20) =$$
$$= 0.0054347t^3 + 0.13204t^2 + 21.265t - 4.2541, \qquad 10 \le \mathrm{t} \le 22.5$$

b) Raketaning $t = 11\,\text{s}$ va $t = 16\,\text{s}$ vaqtlar oralig‘ida bosib o‘tgan masofasini topamiz. Buning uchun $t = 10\,\text{s}$ va $t = 22.5\,\text{s}$ vaqtlar oralig‘ida aniqlangan interpolyatsiya formulasidan foydalanish mumkin:

$$s(16) - s(11) = \int_{11}^{16} v(t)dt = \int_{11}^{16} (0.0054347t^3 + 0.13204t^2 + 21.265t - 4.2541)dt =$$

$$= \left(0.0054347\frac{t^4}{4} + 0.13204\frac{t^3}{3} = 21.265\frac{t^2}{2} - 4.2541t \right)\Bigg|_{11}^{16} = 1605\,\text{m.}$$

Demak, raketa 11-sekunddan 16-sekundgacha 1605 m masofani bosib o‘tar ekan.

c) Raketaning $t=16$ sekunddagi tezlanishini aniqlaymiz. Bilimizki, tezlikdan hosila olsak, tezlanish hosil bo‘ladi:

$$a(t)=v'(t)=(0.0054347t^3+0.13204t^2+21.265t-4.2541)'=$$
$$=0.016304t^2+0.26408t+21.265$$

$$a(16)=0.016304\cdot 16^2+0.26408\cdot 16+21.265=29.664\,\text{m/s}^2$$

Shunday qilib, $t=16$ sekundda raketa 29.664 m/s^2 tezlanish bilan harakatlanayotgan ekan.◄

14-misol. Jadval bilan berilgan $y=f(x)$ funksiya ko‘rinishini tiklang va uning qiymatini $x=0.4$ bo‘lgan hol uchun Nyuton interpolyatsion formulasidan foydalanib hisoblang:

n	x_n	f_n
0	0	–0,5
1	0,1	0
2	0,3	0,2
3	0,5	1

Yechilishi: ► 4 tugun nuqta uchun Nyuton formulasidan foydalanamiz:

$$f_3(x)=f[x_0]+f[x_1,x_0](x-x_0)+f[x_2,x_1,x_0](x-x_0)(x-x_1)+$$

$$+f[x_3,x_2,x_1,x_0](x-x_0)(x-x_1)(x-x_2)$$

Bo‘lingan ayirmalarni hosil qilamiz:

$$f[x_1,x_0]=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{0-(-0{,}5)}{0{,}1}=5;$$

$$f[x_2,x_1]=\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{0{,}2-0}{0{,}3-0{,}1}=1;$$

$$f[x_3,x_2]=\frac{f(x_3)-f(x_2)}{x_3-x_2}=\frac{1-0{,}2}{0{,}5-0{,}3}=4\ ;$$

$$f[x_2, x_1, x_0] = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{1-5}{0,3-0} = -\frac{40}{3};$$

$$f[x_3, x_2, x_1] = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = \frac{4-1}{0,5-0,1} = \frac{15}{2};$$

$$f[x_3, x_2, x_1, x_0] = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0} = \frac{\frac{15}{2} + \frac{40}{3}}{0,5} = \frac{125}{3}.$$

Hisoblash natijalarini jadvalga kiritamiz:

n	x_n	f_n	$f(x_n, x_{n+1})$	$f(x_n, x_{n+1}, x_{n+2})$	$f(x_n, x_{n+1}, x_{n+2}, x_{n+3})$
0	0	–0,5			
1	0,1	0	5	–40/3	125/3
2	0,3	0,2	1	15/2	
3	0,5	1	4		

Birinchi ustundagi bo‘lingan ayirmalardan foydalanib, quyidagini hosil qilamiz:

$$f_3(x) = -0.5 + (x-0)\cdot 5 + (x-0)(x-0.1)\cdot\left(-\frac{40}{3}\right) + (x-0)(x-0.1)(x-0.3)\cdot\frac{125}{3} =$$

$$= \frac{125}{3}x^3 - 30x^2 + \frac{91}{12}x - 0,5.$$

Demak, $y = \frac{125}{3}x^3 - 30x^2 + \frac{91}{12}x - 0,5.$

$$y(0,4) = \frac{125}{3}\cdot 0,4^3 - 30\cdot 0,4^2 + \frac{91}{12}\cdot 0,4 - 0,5 \approx 0.3999. \blacktriangleleft$$

1-vazifa. O‘quv yillari va yakuniy baho.

a) Ma‘lumotlarga mos keluvchi regressiya tenglamasini tuzing va grafigini chizing.

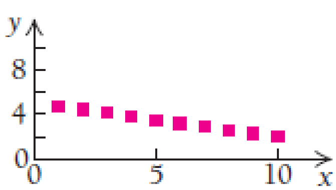
b) Agar dars soatlari 11 soat bo‘lsa, o‘quvchining baho ko‘rsatkichi qanday bo‘ladi?

Dars soatlari	Yakuniy baho
7	83
8	85
9	88
10	91
11	?

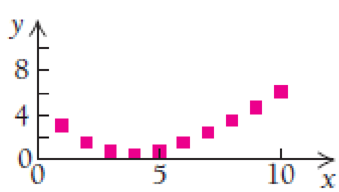
MUSTAQIL YECHISH UCHUN MISOLLAR:

1. Matematik model tanlash.

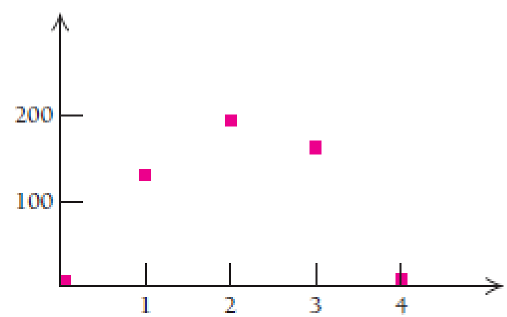
Nuqtalarning geometrik o‘rnidan foydalanib, qaysi turdagi funksiya bo‘lishini aniqlang: chiziqli, kvadratik, kubik, …



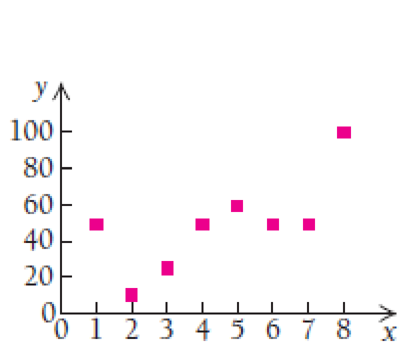
a)

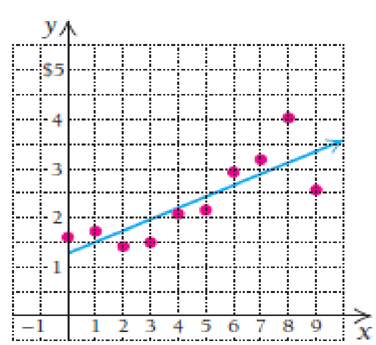


b)

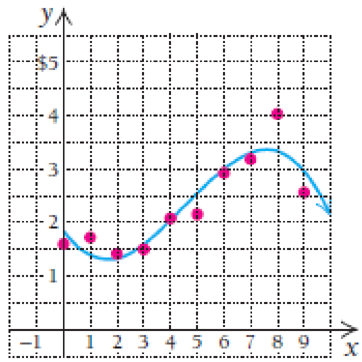


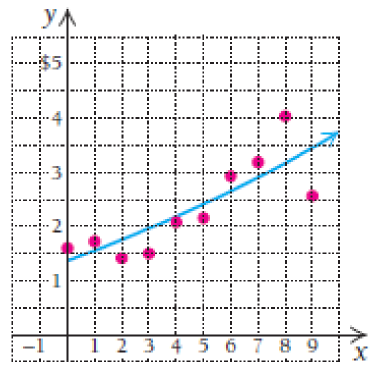
c)





d) e)





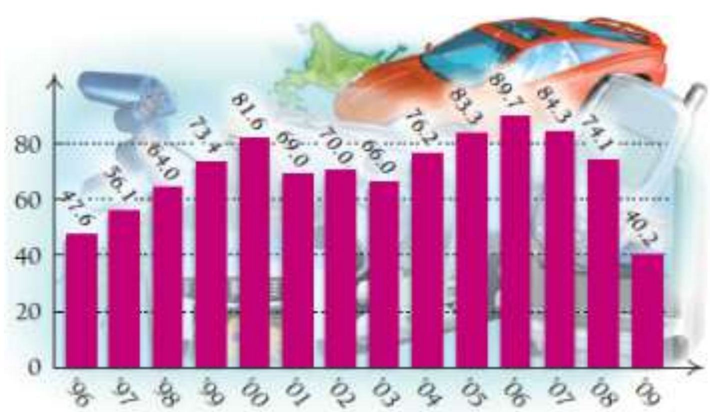
f) k)

2. Tadbirkorlik va iqtisod. O‘zbekistonda 1996-2009 yillarda Yaponiyaning Nissan avtomobilining sotilishi diagrammasi keltirilgan. Gorizontal o‘qda yillar, vertikal o‘qda avtomobil soni (ming hisobida) keltirilgan.

a) Ma‘lumotlar asosida grafik chizing.

b) Grafik asosida modelni tanlang va funksiya tenglamasini tuzing.

c) Modeldan foydalanib, 2020 yilda nechta avtomobil sotilishini bashorat qiling.

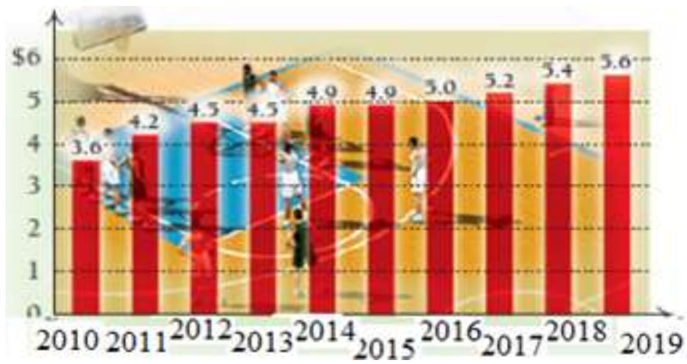


3. Ishchining 2000-2009 yillardagi kunlik o‘rtacha maoshi diagrammasi keltirilgan.

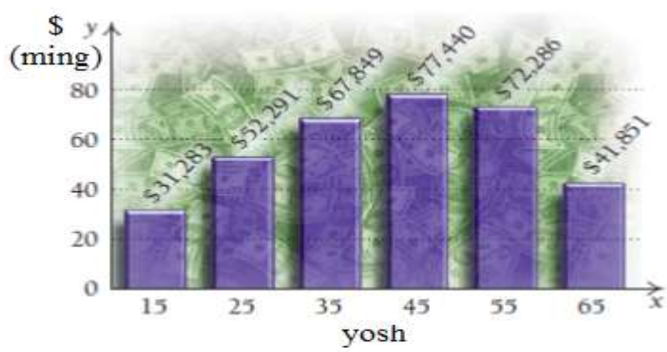
a) Ma‘lumotlar asosida grafik chizing.

b) Grafikka mos funksiya tenglamasini tuzing.

c) Modeldan foydalanib, 2025 yilda ishchining kunlik maoshi qancha bo‘lishini aniqlang.



4. 15 yoshdan 65 yoshgacha bo‘lgan davrda yiliga o‘rtacha pul ishlab topish ko‘rsatkichi diagrammasi keltirilgan.



a) Ma‘lumotlar asosida grafik chizing.

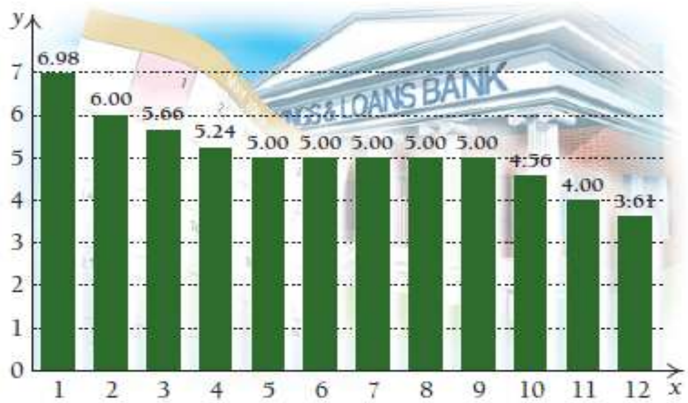
b) Grafikka mos funksiya tenglamasini tuzing.

c) Modeldan foydalanib, 40 va 72 yoshda 1 yilda qancha mablag‘ topish mumkinligini aniqlang.

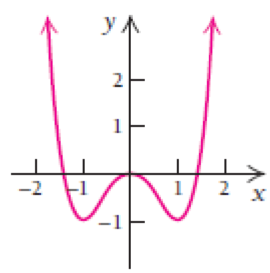
5. 2018 yilda neft narxining pasayishi kuzatildi. Agar har oyda neft narxi necha dollarga arzonlashgani diagrammada keltirilgan bo‘lsa, quyidagilarni aniqlang:

a) Ma‘lumotlar asosida grafik chizing.

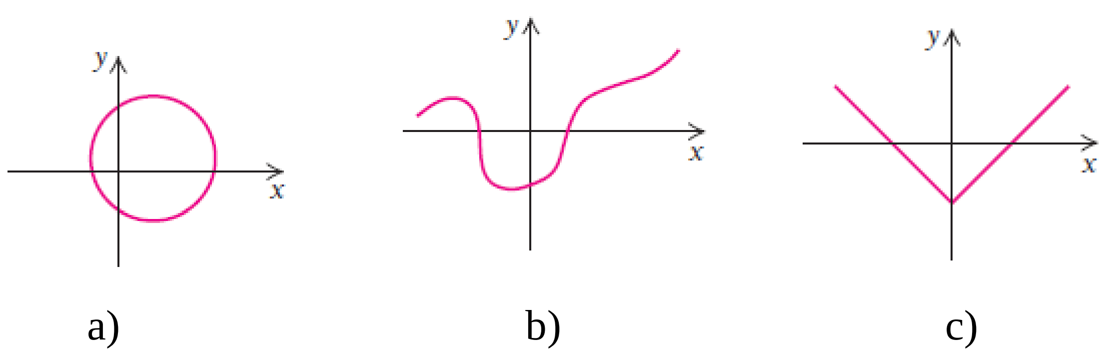
b) Grafikka mos funksiya tenglamasini tuzing.

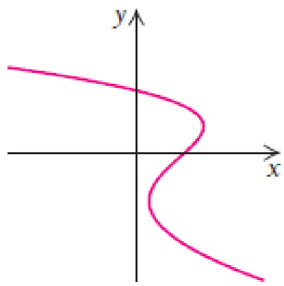


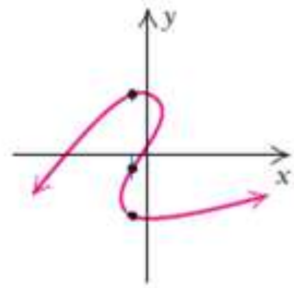
6. Berilgan grafik $y = x^4 - 2x^2$ funksiyaning grafigi bo‘la oladimi?

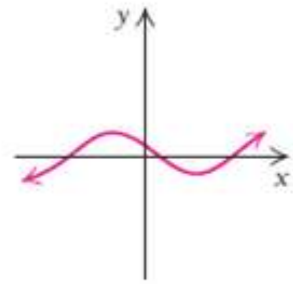


7. Quyidagi grafiklarni tekshirib, funksiya bo‘ladimi yoki yo‘qmi aniqlang. Javobingizni tushuntiring.

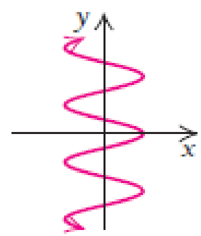






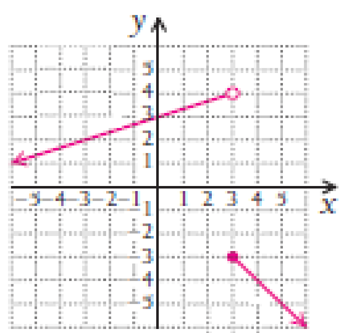


d) e) f)

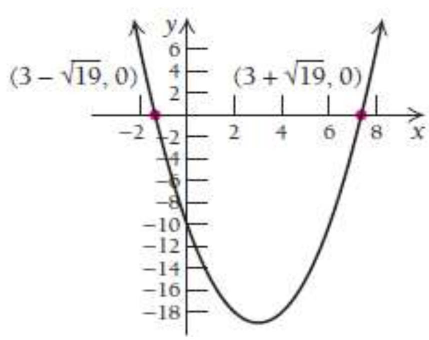


k) l)

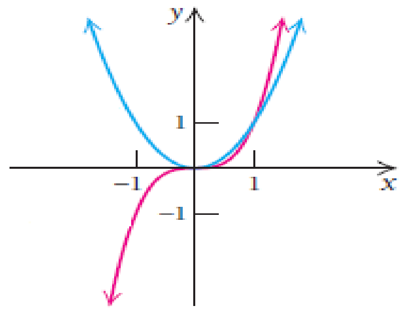
8. Bo‘lakli berilgan funksiya tenglamasini tuzing.



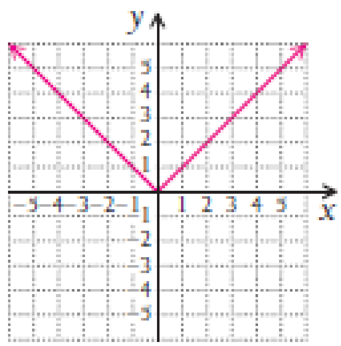
9. Berilgan grafikning tenglamasini tuzing.



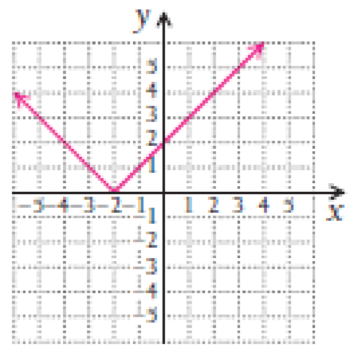
10. Grafiklarni o‘rganib, tenglamalarini yozing.



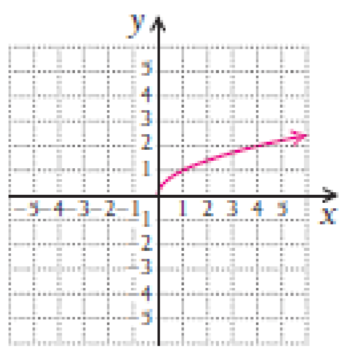
11. Grafiklarga mos funksiya tenglamalarini tuzing:



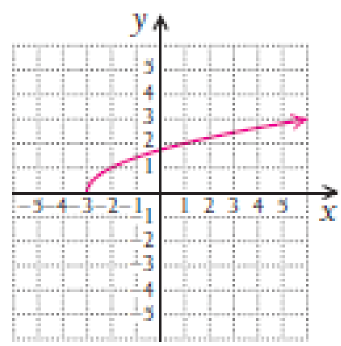
a)



b)



c)



d)

II BOB. DIFFERENSIALLASH

2.1. Sonli ketma-ketlik. Ketma-ketlik limiti

2.2. Funksiya limiti: sonli va grafik yondoshuv

2.3. Funksiya limitining xossalari va uzluksizlik

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2.7. Differensiallash qoidalari: ko'paytma va bo'linmaning hosilasi

2.8. Murakkab funksiyaning hosilasi

2.9. Yuqori tartibli hosilalar

Bu bobni o'rganish nima uchun kerak?

II bobdan biz hisob fanining tadqiqotlarini boshlaymiz. Bu bobda limit va uzluksizlik asosiy tushunchalar hisoblanadi. Limit va uzluksizlik tushunchalaridan foydalanib, asosiy maqsadimiz bo'lgan differensiallash amalini bajaramiz.

Differensiallash – bu hosila olish jarayonidir. Funksiyaning hosilasi bizga berilgan funksiyaga biror nuqtada o'tkazilgan urinmani va bu urinmaning og'ish burchagini topishga yordam beradi. Funksiyaning nuqtadagi hosilasiga hosilaning geometrik ma'nosi deyiladi. Shuningdek, hosila yordamida o'tilgan yo'lning (bir lahzadagi) oniy hosilasini ham topish mumkin. Oniy hosilaga hosilaning fizik ma'nosi deyiladi.

Ushbu bobda hosilani hisoblashning turli usullarini o'rganamiz.

2.1. Sonli ketma-ketlik. Ketma-ketlik limiti

Ushbu bo'limda limit tushunchasini o'rganamiz.

Faraz qiling, futbol oyinida raqib komanda o'yinchisi o'yin qoidasini buzdi. Hakam bu komandaga nisbatan 11 metrlik jarima belgiladi. Agar bu komanda takroran 2-marta o'yin qoidasini buzsa, masofaning yarmiga teng, ya'ni 5.5 metrlik jarima oladi. Agar takror va takror o'yin qoidalari buzilsa, u holda masofa ham yarmiga qisqarib boraveradi: $\frac{5.5}{2} = 2{,}75$ m, $\frac{2.75}{2} = 1.375$ m va h.k., lekin hech qachon darvoza chizig'ini kesib o'tmaydi. Aytmoqchimizki, limit – to'p bilan darvoza chizig'i orasidagi masofa nolga teng bo'lgandagi holat.

Hisob fani tadqiq qiladigan muhim narsa shuki, kirish qiymatlari o'zgarganda, funksiya qiymati (natija) qanday o'zgarishini baholaydi. Bu tadqiqotda limit asosiy o'rin tutadi.

2.1.1. Sonli ketma-ketliklar

Natural sonlar to‘plamida aniqlangan funksiya, ya’ni $x = f(n),\ n \in N$ funksiya **sonli ketma-ketlik** deyiladi.

Agar n ga $1,\ 2,\ 3,\ldots,n,\ldots$ va hokazo qiymatlar bersak, bu funksiyaning xususiy qiymatlarini topamiz, ularga **ketma-ketlikning hadlari** deyiladi: $x_1 = f(1),\ x_2 = f(2),\ \ldots,\ x_n = f(n),\ldots$

Sonli ketma-ketlik $\{x_n\}$ yoki $\{f(n)\}$ orqali belgilanadi. Ketma-ketlikning $n-$hadi uning **umumiy hadi** deyiladi. Ketma-ketlikning umumiy hadi ma’lum bo‘lsa, ketma-ketlik berilgan hisoblanadi.

1- misol. $x = \dfrac{n}{n+1}$ funksiyani ketma-ketlik shaklida yozing.

Yechilishi: ► $x = \dfrac{n}{n+1}$ da $n = 1,\ 2,\ 3,\ldots$ natural sonlarni qo‘yib, to‘g‘ri kasrlar ketma-ketligini hosil qilamiz:

$$\{x_n\} = \left\{\frac{n}{n+1}\right\} = \left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots,\ \frac{n}{n+1},\ \ldots\right\}\ . \ ◄$$

Bu misolda $n \in N$ ketma-ketlik **cheksiz ketma-ketlikdir**, ya’ni uning oxirgi hadi mavjud emas.

Barcha hadlari bir xil qiymat qabul qiladigan $\{x_n\}$ ketma-ketlik **o‘zgarmas ketma-ketlik** deyiladi.

Shunday M son mavjud bo‘lsaki, barcha $n \in N$ uchun $x_n < M$ tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlik **yuqoridan chegaralangan ketma-ketlik** deyiladi.

Shunday $M > 0$ son mavjud bo‘lsaki, istalgan $n \in N$ uchun $x_n > M$ tengsizlik bajarilsa, $\{x_n\}$ ketma-ketlik **quyidan chegaralangan ketma-ketlik** deyiladi.

Ham quyidan, ham yuqoridan chegaralangan $\{x_n\}$ ketma-ketlik **chegaralangan ketma-ketlik** deyiladi. Bu holda shunday $M > 0$ son mavjud bo‘ladiki, istalgan $n \in N$ uchun $|x_n| < M$ tengsizlik bajariladi.

Agar istalgan $n \in N$ uchun $x_n < x_{n+1}$ $(x_n > x_{n+1})$ tengsizlik bajarilsa, $\{x_n\}$ **monoton o‘suvchi (kamayuvchi) ketma-ketlik** deyiladi.

Agar istalgan $n \in N$ uchun $x_n \geq x_{n+1}$ $(x_n \leq x_{n+1})$ tengsizlik bajarilsa, $\{x_n\}$ **o‘smaydigan (kamaymaydigan) ketma-ketlik** deyiladi.

2-misol.

1) $\{x_n\} = \{n\} = \{1, 2, 3, ..., n, ...\}$ - o‘suvchi, quyidan chegaralangan;

2) $\{x_n\} = \{1 - 2n\} = \{-1, -3, -5, ...\}$ -kamayuvchi, yuqoridan chegaralangan;

3) $\{x_n\} = \left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right\}$ kamayuvchi, yuqoridan chegaralangan ketma-ketlik.

2.1.2. Ketma-ketlikning limiti

a o‘zgarmas son va $\{x_n\}$ ketma-ketlik berilgan bo‘lsin.

Agar istalgan $\varepsilon > 0$ son uchun shunday $N = N(\varepsilon) > 0$ son mavjud bo‘lsaki, barcha $n \geq N$ lar uchun $|x_n - a| < \varepsilon$ tengsizlik bajarilsa, a

o‘zgarmas son $\{x_n\}$ **ketma-ketlikning limiti** deyiladi va bu quyidagicha yoziladi:

$$\lim_{n\to\infty} x_n = a.$$

Agar $\{x_n\}$ ketma-ketlik chekli limitga ega bo‘lsa, u **yaqinlashuvchi ketma-ketlik**, aks holda esa **uzoqlashuvchi ketma-ketlik** deyiladi.

$|x_n - a| < \varepsilon$ tengsizlik $a - \varepsilon < x_n < a + \varepsilon$ tengsizliklarga teng kuchli ekanini bilamiz. Buni hisobga olsak, limit tushunchasini geometrik nuqtai nazardan bunday tushuntirish mumkin:

Ta’rif. Agar istalgan $\varepsilon > 0$ son uchun shunday $N = N(\varepsilon) > 0$ son topilsaki, $\{x_n\}$ ketma-ketlikning $n \geq N$ dan boshlab barcha hadlari a nuqtaning $\varepsilon -$ atrofiga tushsa, a o‘zgarmas son $\{x_n\}$ **ketma-ketlikning limiti** deyiladi.

3-misol. 0 soni $\{x_n\} = \left\{\frac{1}{n}\right\}$ ketma-ketlikning limiti ekanligi, ya’ni $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ ni isbotlang.

Yechilishi: ► **I usul.** Ixtiyoriy $\varepsilon > 0$ sonni olaylik. $\left|\frac{1}{n} - 0\right| < \varepsilon$ yoki $\left|\frac{1}{n}\right| < \varepsilon$ tengsizlikni tuzamiz. Biroq $n > 0$, shuning uchun $\frac{1}{n} < \varepsilon$ yoki $n < \frac{1}{\varepsilon}$. Bundan ko‘rinadiki, $N = N(\varepsilon)$ sifatida $\frac{1}{\varepsilon}$ dan katta istalgan son, ya’ni $N(\varepsilon) > \frac{1}{\varepsilon}$ olinsa, u holda barcha $n > N(\varepsilon)$ uchun $\left|\frac{1}{n}\right| < \varepsilon$ yoki $\left|\frac{1}{n} - 0\right| < \varepsilon$

tengsizlik bajariladi. Bu esa $\lim_{n\to\infty}\frac{1}{n}=0$ ekanini bildiradi. Masalan, $\varepsilon =$ 0,01 uchun $N(\varepsilon) = 100$ va $n > 100$ uchun $\left|\frac{1}{n}\right| \leq 0{,}01$.

II usul. Jadval yordamida ketma-ketlik limiti 0 ga yaqinlashishini ko‘rsatamiz.

x	$\lim_{n\to\infty}\frac{1}{n}$
1	1
10	0.1
100	0.01
1000	0.001
10000	0.0001
100000	0.00001

◄

4-misol. $x_n = \frac{1-2n^2}{2+4n^2}$ ketma-ketlikning limiti $a=-\frac{1}{2}$ ekanini isbotlang.

Yechilishi: ► $\forall\varepsilon>0$ son uchun unga mos $N=N(\varepsilon)>0$ son mavjudligini ko‘rsatamiz: barcha $n\geq N$ larda $|x_n - a| < \varepsilon$ shart bajarilishi kerak. $|x_n - a| = \left|\frac{1-2n^2}{2+4n^2}+\frac{1}{2}\right| < \varepsilon$ yoki

$$\left|\frac{1-2n^2}{2(1+2n^2)}+\frac{1}{2}\right| < \varepsilon,$$

$$\left|\frac{1-2n^2+1+2n^2}{2+4n^2}\right| < \varepsilon,$$

$$\left|\frac{2}{2(1+2n^2)}\right| < \varepsilon,$$

$$\frac{1}{1+2n^2}<\varepsilon.$$

Tengsizlikdan n ni topib olamiz: $\frac{1}{\varepsilon}<1+2n^2$,

$$\frac{1}{\varepsilon}-1<2n^2,$$

$$\frac{1}{2\varepsilon}-\frac{1}{2}<n^2.$$

Har ikki tonondan ildiz chiqaramiz: $n>\sqrt{\frac{1}{2\varepsilon}-\frac{1}{2}}$, $N=\left[\sqrt{\frac{1}{2\varepsilon}-\frac{1}{2}}\right]$ butun qismidan iborat bo‘ladi. Demak, $N(\varepsilon)=\left[\sqrt{\frac{1}{2\varepsilon}-\frac{1}{2}}\right]$ deb tanlash yetarli. ◄

MUSTAQIL YECHISH UCHUN MISOLLAR

1-20 misollarda $\lim\limits_{n\to\infty} a_n = a$ **ekanligini isbotlang (**$N(\varepsilon)$**ni ko‘rsating).**

1. $a_n=\frac{3n+2}{2n-1}, \quad a=\frac{3}{2}.$

2. $a_n=\frac{6n-1}{3n+1}, \quad a=2.$

3. $a_n=\frac{5n+4}{2n+1}, \quad a=\frac{5}{2}.$

4. $a_n=\frac{3n-5}{2n+1}, \quad a=\frac{3}{2}.$

5. $a_n=\frac{n-1}{7n+1}, \quad a=\frac{1}{7}.$

6. $a_n=\frac{3n^2+1}{4n^2+2}, \quad a=\frac{3}{4}.$

7. $a_n=\frac{9+n^3}{1-2n^3}, \quad a=-\frac{1}{2}.$

8. $a_n=\frac{6n+2}{3n-1}, \quad a=2.$

9. $a_n=\frac{1+2n^2}{2-4n^2}, \quad a=-\frac{1}{2}.$

10. $a_n=-\frac{n}{5n+1}, \quad a=-\frac{1}{5}.$

11. $a_n = \dfrac{2n+1}{1-n}, \quad a = -2.$

12. $a_n = \dfrac{2n+1}{3n-5}, \quad a = \dfrac{2}{3}.$

13. $a_n = \dfrac{1-2n^2}{n^2+3}, \quad a = -2.$

14. $a_n = \dfrac{3n^2}{2-n^2}, \quad a = -3.$

15. $a_n = \dfrac{2n}{6n-1}, \quad a = \dfrac{1}{3}.$

16. $a_n = \dfrac{3n^3}{1-n^3}, \quad a = -3.$

17. $a_n = \dfrac{4-2n}{1+3n}, \quad a = -\dfrac{2}{3}.$

18. $a_n = \dfrac{n+15}{6-n}, \quad a = -1.$

19. $a_n = \dfrac{3-2n^2}{1+n^2}, \quad a = -2.$

20. $a_n = \dfrac{2n-1}{2-3n}, \quad a = -\dfrac{2}{3}.$

21-40 misollarda sonli ketma-ketliklarning limitlarini hisoblang.

21. $\lim\limits_{n\to\infty} \dfrac{n\sqrt[3]{5n^2} + \sqrt[4]{9n^8+1}}{\left(n+\sqrt{n}\right)\sqrt{7-n+n^2}}.$

22. $\lim\limits_{n\to\infty} \dfrac{\sqrt{n-1} - \sqrt{n^2+1}}{\sqrt[3]{3n^3+3} + \sqrt[4]{n^5+1}}.$

23. $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^3+1} - \sqrt{n-1}}{\sqrt[3]{n^3+1} - \sqrt{n-1}}.$

24. $\lim\limits_{n\to\infty} \dfrac{\sqrt[3]{n^2-1} + 7n^3}{\sqrt[4]{n^{12}+n+1} - n}.$

25. $\lim\limits_{n\to\infty} \dfrac{\sqrt{3n-1} - \sqrt[3]{125n^3+n}}{\sqrt[5]{n} - n}.$

26. $\lim\limits_{n\to\infty} \dfrac{n\sqrt[5]{n} - \sqrt[3]{27n^6+n^2}}{\left(n+\sqrt[4]{n}\right)\sqrt{9+n^2}}.$

27. $\lim\limits_{n\to\infty} \dfrac{\sqrt{n+2} - \sqrt{n^2+2}}{\sqrt[4]{4n^4+1} - \sqrt[3]{n^4-1}}.$

28. $\lim\limits_{n\to\infty} \dfrac{6n^3 - \sqrt{n^5+1}}{\sqrt{4n^6+3} - n}.$

29. $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^4+2} + \sqrt{n-2}}{\sqrt[4]{n^4+2} + \sqrt{n-2}}.$

30. $\lim\limits_{n\to\infty} \dfrac{\sqrt{5n+2} - \sqrt[3]{8n^3+5}}{\sqrt[4]{n+7} - n}.$

31. $\lim\limits_{n\to\infty} \dfrac{n\sqrt[4]{3n+1} + \sqrt{81n^4-n^2+1}}{\left(n+\sqrt[3]{n}\right)\sqrt{5-n+n^2}}.$

32. $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^5+3} - \sqrt{n-3}}{\sqrt[5]{n^5+3} + \sqrt{n-3}}.$

33. $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n}-9n^2}{3n-\sqrt[4]{9n^8+1}}$.

34. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n+3}-\sqrt{n^2-3}}{\sqrt[3]{n^5-4}-\sqrt[4]{n^4+1}}$.

35. $\lim\limits_{n\to\infty}\dfrac{\sqrt{4n+1}-\sqrt[3]{27n^3+4}}{\sqrt[4]{n}-\sqrt[3]{n^5+n}}$.

36. $\lim\limits_{n\to\infty}\dfrac{n\sqrt[3]{7n}-\sqrt[4]{81n^8-1}}{\left(n+4\sqrt{n}\right)\sqrt{n^2-5}}$.

37. $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n^3-7}+\sqrt[3]{n^2+4}}{\sqrt[4]{n^5+5}+\sqrt{n}}$.

38. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^6+4}+\sqrt{n-4}}{\sqrt[5]{n^6+6}-\sqrt{n-6}}$.

39. $\lim\limits_{n\to\infty}\dfrac{4n^2-\sqrt[4]{n^3}}{\sqrt[3]{n^6+n^3+1}-5n}$.

40. $\lim\limits_{n\to\infty}\dfrac{\sqrt{n+3}-\sqrt[3]{8n^3+3}}{\sqrt[4]{n+4}-\sqrt[5]{n^5+5}}$.

41. $\lim\limits_{n\to\infty}\sqrt{n^3+8}\left(\sqrt{n^3+2}-\sqrt{n^3-1}\right)$.

42. $\lim\limits_{n\to\infty}\left(\sqrt{n\ (n+5)}-n\right)$.

43. $\lim\limits_{n\to\infty}\dfrac{\sqrt{\left(n^3+1\right)\left(n^2+3\right)}-\sqrt{n\left(n^4+2\right)}}{2\sqrt{n}}$.

44. $\lim\limits_{n\to\infty}\sqrt[3]{n}\left(\sqrt[3]{n^2}-\sqrt[3]{n(n-1)}\right)$.

2.2. Funksiya limiti: sonli va grafik yondoshuv

2.2.1. Funksiya limiti. Chap va o‘ng limitlar

Aytaylik, f funksiya berilgan va faraz qilingki, x qiymat (kirish) biror a soniga asta sekin yaqinlashib bormoqda. Agar mos ravishda funksiyaning qiymatlari ham qandaydir L soniga yaqinlashib borsa, u holda L soni $x \to a$ dagi limit bo‘ladi.

Limitni jadval shaklida ifodalash uning **sonli yondoshuvi**, grafik shaklda ifodalash esa limitni **grafik yondoshuvi** bo‘ladi.

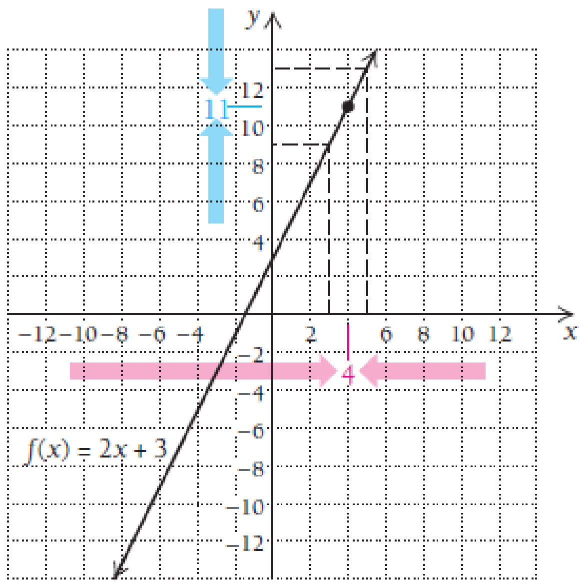
5-misol. $f(x) = 2x + 3$ funksiya berilgan bo‘lsin va $x-$qiymat 4 ga asta-sekin yaqinlashib borsin.

Jadval va grafikdan ko‘rib turibmizki, agar x qiymat 4 ga chapdan yaqinlashsa, funksiyaning qiymati 11 ga yaqinlashmoqda, agar x qiymat 4 ga o‘ngdan yaqinlashsa, funksiyaning qiymati yana 11 ga yaqinlashmoqda. Shunga ko‘ra, 4 ga ikkala tomondan yaqinlashganda ham funksiyaning qiymati 11 ga teng deyish mumkin.

Strelka “$\to$” belgisi yaqinlashadi degan ma’noni bildiradi:

$x \to 4$ da $f(x) = 2x + 3 \to 11$. Buni qisqacha $\lim\limits_{x\to 4}(2x+3) = 11$ deb yozamiz. O‘qilishi: x qiymat 4 ga intilganda, $2x+3$ qiymat 11 ga intiladi.

x	$f(x)$
2	7
3.6	10.2
3.9	10.8
3.99	10.98
3.999	10.998
← . . .	. . . →
4.001	11.002
4.01	11.02
4.1	11.2
4.8	12.6
5	13



Ta'rif. Agar x ning qiymatlari biror a soniga yetarlicha yaqin bo‘lib, lekin teng bo‘lmasa va bunda f funksiyaning barcha qiymatlari L soniga yaqinlashib borsa, u holda L soni x ning a ga intilgandagi **funksiya limiti** deyiladi va quyidagicha belgilanadi:

$$\lim_{x\to a} f(x) = L \qquad \text{yoki} \qquad f(x) \underset{x\to a}{\to} L.$$

Agar limitni $\lim_{x\to a} f(x) = L$ ko‘rinishida yozsak, x qiymat a soniga ikki tomondan ham yaqinlashishini bildiradi.

$\lim_{x\to a^-} f(x)$ yozuv x qiymat a soniga chap tomondan yaqinlashishini, bunda $x < a$ ekanini,

$\lim_{x\to a^+} f(x)$ yozuv x qiymat a soniga o‘ng tomondan yaqinlashishini va bunda $x > a$ ekanini bildiradi. Ularni **chap va o‘ng limitlar** deb yuritiladi. Limit mavjud bo‘lishi uchun chap va o‘ng limitlar mavjud bo‘lishi va teng bo‘lishi kerak.

Teorema. Agar funksiyaning chap va o‘ng limitlari mavjud hamda o‘zaro teng bo‘lsa, $x \to a$ da $f(x)$ ning limiti mavjud va L ga teng bo‘ladi, ya’ni

agar $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = L$ bo‘lsa, u holda $\lim\limits_{x\to a} f(x) = L$ bo‘ladi.

Teoremaning teskari tasdig‘i ham o‘rinli: agar $\lim\limits_{x\to a} f(x) = L$ mavjud bo‘lsa, u holda $\lim\limits_{x\to a^-} f(x)$ chap va $\lim\limits_{x\to a^+} f(x)$ o‘ng limit mavjud, shuning bilan birga L ga teng.

6-misol. $f(x) = \dfrac{x^2 - 1}{x - 1}$ funksiya berilgan.

a) $f(1)$ ning qiymati nimaga teng?

b) $x \to 1$ da $f(x)$ funksiya limitini hisoblang.

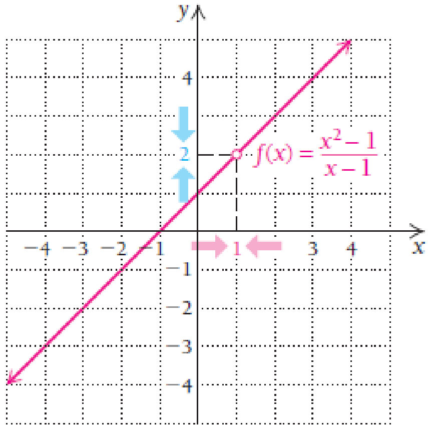
Yechilishi: ► a) $f(1) = \dfrac{1^2 - 1}{1 - 1} = \dfrac{0}{0}$ kasrning maxrajida 0 hosil bo‘ldi. Shunga ko‘ra, $x = 1$ da funksiyaning qiymati mavjud emas.

b) x ga 1 ning o‘ng va chap tomonlaridan unga yaqin qiymatlarni qo‘yib ko‘ramiz. U holda $f(x)$ ning qiymatlari 2 ga yaqinlashmoqda:

$$\lim_{x\to 1} f(x) = 2.$$

$x \to 1^-$ $(x < 1)$	$f(x)$
0.9	1.9
0.99	1.99
0.999	1.999

$x \to 1^+$ $(x > 1)$	$f(x)$
1.1	2.1
1.01	2.01
1.001	2.001



Grafikda (1,2) nuqtada teshikcha hosil bo'lgan. Funksiya $x = 1$ nuqtada aniqlanmagan, lekin funksiyaning $x \to 1$ da limiti mavjud. ◀

1-vazifa. $f(x) = \dfrac{x^2 - 16}{x - 4}$ funksiya berilgan.

a) $f(4)$ nimaga teng?

b) $x \to 4$ da $f(x)$ funksiya limitini hisoblang.

7-misol. $H(x) = \begin{cases} 2x + 2, & \text{agar} \quad x < 1 \\ 2x - 4, & \text{agar} \quad x \geq 1 \end{cases}$ bo'lakli aniqlangan funksiya berilgan. Funksiya grafigini chizing va quyidagi limitlarni, agar ular mavjud bo'lsa, a) $\lim\limits_{x \to 1} H(x)$; b) $\lim\limits_{x \to -3} H(x)$ larni toping.

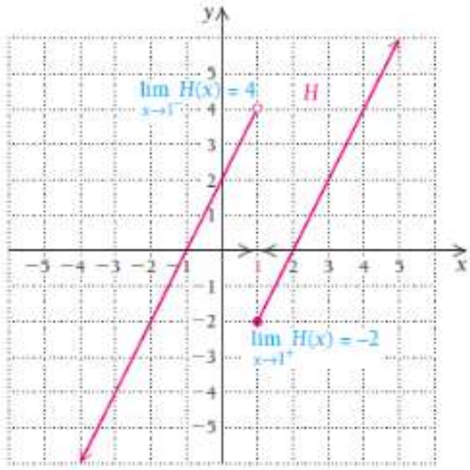
Yechilishi: ▶ a) Sonli usulda chap va o'ng limitlarni hisoblaymiz va kirish-chiqish jadvalini to'ldiramiz. Shunga ko'ra grafigini chizamiz. $x \to 1$ ga chapdan intilsa, funksiya qiymati 4 ga intilmoqda: $\lim\limits_{x \to 1^-} H(x) = 4$,

$x \to 1$ ga o‘ngdan intilsa, funksiya qiymati -2 ga intilmoqda: $\lim\limits_{x \to 1^+} H(x) = -2.$

Bundan kelib chiqadiki, funksiya $x = 1$ nuqtada aniqlangan, lekin $x = 1$ nuqtada $\lim\limits_{x \to 1} H(x)$ limit mavjud emas.

$x \to 1^-$ $(x < 1)$	$H(x)$
0	2
0.5	3
0.8	3.6
0.9	3.8
0.99	3.98
0.999	3.998

$x \to 1^+$ $(x > 1)$	$H(x)$
2	0
1.8	−0.4
1.1	−1.8
1.01	−1.98
1.001	−1.998
1.0001	−1.9998

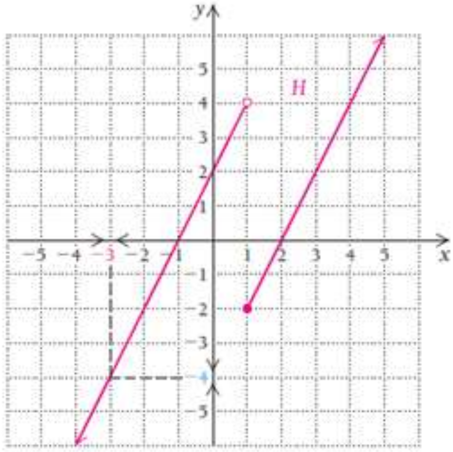


b) $x \to -3$ ga chapdan intilsa, funksiya qiymati -4 ga intilyapti:

$$\lim_{x \to -3^-} H(x) = -4,$$

$x \to -3^-$ $(x < -3)$	$H(x)$
-4	-6
-3.5	-5
-3.1	-4.2
-3.01	-4.02
-3.001	-4.002

$x \to -3^+$ $(x > -3)$	$H(x)$
-2	-2
-2.5	-3
-2.9	-3.8
-2.99	-3.98
-2.999	-3.998



$x \to -3$ ga o‘ngdan intilganda ham funksiya qiymati -4 ga intilyapti:

$$\lim_{x \to -3^+} H(x) = -4 .$$

Demak, funksiya $x = -3$ nuqtada aniqlangan va $\lim\limits_{x \to -3} H(x) = -4$ limit mavjud.◄

Xulosa: Funksiyaning biror nuqtada limiti mavjud bo‘lishi yoki mavjud bo‘lmasligi funksiyaning shu nuqtadagi qiymatiga bog‘liq emas.

2-vazifa. $f(x) = \begin{cases} -x+4, & \text{agar } x \le 3 \\ 2x+1, & \text{agar } x > 3 \end{cases}$ funksiya berilgan.

a) $\lim\limits_{x \to 3^-} f(x)$, $\lim\limits_{x \to 3^+} f(x)$ va $\lim\limits_{x \to 3} f(x)$ aniqlang;

b) $\lim\limits_{x \to 1} f(x)$ ni toping.

8-misol. $G(x)=\begin{cases} 5, & \text{agar } x=1 \\ x+1, & \text{agar } x\neq 1 \end{cases}$ bo‘lakli aniqlangan funksiya berilgan. Funksiya grafigini chizing va quyidagi limitlarni, agar ular mavjud bo‘lsa, toping: a) $\lim\limits_{x\to 1} G(x)$; b) $\lim\limits_{x\to -2} G(x)$

Yechilishi: ►

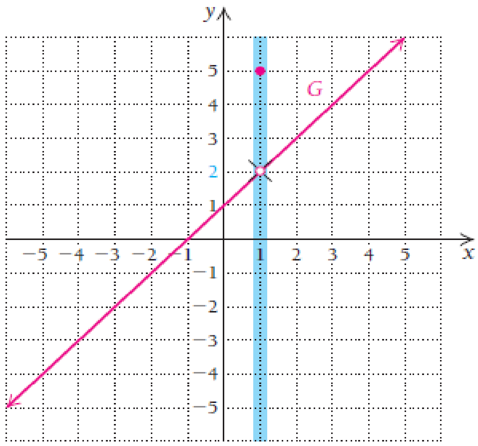
a) $x\to 1$ ga chapdan intilganda $\lim\limits_{x\to 1^-} G(x)=2$, o‘ngdan intilganda ham $\lim\limits_{x\to 1^+} G(x)=2$ ekanligidan $\lim\limits_{x\to 1} G(x)=2$ bo‘lishi kelib chiqadi.

Funksiyaning limit qiymati bo‘lgan 2 soni funksiyaning qiymati emas, chunki $x=1$ da $G(1)=5$.

$x\to 1^-$ $(x<1)$	$G(x)$
0	1
0.5	1.5
0.9	1.9
0.99	1.99

$\lim\limits_{x\to 1} G(x) = 2$

$x\to 1^+$ $(x>1)$	$G(x)$
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

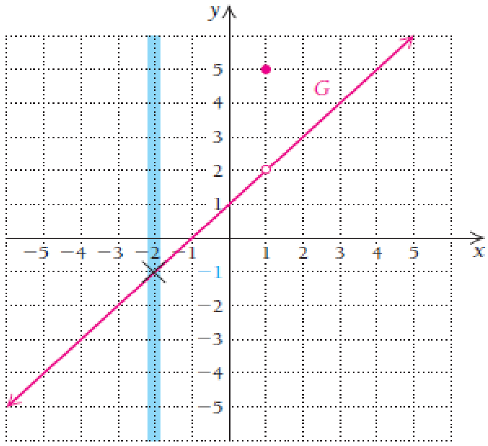


b) $x\to -2$ da $\lim\limits_{x\to -2} G(x)=-1$, funksiyaning qiymati ham $G(-2)=-1$ ga teng.

$x \to -2^-$ $(x < -2)$	$G(x)$
-3	-2
-2.5	-1.5
-2.1	-1.1
-2.01	-1.01

$\lim_{x \to -2} G(x) = -1$

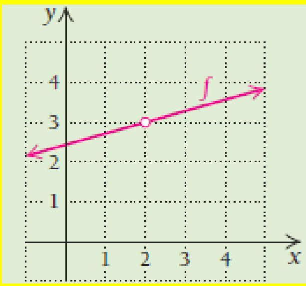
$x \to -2^+$ $(x > -2)$	$G(x)$
-1.5	-0.5
-1.9	-0.9
-1.99	-0.99
-1.999	-0.999



◄

3-vazifa. Grafikdan foydalanib, $f(x)$ funksiya uchun

a) $\lim_{x \to 2^-} f(x)$; b) $\lim_{x \to 2^+} f(x)$; c) $\lim_{x \to 2} f(x)$ aniqlang.



2.2.2. Cheksiz katta va cheksiz kichik funksiyalar

Limitlar bizga ba'zi funksiyalarga nisbatan cheksizlikning rolini tushunishimizga yordam beradi.

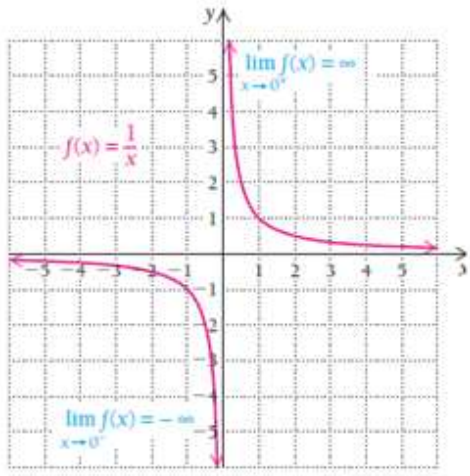
9-misol. $f(x) = \dfrac{1}{x}$ funksiya berilgan. Hisoblang:

a) $\lim\limits_{x\to 0^-} f(x)$; b) $\lim\limits_{x\to 0^+} f(x)$; c) $\lim\limits_{x\to 0} f(x)$

Yechilishi: ▶ Dastlab $f(0)$ ni tekshiramiz. Funksiya $x=0$ nuqtada ma'noga ega emas. Funksiya grafigi $x=0$ dan o'tmaydi.

$x\to 0^-$ $(x<0)$	$f(x)$
−0.1	−10
−0.01	−100
−0.001	−1,000
−0.0001	−10,000

$x\to 0^+$ $(x>0)$	$f(x)$
0.1	10
0.01	100
0.001	1,000
0.0001	10,000



Jadvaldan va grafikdan ko'rinadiki, x o'zgaruvchi 0 ga chapdan intilganda funksiya qiymati juda kichiklashib, $-\infty$ ga ketadi:

$$\lim_{x\to 0^-} f(x) = -\infty .$$

x o'zgaruvchi 0 ga o'ngdan intilganda esa funksiya qiymati juda kattalashib, ∞ ga ketadi: $\lim\limits_{x\to 0^+} f(x) = \infty$.

Shunday qilib, $x\to 0$ da funksiya $\lim\limits_{x\to 0} f(x)$ limitga ega emas, chunki o'ng va chap limitlar teng emas.◀

10-misol. $f(x)=\dfrac{1}{x}$ funksiya uchun

a) $\lim\limits_{x\to \infty} f(x)$; b) $\lim\limits_{x\to -\infty} f(x)$ ni hisoblang.

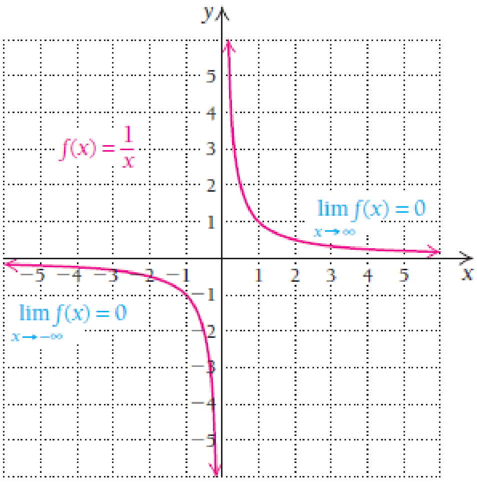
Yechilishi: ▶ Jadval hamda grafikdan ko'rinadiki, x ning qiymati kattalashgani sari funksiya 0 ga intiladi, ya'ni $\lim\limits_{x\to\infty} f(x)=0$.

x ning qiymatlari manfiy tomonga qarab o'sganda ham funksiya 0 ga intiladi, ya'ni $\lim\limits_{x\to-\infty} f(x)=0$.

$x\to\infty$	$f(x)$
10	0.1
100	0.01
1,000	0.001
10,000	0.0001

$\lim\limits_{x\to\infty} f(x) = 0$

$x\to-\infty$	$f(x)$
−10	−0.1
−100	−0.01
−1,000	−0.001
−10,000	−0.0001



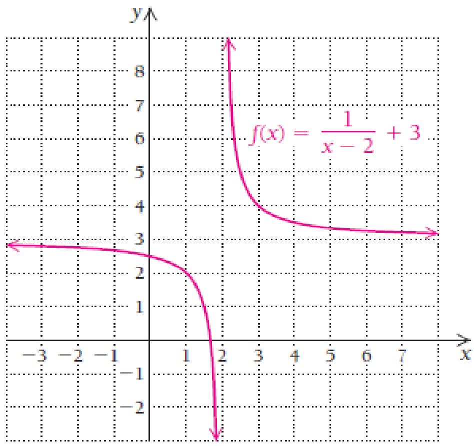
◀

11-misol. $f(x)=\dfrac{1}{x-2}+3$ funksiya grafigini chizing va

a) $\lim\limits_{x\to3} f(x)$;

b) $\lim\limits_{x\to2} f(x)$ limitlarni hisoblang.

Yechilishi: ▶ Ushbu funksiya ham $f(x)=\dfrac{1}{x}$ funksiyaga o'xshash, faqat bu $f(x)=\dfrac{1}{x-2}+3$ funksiya 2 birlik o'ngga va 3 birlik tepaga siljiydi.

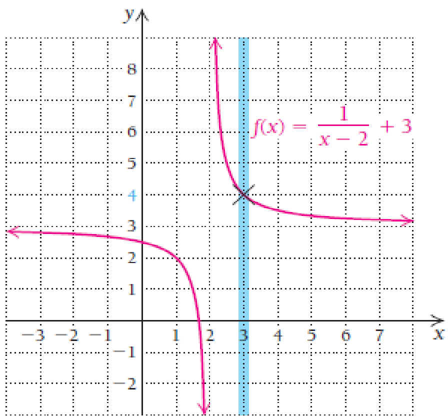


a) $\lim\limits_{x \to 3} f(x)$ ni aniqlash uchun $x \to 3^-$ va $x \to 3^+$ intilgandagi chap va o‘ng limitlarni hisoblaymiz. $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = 4$ ekanini topamiz:

$x \to 3^-$ $(x < 3)$	$f(x)$
2.1	13
2.5	5
2.9	$4.\overline{1}$
2.99	$4.\overline{01}$

$\lim\limits_{x \to 3} f(x) = 4$

$x \to 3^+$ $(x > 3)$	$f(x)$
3.5	$3.\overline{6}$
3.2	$3.8\overline{3}$
3.1	$3.\overline{90}$
3.01	$3.\overline{9900}$



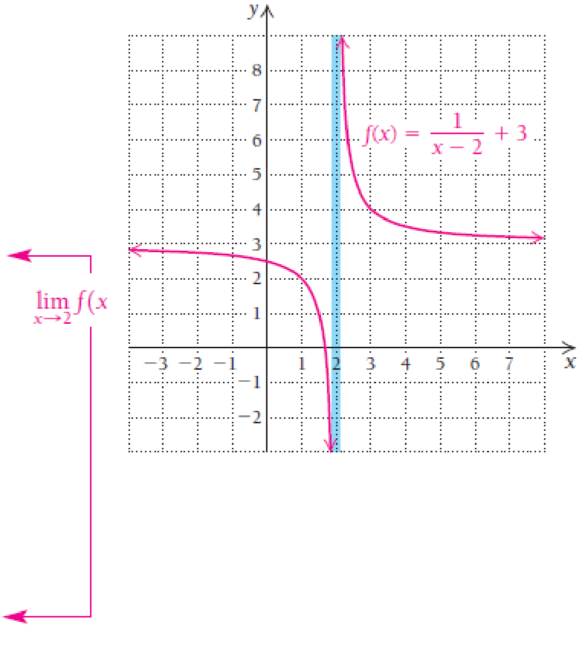
b) $\lim\limits_{x \to 2} f(x)$ ni hisoblash uchun 2 ga chapdan va o‘ngdan yaqinlashamiz. Jadval va grafikdan ko‘rish mumkinki, x 2 ga chapdan intilganda funksiya qiymati $-\infty$ ga ketadi: $\lim\limits_{x \to 0^-} f(x) = -\infty$.

x 2 ga o‘ngdan intilganda esa funksiya qiymati ∞ ga ketadi: $\lim\limits_{x \to 0^+} f(x) = \infty$.

Shunday qilib, $x \to 2$ da funksiya $\lim_{x\to 2} f(x)$ limitga ega emas, chunki o‘ng va chap limitlar teng emas.

$x \to 2^-$ $(x < 2)$	$f(x)$
1.5	1
1.9	−7
1.99	−97
1.999	−997

$x \to 2^+$ $(x > 2)$	$f(x)$
2.5	5
2.1	13
2.01	103
2.001	1003

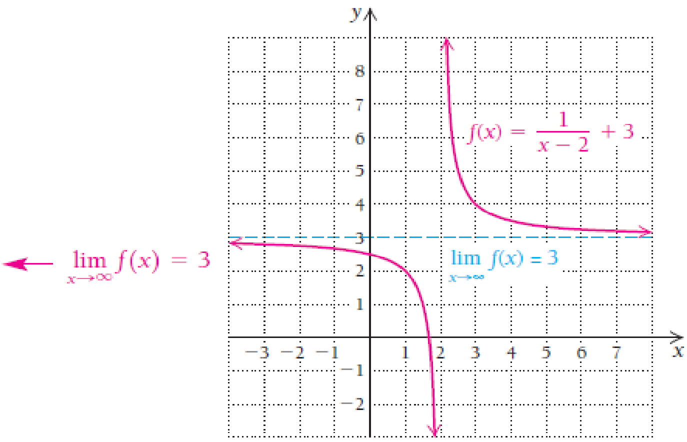


◀

12-misol. $f(x) = \dfrac{1}{x-2} + 3$ funksiya uchun $\lim_{x\to\infty} f(x)$ ni hisoblang.

Yechilishi: ▶ x ga katta qiymatlar berganimiz sari $f(x)$ ning qiymati 3 ga tobora yaqinlashib boraveradi: $\lim_{x\to+\infty} f(x) = 3$.

$x \to \infty$	$f(x)$
5	$3.\overline{3}$
10	3.125
100	3.0102
1000	3.0010



Xuddi shuningdek, x ning o‘rniga qo‘yiladigan manfiy sonlarni $-\infty$ ga tomon kattalashtirib borsak, $f(x)$ ning qiymati yana 3 ga tobora yaqinlashib boraveradi: $\lim_{x\to -\infty} f(x)=3$. Shunga ko‘ra, $\lim_{x\to\infty} f(x)=3$ deb xulosa qilamiz.◄

4-vazifa. $g(x)=\dfrac{1}{x-1}+5$ funksiya grafigini chizing va

a) $\lim_{x\to 1} f(x)$; b) $\lim_{x\to 2} f(x)$; c) $\lim_{x\to\infty} f(x)$ aniqlang.

Agar $y=f(x)$ funksiya $x=a$ nuqtaning biror atrofida aniqlangan va istalgan $M>0$ son uchun shunday $\delta>0$ son mavjud bo‘lsaki, $|x-a|<\delta$ tengsizlikni qanoatlantiradigan barcha $x\neq a$ nuqtalar uchun $|f(x)|>M$ tengsizlik bajarilsa, $x\to a$ da $y=f(x)$ **funksiya cheksizlikka intiladi** deyiladi va bu quyidagicha yoziladi:

$$\lim_{x\to\infty} f(x)=\infty$$

Misol uchun: $\lim_{x\to 1}\dfrac{1}{x-1}=\infty.$

Agar $\lim_{x\to a} f(x)=\infty$ $\left(\lim_{x\to\infty} f(x)=\infty\right)$ bo‘lsa, u holda $f(x)$ funksiya $x\to a$ da (yoki $x\to\infty$ da) **cheksiz katta funksiya** deyiladi.

Bu ta’rifdan ko‘rindiki, agar $f(x)$ funksiya cheksiz katta funksiya bo‘lsa, u holda istalgan $M>0$ uchun shunday $\delta>0$ topiladiki, $|x-a|<\delta$ tengsizlikni qanoatlantiradigan barcha x lar uchun $|f(x)|>M$ tengsizlik bajariladi. Bundan cheksiz katta funksiya chegaralanmagan funksiya ekani kelib chiqadi.

Agar $\lim\limits_{x\to a} f(x) = 0$ $\left(\text{yoki} \lim\limits_{x\to\infty} f(x) = 0\right)$ bo‘lsa, $f(x)$ funksiya $x \to a$ da (yoki $x \to \infty$ da) **cheksiz kichik funksiya** deyiladi.

Bu ta’rifdan ko‘rinadiki, $f(x)$ funksiya masalan, $x \to a$ da cheksiz kichik funksiya bo‘lsa, u holda istalgan kichik $\varepsilon > 0$ son uchun shunday $\delta > 0$ son topilsaki, $|x - a| < \delta$ tengsizlik qanoatlantiradigan barcha x lar uchun $|f(x)| < \varepsilon$ tengsizlik o‘rinli bo‘ladi.

Teorema. 1) Agar $f(x)$ funksiya $x \to a$ da ($x \to \infty$ da) cheksiz kichik funksiya bo‘lsa, u holda $\frac{1}{f(x)}$ funksiya $x \to a$ da ($x \to \infty$ da) cheksiz katta funksiyadir.

2) Agar $\varphi(x)$ funksiya $x \to a$ da ($x \to \infty$ da) cheksiz katta funksiya bo‘lsa, u holda $\frac{1}{\varphi(x)}$ funksiya $x \to a$ da ($x \to \infty$ da)cheksiz kichik funksiyadir.

Cheksiz kichik funksiyalarning asosiy xossalari:

1^0. Chekli sondagi cheksiz kichik funksiyalarning algebraik yig‘indisi cheksiz kichik funksiyadir.

2^0. Cheksiz kichik funksiyaning chegaralangan funksiyaga ko‘paytmasi cheksiz kichik funksiyadir.

3^0. Cheksiz kichik funksiyalarning ko‘paytmasi cheksiz kichik funksiyadir.

4^0. Cheksiz kichik funksiyaning noldan farqli limitiga ega bo‘lgan funksiya cheksiz kichik funksiyadir.

5^0.

1) Agar $y = f(x)$ funksiya $x \to a$ da limitga ega bo‘lsa, u holda uni bu limitga teng o‘zgarmas son va cheksiz kichik funksiya yig‘indisi ko‘rinishda ifodalash mumkin.

2) Agar $y = f(x)$ funksiya o‘zgarmas son bilan va $x \to a$ da cheksiz kichik funksiyaning yig‘indisi ko‘rinishda ifodalash mumkin bo‘lsa, u holda o‘zgarmas qo‘shiluvchi bu funksiyaning $x \to a$ dagi limiti bo‘ladi.

2.2.3. Ekvivalent cheksiz kichik funksiyalar. Cheksiz kichik funksiyalarni taqqoslash

Aytaylik, bir vaqtda bir necha α, β, γ, … cheksiz kichik miqdorlar birgina x argumentning funksiyalaridan iborat bo‘lib, x biror a limitga yoki cheksizlikka intilganda ular nolga intilsin.

Ta’rif. Agar $\frac{\beta}{\alpha}$ nisbat chekli va noldan farqli limitga ega, ya’ni $\lim \frac{\beta}{\alpha} = A \neq 0$ yoki $\lim \frac{\alpha}{\beta} = \frac{1}{A} \neq 0$ bo‘lsa, u holda β va α cheksiz kichik miqdorlar **bir xil tartibli cheksiz kichik miqdorlar** deyiladi.

13-misol. $\alpha = x,\ \beta = sin2x$ bo‘lsin, bu yerda $x \to 0$ α va β bir xil tartibli cheksiz kichik miqdordir, chunki $\lim\limits_{x\to 0} \frac{\beta}{\alpha} = \lim\limits_{x\to 0} \frac{\sin 2x}{x} = 2$.

14-misol. $x \to 0$ da $f(x) = \cos 2x - \cos^3 2x$ va $\varphi(x) = 3x^2 - 5x^3$ funksiyalar bir xil tartibli cheksiz kichik miqdorlar ekanligini isbotlang.

Yechilishi: ► $f(x)$ va $\varphi(x)$ funksiyalar nisbatining $x \to 0$ dagi limitini topamiz.

$$\lim_{x\to 0}\frac{f(x)}{\varphi(x)} = \lim_{x\to 0}\frac{\cos 2x - \cos^3 2x}{3x^2 - 5x^3} = \lim_{x\to 0}\frac{\cos 2x\left(1 - \cos^2 2x\right)}{3x^2 - 5x^3} =$$

$$= \lim_{x\to 0}\left(2\cdot\frac{\cos 2x \cdot \sin^2 2x}{x^2(3-5x)}\right) = \lim_{x\to 0}\frac{8\cos 2x\cdot \sin 2x \cdot \sin 2x}{2x\cdot 2x(3-5x)} = \frac{8}{3}$$

Limitni qiymati noldan farqli sondan iborat bo`lgani uchun berilgan funksiyalar bir xil tartibli cheksiz kichik miqdorlardir. ◄

Ta'rif. Agar ikkita cheksiz kichik miqdorning nisbati $\frac{\alpha}{\beta}$ nolga intilsa, ya'ni $lim\frac{\alpha}{\beta}=0$ $(lim\frac{\alpha}{\beta}=\infty)$ bo'lsa, u holda β cheksiz kichik miqdor α cheksiz miqdorga nisbatan **yuqori tartibli cheksiz kichik miqdor** deyiladi.

15-misol. $\alpha = x$, $\beta = x^n$, $n>1, x\to 0$ bo'lsin. β cheksiz kichik miqdor α cheksiz kichik miqdorga nisbatan yuqori tartibli cheksiz kichik miqdordir, chunki

$$\lim_{x\to 0}\frac{x^n}{x} = \lim_{x\to 0} x^{n-1} = 0.$$

Bunda α cheksiz kichik miqdor β cheksiz kichik miqdorga nisbatan quyi tartibli cheksiz kichik miqdordir.

Ta'rif. Agar β va α^k bir hil tartibli cheksiz kichik miqdorlar uchun $\lim \frac{\beta}{\alpha^k} = A \neq 0$ bo'lsa, β cheksiz kichik miqdorga nisbatan α ***k* - tartibli cheksiz kichik miqdor** deyiladi.

16-misol. Agar $\alpha = x$, $\beta = x^3$ bo'lsa, u holda $x \to 0$ da β cheksiz kichik miqdor α cheksiz kichik miqdorga nisbatan 3-tartibli cheksiz kichik miqdordir, chunki $\lim\limits_{x\to 0} \frac{\beta}{\alpha^3} = \lim\limits_{x\to 0} \frac{x^3}{(x)^3} = 1$.

Ta'rif. Agarda ikkita cheksiz kichik miqdorning $\frac{\beta}{\alpha}$ nisbati birga intilsa, ya'ni $lim \frac{\alpha}{\beta} = 1$ bo'lsa, u holda β va α cheksiz kichik miqdorlar **ekvivalent cheksiz kichik miqdorlar** deyiladi va $\alpha \sim \beta$ shaklida yoziladi.

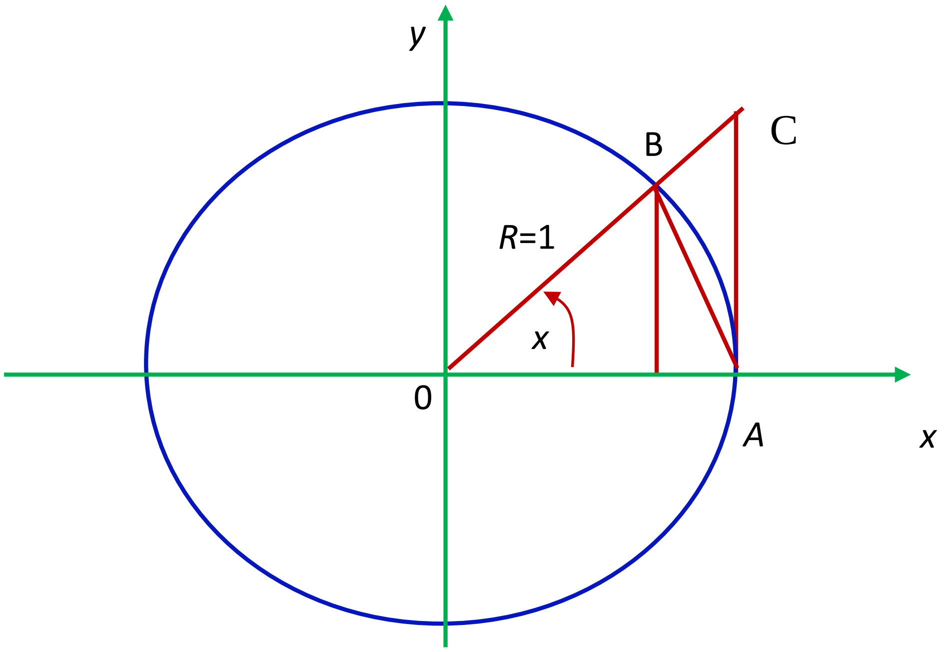
2.2.4. Birinchi ajoyib limit

Ushbu muhim $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ limit munosabatni keltirib chiqaramiz. Bu limit **birinchi ajoyib limit** deb ataladi.

Teorema. $\frac{\sin x}{x}$ funksiya $x \to 0$ da 1 ga teng limitga ega.

Isboti. ► R radiusli aylana olamiz, radianlarda ifodalangan x

burchak $0 < x < \frac{\pi}{2}$ oraliqda yotadi deb faraz qilaylik.



Shakldan ko‘rinadiki, $S_{\Delta BOA} < S_{\Delta BOA\ \text{сектор}} < S_{\Delta COA}$.

Biroq, $S_{\Delta BOA} = \frac{1}{2} OA \cdot BO \cdot sinx = \frac{R^2}{2} \cdot sinx$

$$S_{\Delta BOA\ \text{сектор}} = \frac{1}{2} OA^2 \cdot \overset{\frown}{AB} = \frac{R^2}{2} \cdot x\ ,$$

$$S_{\Delta COA} = \frac{1}{2} OA \cdot AC = \frac{R^2}{2} \cdot tgx$$

Shu sababli tengsizliklar ushbu ko‘rinishni oladi:

$$\frac{R^2}{2} \cdot sinx < \frac{R^2}{2} \cdot x < \frac{R^2}{2} \cdot tgx \quad \text{yoki} \quad sinx < x < tgx$$

Barcha hadlarni $sinx > 0$ ga bo‘lamiz $\left(0 < x < \frac{\pi}{2}\right)$;

$$1 < \frac{x}{sinx} < \frac{1}{cosx} \quad \text{yoki} \quad cosx < \frac{sinx}{x} < 1.$$

$\frac{\sin x}{x}$ funksiya bir xil limitga $\lim_{x\to 0} \cos x = 1$ va $\lim_{x\to 0} 1 = 1$

ga ega bo‘lgan funksiyalar bilan chegaralangan. Oraliq funksiyaning limiti haqidagi teoremaga asosan $\lim\limits_{x\to 0}\frac{sinx}{x}=1.$ ◄

Quyidagi limitlar o‘zaro teng kushlidir:

1) $\lim\limits_{x\to 0}\frac{\sin x}{x}=\lim\limits_{x\to 0}\frac{tgx}{x}=\lim\limits_{x\to 0}\frac{shx}{x}=\lim\limits_{x\to 0}\frac{thx}{x}=1$

2) $\lim\limits_{x\to 0}\frac{\sin\alpha x}{x}=\alpha,\left(\alpha\in R\right)$

17-misol. $\lim\limits_{x\to 0}\frac{\sin 3x}{x}$ limitni hisoblang.

Yechilishi: ► $\lim\limits_{x\to 0}\frac{sin3x}{x}=\lim\limits_{x\to 0}\frac{3\cdot sin3x}{3x}=3\cdot\lim\limits_{x\to 0}\frac{sin3x}{3x}=3.$ ◄

18-misol. $\lim\limits_{x\to 0}\frac{1-\cos 5x}{x^2}$ limitni hisoblang.

Yechilishi: ► $\lim\limits_{x\to 0}\frac{1-\cos 5x}{x^2}$ limitni hisoblash uchun uni 1-ajoyib limit ko`rinishiga keltirishimiz kerak. Buning uchun $\sin^2\frac{x}{2}=\frac{1-\cos 2x}{2}$ tenglikdan foydalanamiz.

$$\lim\limits_{x\to 0}\frac{1-\cos 5x}{x^2}=\lim\limits_{x\to 0}\frac{2\sin^2\frac{5x}{2}}{x^2}=2\lim\limits_{x\to 0}\left(\frac{\sin\frac{5x}{2}}{x}\right)^2=2\cdot\left(\frac{5}{2}\right)^2=\frac{25}{2}$$ ◄

2.2 bo‘limda nimalarni o‘rgandik?

- x ning a ga intilgandagi f **funksiyaning limiti** $\lim_{x\to a} f(x) = L$ deb yoziladi. Bu yozuv x ning qiymatlari a ga intilganda, mos ravishda $f(x)$ ning qiymatlari L ga intiladi degan ma’noni bildiradi. L qiymat yagona va chekli son bo‘lishi kerak.
- **Chap limit** $\lim_{x\to a^-} f(x)$ deb yoziladi. x ning qiymatlari a ga chap tomondan yaqinlashadi, ya’ni $x < a$.
- **O‘ng limit** $\lim_{x\to a^+} f(x)$ deb yoziladi. x ning qiymatlari a ga o‘ng tomondan yaqinlashadi, ya’ni $x > a$.
- $x \to a$ da chap va o‘ng limitlar bir-biriga teng bo‘lsa, $f(x)$ **funksiya limitga ega bo‘ladi**. $x \to a$ da chap va o‘ng limitlar bir-biriga teng bo‘lmasa, $f(x)$ **funksiya limitga ega bo‘lmaydi.**
- Funksiyaning a nuqtadagi $f(a)$ qiymati mavjud bo‘lmasa ham $\lim_{x\to a} f(x)$ **limit mavjud bo‘lishi mumkin.**
- $\lim_{x\to a} f(x)$ limit mavjud bo‘lishi va funksiyaning a nuqtadagi $f(a)$ qiymatidan **farq qilishi mumkin.**
- **Jadval va grafik** – limitni hisoblashda yordam beradigan vositalardir.

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-10 misollarda chiziqchalar o‘rniga mos javoblarni toping:

1. x 3 ga intilganda, $5x-7$ ning qiymatlari ______ ga intiladi.

2. x -5 ga intilganda, $2x+6$ ning qiymatlari ______ ga intiladi.

3. x ning qiymatlari _____ ga intilganda, $5x$ ning qiymatlari 6 ga intiladi.

4. x ning qiymatlari _____ ga intilganda, $x-2$ ning qiymatlari 12 ga intiladi.

5. $\lim_{x\to 5} f(x)$ yozuv _______ deb o‘qiladi.

6. $\lim_{x\to 3} h(x)$ yozuv _______ deb o‘qiladi.

7. $\lim_{x\to 5^-} g(x)$ yozuv _______ deb o‘qiladi.

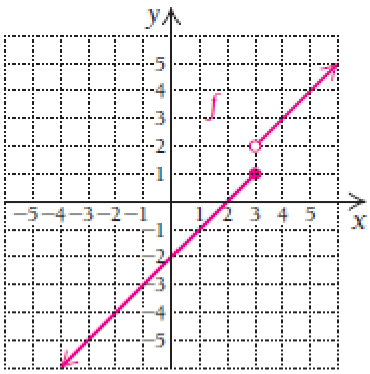
8. $\lim_{x\to -2^+} f(x)$ yozuv _______ deb o‘qiladi.

9. _______ yozuv, x 3 ga o‘ngdan intilgandagi $f(x)$ funksiyaning o‘ng limiti deb o‘qiladi.

10. _______ yozuv, x 3 ga chapdan intilgandagi $f(x)$ funksiyaning chap limiti deb o‘qiladi.

11-18 misollarda bo‘lakli berilgan $f(x)$ funksiya uchun quyidagi limitlarni hisoblang (grafigi va analitik ko‘rinishi keltirilgan):

$$f(x)=\begin{cases} x-2, & \text{agar} \quad x\le 3 \\ x-1, & \text{agar} \quad x>3 \end{cases}$$



11. $\lim\limits_{x\to 3^+} f(x)$; **12.** $\lim\limits_{x\to 3^-} f(x)$; **13.** $\lim\limits_{x\to -1^+} f(x)$;

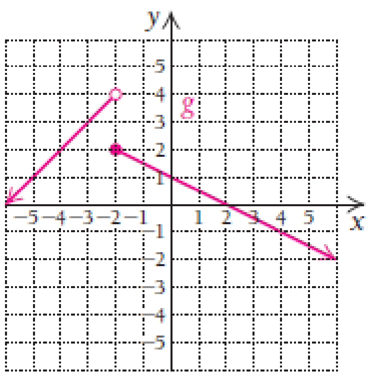
14. $\lim\limits_{x\to -1^-} f(x)$; **15.** $\lim\limits_{x\to -1} f(x)$; **16.** $\lim\limits_{x\to 3} f(x)$;

17. $\lim\limits_{x\to 4} f(x)$; **18.** $\lim\limits_{x\to 2} f(x)$; **19.** $\lim\limits_{x\to -\infty} f(x)$;

20. $\lim\limits_{x\to \infty} f(x)$; **21.** $\lim\limits_{x\to 0} f(x)$.

22-32 misollarda bo‘lakli berilgan $g(x)$ funksiya uchun quyidagi limitlarni hisoblang (grafigi va analitik ko‘rinishi keltirilgan):

$$g(x)=\begin{cases} x+6, & \text{agar} \quad x<-2 \\ -\dfrac{x}{2}+1, & \text{agar} \quad x>-3 \end{cases}$$



Agar limit mavjud bo‘lmasa, buning sababini tushuntiring.

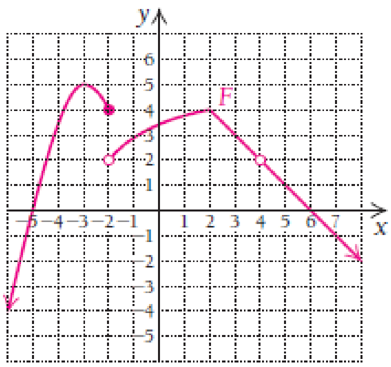
22. $\lim\limits_{x\to -2^+} g(x)$; **23.** $\lim\limits_{x\to -2^-} g(x)$; **24.** $\lim\limits_{x\to -4^+} g(x)$;

25. $\lim\limits_{x\to 4^-} g(x)$; **26.** $\lim\limits_{x\to -2} g(x)$; **27.** $\lim\limits_{x\to 4} g(x)$;

28. $\lim_{x\to 2} g(x)$; **29.** $\lim_{x\to -4} g(x)$; **30.** $\lim_{x\to 0} g(x)$;

31. $\lim_{x\to -\infty} g(x)$; **32.** $\lim_{x\to \infty} g(x)$.

33-41 misollarda $F(x)$ funksiya grafigidan foydalanib quyidagi limitlarni hisoblang:

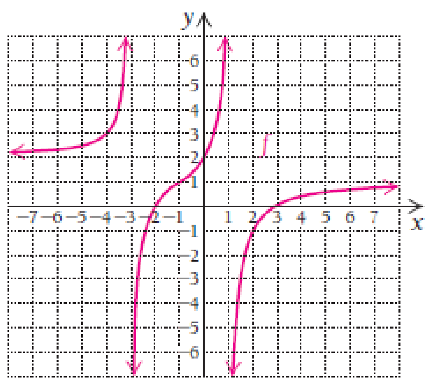


33. $\lim_{x\to -2} F(x)$; **34.** $\lim_{x\to -3} F(x)$; **35.** $\lim_{x\to 2} F(x)$;

36. $\lim_{x\to -5} F(x)$; **37.** $\lim_{x\to 4} F(x)$; **38.** $\lim_{x\to 6} F(x)$;

39. $\lim_{x\to -2^-} F(x)$; **40.** $\lim_{x\to -2^+} F(x)$; **41.** $\lim_{x\to \infty} F(x)$;

42-51 misollarda $f(x)$ funksiya grafigidan foydalanib quyidagi limitlarni hisoblang. Agar limit mavjud bo‘lmasa, buning sababini tushuntiring:



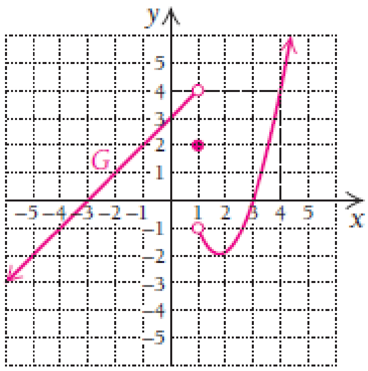
42. $\lim_{x\to 2} f(x)$; **43.** $\lim_{x\to -2} f(x)$; **44.** $\lim_{x\to -1} f(x)$;

45. $\lim\limits_{x\to-3} f(x)$; **46.** $\lim\limits_{x\to3} f(x)$; **47.** $\lim\limits_{x\to0} f(x)$;

48. $\lim\limits_{x\to1} f(x)$; **49.** $\lim\limits_{x\to-4} f(x)$; **50.** $\lim\limits_{x\to-\infty} f(x)$;

51. $\lim\limits_{x\to\infty} f(x)$.

52-61 misollarda $G(x)$ funksiya grafigidan foydalanib quyidagi limitlarni hisoblang. Agar limit mavjud bo'lmasa, buning sababini tushuntiring:



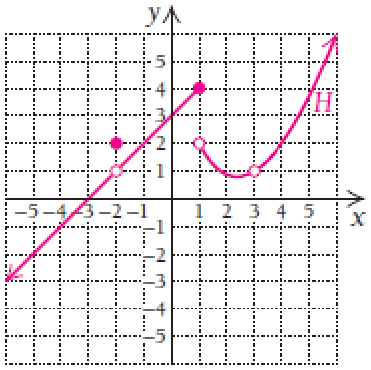
52. $\lim\limits_{x\to0} G(x)$; **53.** $\lim\limits_{x\to-2} G(x)$; **54.** $\lim\limits_{x\to1^-} G(x)$;

55. $\lim\limits_{x\to1^+} G(x)$; **56.** $\lim\limits_{x\to1} G(x)$; **57.** $\lim\limits_{x\to3^-} G(x)$;

58. $\lim\limits_{x\to3^+} G(x)$; **59.** $\lim\limits_{x\to3} G(x)$; **60.** $\lim\limits_{x\to\infty} G(x)$;

61. $\lim\limits_{x\to-\infty} G(x)$.

62-73 misollarda $H(x)$ funksiya grafigidan foydalanib quyidagi limitlarni hisoblang. Agar limit mavjud bo'lmasa, buning sababini tushuntiring:



62. $\lim\limits_{x\to 0} H(x)$; **63.** $\lim\limits_{x\to 1^+} H(x)$; **64.** $\lim\limits_{x\to 1^-} H(x)$;

65. $\lim\limits_{x\to 1} H(x)$; **66.** $\lim\limits_{x\to 3^+} H(x)$; **67.** $\lim\limits_{x\to 3^-} H(x)$;

68. $\lim\limits_{x\to 3} H(x)$; **69.** $\lim\limits_{x\to -2^+} H(x)$; **70.** $\lim\limits_{x\to -2^-} H(x)$;

71. $\lim\limits_{x\to -2} H(x)$; **72.** $\lim\limits_{x\to \infty} H(x)$. **73.** $\lim\limits_{x\to -\infty} H(x)$.

74-86 misollarda har bir funksiya grafigini chizing va grafikdan foydalanib limitlarni hisoblang. Agar limit mavjud bo‘lmasa, buning sababini tushuntiring:

74. $f(x)=x^2$; $\lim\limits_{x\to 0} f(x)$ va $\lim\limits_{x\to -4} f(x)$ ni hisoblang.

75. $f(x)=x^2-5$; $\lim\limits_{x\to 0} f(x)$ va $\lim\limits_{x\to -1} f(x)$ ni hisoblang.

76. $h(x)=|x|$; $\lim\limits_{x\to 0} h(x)$ va $\lim\limits_{x\to -2} h(x)$ ni hisoblang.

77. $h(x)=|x|+2$; $\lim\limits_{x\to 0} h(x)$ va $\lim\limits_{x\to -3} h(x)$ ni hisoblang.

78. $g(x)=\dfrac{1}{x-2}$; $\lim\limits_{x\to 2} g(x)$ va $\lim\limits_{x\to 3} g(x)$ ni hisoblang.

79. $g(x)=\dfrac{1}{x+3}$; $\lim\limits_{x\to -2} g(x)$ va $\lim\limits_{x\to -3} g(x)$ ni hisoblang.

80. $g(x)=\dfrac{1}{x}-3$; $\lim\limits_{x\to 0} g(x)$ va $\lim\limits_{x\to \infty} g(x)$ ni hisoblang.

81. $g(x)=\frac{1}{x}+2$; $\lim\limits_{x\to 0} g(x)$ va $\lim\limits_{x\to\infty} g(x)$ ni hisoblang.

82. $f(x)=\frac{1}{x+3}+5$; $\lim\limits_{x\to -3} f(x)$ va $\lim\limits_{x\to\infty} f(x)$ ni hisoblang.

83. $f(x)=\frac{1}{x-2}+3$; $\lim\limits_{x\to 2} f(x)$ va $\lim\limits_{x\to\infty} f(x)$ ni hisoblang.

84. $F(x)=\begin{cases} 2x+1, & \text{agar } x<1 \\ x, & \text{agar } x\ge 1 \end{cases}$ uchun $\lim\limits_{x\to 1^-} F(x)$, $\lim\limits_{x\to 1^+} F(x)$ va $\lim\limits_{x\to 1} F(x)$.

85. $F(x)=\begin{cases} -x+3, & \text{agar } x<2 \\ x+1, & \text{agar } x\ge 2 \end{cases}$ uchun $\lim\limits_{x\to 2^-} F(x)$, $\lim\limits_{x\to 2^+} F(x)$ va $\lim\limits_{x\to 2} F(x)$.

86. $g(x)=\begin{cases} -x+4, & \text{agar } x<3 \\ x-3, & \text{agar } x>3 \end{cases}$ uchun $\lim\limits_{x\to 3^-} g(x)$, $\lim\limits_{x\to 3^+} g(x)$ va $\lim\limits_{x\to 3} g(x)$.

87. $g(x)=\begin{cases} 3x-4, & \text{agar } x<1 \\ x-2, & \text{agar } x>1 \end{cases}$ uchun $\lim\limits_{x\to 1^-} g(x)$, $\lim\limits_{x\to 1^+} g(x)$ va $\lim\limits_{x\to 1} g(x)$.

88. $H(x)=\begin{cases} x^2, & \text{agar } x<-1 \\ x+2, & \text{agar } x>-1 \end{cases}$ uchun $\lim\limits_{x\to -1} H(x)$ ni hisoblang.

89. $H(x)=\begin{cases} -2x-3, & \text{agar } x<-1 \\ x^3, & \text{agar } x>-1 \end{cases}$ uchun $\lim\limits_{x\to -1} H(x)$ ni hisoblang.

90. $f(x)=\begin{cases} x+1, & \text{agar } x<0 \\ 2, & \text{agar } 0\le x<1 \\ 3-x, & \text{agar } x\ge 1 \end{cases}$ uchun $\lim\limits_{x\to 0} f(x)$ va $\lim\limits_{x\to 1} f(x)$.

91. $f(x)=\begin{cases} 2+x, & \text{agar } x\le -1 \\ x^2, & \text{agar } -1<x<3 \\ 9, & \text{agar } x\ge 3 \end{cases}$ uchun $\lim\limits_{x\to 3} f(x)$ va $\lim\limits_{x\to -1} f(x)$.

Limitning amaliy tatbiqlari

Tadbirkorlik va iqtisod.

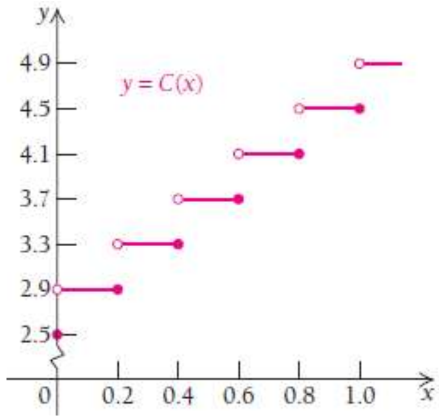
92. Taksiga kira haqi to‘lash. Toshkent shahrida yo‘lovchilar taksiga o‘tirishi bilan 2500 so‘m kirim qilinadi, so‘ngra har 200 metrga 400 so‘mdan hisoblanadi. Agar x masofani (km), $C(x)$ esa yo‘l kira haqini bildirsa, u holda

$C(x) = 2500$, agar $x = 0$ bo‘lsa;

$C(x) = 2900$, agar $0 < x \leq 0.2$ bo‘lsa;

$C(x) = 3300$, agar $0.2 < x \leq 0.4$ bo‘lsa;

$C(x) = 3700$, agar $0.4 < x \leq 0.6$ bo‘lsa va h.k.



$C(x)$ funksiya grafigidan foydalanib, quyidagi limitlarni hisoblang:

a) $\lim\limits_{x\to 0.25^-} C(x)$, $\lim\limits_{x\to 0.25^+} C(x)$ va $\lim\limits_{x\to 0.25} C(x)$.

b) $\lim\limits_{x\to 0.2^-} C(x)$, $\lim\limits_{x\to 0.2^+} C(x)$ va $\lim\limits_{x\to 0.2} C(x)$.

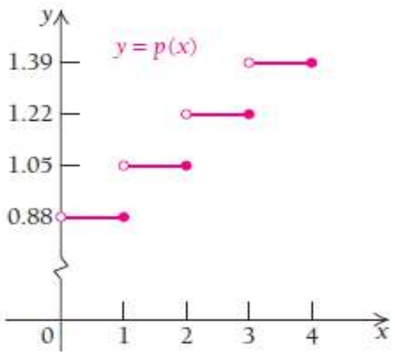
c) $\lim\limits_{x\to 0.6^-} C(x)$, $\lim\limits_{x\to 0.6^+} C(x)$ va $\lim\limits_{x\to 0.6} C(x)$.

93. Pochta jo‘natmasi funksiyasi. O‘zbekistonning tezkor pochtasi orqali katta hajmdagi jo‘natmani yuborish uchun dastlabki 100 grammiga 880 so‘m to‘lanadi. Keyingi har 100 grammiga 170 so‘mdan qo‘shib boriladi. Agar x jo‘natmaning massasi (100 gr), $p(x)$ esa unga to‘lanadigan badal bo‘lsa, u holda

$p(x)=0.88$, agar $0<x\le 1$ bo‘lsa;

$p(x)=1.05$, agar $1<x\le 2$ bo‘lsa;

$p(x)=1.22$, agar $2<x\le 3$ bo‘lsa;



$p(x)$ funksiya grafigidan foydalanib, quyidagi limitlarni hisoblang:

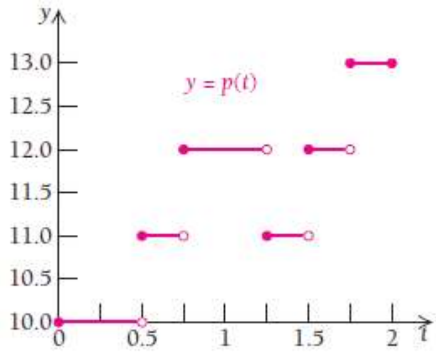
a) $\lim\limits_{x\to 1^-} p(x)$, $\lim\limits_{x\to 1^+} p(x)$ va $\lim\limits_{x\to 1} p(x)$;

b) $\lim\limits_{x\to 2^-} p(x)$, $\lim\limits_{x\to 2^+} p(x)$ va $\lim\limits_{x\to 2} p(x)$;

c) $\lim\limits_{x\to 2.6^-} p(x)$, $\lim\limits_{x\to 2.6^+} p(x)$ va $\lim\limits_{x\to 2.6} p(x)$;

d) $\lim\limits_{x\to 3} p(x)$, $\lim\limits_{x\to 3.4} p(x)$.

94. Fermer xo‘jaligining kengayishi. Parrandachilik bilan shug‘ullanuvchi fermer xo‘jaligida parranda sonining o‘sishi $p(t)$ vaqt funksiyasi bilan ifodalangan.



Har yarim yilda parranda soni (100 000 hisobida) grafikda ko‘rsatilgandek ortib borsa, quyidagilarni hisoblang:

a) $\lim\limits_{t\to 1.5^-} p(t)$, $\lim\limits_{t\to 1.5^+} p(t)$ va $\lim\limits_{t\to 1.5} p(t)$;

b) $\lim\limits_{t\to 1.75^-} p(t)$, $\lim\limits_{t\to 1.75^+} p(t)$ va $\lim\limits_{t\to 1.75} p(t)$;

c) limit mavjud bo‘lmaydigan holat bo‘lishi mumkinmi?

95. Sintez qilish. Quyidagi misollarda chiziqchalar o‘rnini shunday to‘ldiringki, $\lim\limits_{x\to 2} f(x)$ limit mavjud bo‘lsin:

a) $f(x)=\begin{cases} \dfrac{x}{2}+\_\_\_, & \text{agar} \quad x<2 \\ -x+6, & \text{agar} \quad x>2 \end{cases}$;

b) $f(x)=\begin{cases} -\dfrac{x}{2}+1, & \text{agar} \quad x<2 \\ \dfrac{3}{2}x+\_\_\_, & \text{agar} \quad x>2 \end{cases}$

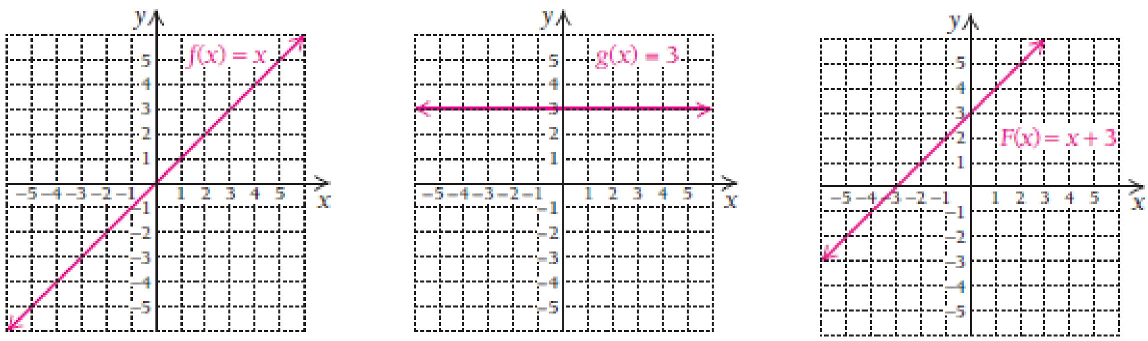
c) $f(x)=\begin{cases} x^2-9, & \text{agar} \quad x<2 \\ -x^2+\_\_\_\_, & \text{agar} \quad x>2 \end{cases}$.

2.3. Funksiya limitining xossalari va uzluksizlik

Funksiya limitini topish uchun uning jadvalini tuzish va grafigini yasash juda ko‘p vaqt talab qiladi. Ushbu bo‘limda funksiyalarning har xil qiymatlari uchun limitlarini tezroq hisoblash usullarini o‘rganamiz. Shuningdek, limitlardan foydalanib, hisob fani uchun juda muhim bo‘lgan uzluksizlik tushunchasini bilib olamiz.

2.3.1. Funksiya limitining algebraik shakli

Bizga $f(x)=x$, $g(x)=3$ va $F(x)=x+3$ funksiyalar grafiklari bilan berilgan bo‘lsin. E’tibor bering, $F(x)$ funksiya $f(x)$ va $g(x)$ funksiyalarning yig‘indisidan iborat:



$x\to 2$ dagi $f(x)$, $g(x)$ va $F(x)$ funksiyalarning limitlarini topish kerak bo‘lsin. 2.1 bo‘limda funksiya limitini jadval hamda grafikdan foydalanib aniqlashni o‘rgangan edik. Shunga ko‘ra,

$$\lim_{x\to 2} f(x)=2,\qquad \lim_{x\to 2} g(x)=3,\qquad \lim_{x\to 2} F(x)=5.$$

Bu usul a ning ixtiyoriy qiymatida o'rinli. Aytaylik, $a=-1$ bo'lsin. U holda $\lim\limits_{x\to -1} f(x)=-1$, $\lim\limits_{x\to -1} g(x)=3$, $\lim\limits_{x\to -1} F(x)=2$ o'rinli.

Misollar asosida quyidagi xulosaga kelish mumkin:

1. a ning barcha haqiqiy qiymatlarida $\lim\limits_{x\to a} x=a$ tenglik o'rinli.

2. a ning barcha haqiqiy qiymatlarida $\lim\limits_{x\to a} 3=3$ tenglik o'rinli.

$F(x)=f(x)+g(x)$ tenglikdan foydalanib,

3. a ning barcha haqiqiy qiymatlarida $\lim\limits_{x\to a}(x+3)=a+3$ tenglik o'rinli deb xulosa qilamiz.

Funksiyalarni kuzatish va umumiy xulosalar qilish bilan har qanday funksiya uchun limitlar xossalarini aniqlash mumkin.

2.3.2. Funksiya limitining xossalari

Agar $\lim\limits_{x\to a} f(x)=L$ va $\lim\limits_{x\to a} g(x)=M$ hamda c biror o'sgarmas son bo'lsa, quyidagi xossalar o'rinli:

1⁰. O'zgarmas sonning limiti uning o'ziga teng: $\lim\limits_{x\to a} c=c$.

2⁰. Darajaning limiti limitning darajasiga teng, musbat ko'rsatkichli ildizning limiti limitning shu ko'rsatkichli ildiziga teng:

$$\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a} f(x)\right]^n=L^n$$

$\lim\limits_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim\limits_{x\to a} f(x)}=\sqrt[n]{L}$, agar n juft bo'lsa, $L\geq 0$ bo'lishi kerak.

3⁰. Yig‘indi (yoki ayirma) ning limiti limitlar yig‘indi (yoki ayirma) siga teng: $\lim\limits_{x\to a}[f(x)\pm g(x)]=\lim\limits_{x\to a} f(x)\pm\lim\limits_{x\to a} g(x)=L+M$.

4⁰. Ko‘paytmaning limiti limitlar ko‘paytmasiga teng:

$$\lim_{x\to a}[f(x)\cdot g(x)]=[\lim_{x\to a} f(x)]\cdot[\lim_{x\to a} g(x)]=L\cdot M \ .$$

5⁰. Bo‘linmaning limiti limitlar bo‘linmasiga teng:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}=\frac{L}{M}\,,\ \text{bunda}\quad M\neq 0\,.$$

6⁰. O‘zgarmas sonning ko‘paytmasining limiti limitning o‘zgarmas songa ko‘paytmasiga teng: $\lim\limits_{x\to a} c\cdot f(x)=c\cdot\lim\limits_{x\to a} f(x)=c\cdot L$.

6-xossa 1- va 4- xossalardan kelib chiqadi.

1-misol. $\lim\limits_{x\to 5}(x^2-2x+8)$ ni limitning xossalaridan foydalanib hisoblang.

Yechilishi: ► Limitni hisoblash uchun x ning o‘rniga $x=5$ qiymatni qo‘yamiz hamda 3- va 6-xossalardan foydalanamiz:

$$\lim_{x\to 5}(x^2-2x+8)=\lim_{x\to 5} x^2-\lim_{x\to 5} 2x+\lim_{x\to 5} 8=\lim_{x\to 5} 5^2-2\cdot\lim_{x\to 5} 5+\lim_{x\to 5} 8=23$$

yoki $\lim\limits_{x\to 5}(x^2-2x+8)=\lim\limits_{x\to 5}(5^2-2\cdot 5+8)=23$. ◄

1-misoldan quyidagi teorema kelib chiqadi.

Teorema (Ratsional funksiyaning limiti haqida).

Har qanday $F(x)$ ratsional funksiya va uning aniqlanish sohasiga tegishli a soni uchun $\lim\limits_{x\to a} F(x)=F(a)$ tenglik o‘rinli.

Ratsional funksiyalar – barcha funksiyalar oilasidan iborat, unga o‘zgarmas funksiya, chiziqli funksiyalarni ham o‘z ichiga olgan barcha ko‘phad ko‘rinishidagi funksiyalar hamda ularning nisbatlaridan tashkil topgan funksiyalar kiradi. Limitlarning xossalari bizga ratsional funksiyalar limitini jadval va grafiklarsiz tez hisoblashga imkon beradi.

2-misol. $\lim\limits_{x\to 3}(x^4-5x^3+x^2-2x+1)$ hisoblang.

Yechilishi: ► $\lim\limits_{x\to 3}(x^4-5x^3+x^2-2x+1)=3^4-5\cdot 3^3+3^2-2\cdot 3+1=-50$ ◄

3-misol. $\lim\limits_{x\to 0}\sqrt{5x^3+x^2+4}$ hisoblang.

Yechilishi: ► $\lim\limits_{x\to 0}\sqrt{5x^3+x^2+4}=\sqrt{5\cdot 0+0+4}=\sqrt{4}=2$ ◄

1-vazifa. Limitlarni hisoblang va har bir qadamda qaysi xossadan foydalanganingizni yozib boring:

a) $\lim\limits_{x\to -1}(2x^4-x^3+4x^2-5x+1)$;

b) $\lim\limits_{x\to 3}\dfrac{7x^2-3x-11}{2x+17}$; c) $\lim\limits_{x\to -2}\sqrt{7+5x^2}$.

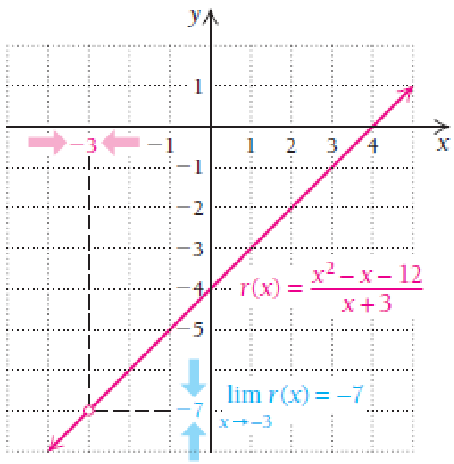
4-misol. $r(x)=\dfrac{x^2-x-12}{x+3}$ funksiya berilgan. $\lim\limits_{x\to -3}r(x)$ ni hisoblang.

Yechilishi: ► Funksiya $r(-3)$ da aniqlanmagan, chunki x ning o‘rniga -3 qiymatni qo‘ysak, maxraj nolga aylanadi. Bo‘linmaning limiti xossasidan to‘g‘ridan to‘g‘ri foydalanib bo‘lmaydi. Shuning uchun jadval va grafikdan foydalanamiz. $x\neq -3$

$x \to -3^-$ $(x < -3)$	$r(x)$
-3.1	-7.1
-3.01	-7.01
-3.001	-7.001

$x \to -3^+$ $(x > -3)$	$r(x)$
-2.9	-6.9
-2.99	-6.99
-2.999	-6.999

$$\lim_{x \to -3} r(x) = -7$$



Jadval va grafikdan $\lim_{x \to -3}\left(\frac{x^2 - x - 12}{x+3}\right) = -7$ ekanini ko‘rish qiyin emas. Endi algebraik yo‘l bilan, ya’ni xossalardan foydalanib, limitni hisoblaymiz: $\lim_{x \to -3} \frac{x^2 - x - 12}{x+3} = \lim_{x \to -3} \frac{(x+3)(x-4)}{x+3} = \lim_{x \to -3}(x-4) = -7$, $x \neq -3$.◀

Grafikdan ko‘rinadiki, (-3, -7) nuqta bo‘yalmagan. $r(-3)$ qiymatda funksiya aniqlanmagan, lekin $x \to 3$ da funksiya limiti mavjud.

$$\lim_{x \to -3} \frac{x^2 - x - 12}{x+3} = \lim_{x \to -3} \frac{(-3)^2 - (-3) - 12}{-3+3} = \lim_{x \to -3} \frac{0}{0}$$

bo‘lgani uchun kasrni qisqartirib, keyin limitni hisoblash kerak degan xulosaga keldik. Bunday **aniqmaslik** kasrning surat va maxrajida umumiy bo‘linuvchi borligini bildiradi, bizning misolda bu $x+3$.

$\frac{0}{0}$ yozuv limit mavjud bo'lishi mumkinligini bildiradi. Bunday hollarda limitni hisoblash uchun oldin algebraik soddalashtirishlar bajarish yoki jadval va grafiklardan foydalanish maqsadga muvofiq.

5-misol. $\lim\limits_{x\to -2}\frac{x^2-x-6}{x+2}$ hisoblang.

Yechilishi: ▶ $\lim\limits_{x\to -2}\frac{x^2-x-6}{x+2}=\lim\limits_{x\to -2}\frac{(x+2)(x-3)}{x+2}=\lim\limits_{x\to -2}(x-3)=-5, \quad x\neq -2$ ◀

6-misol. $\lim\limits_{x\to 4}\frac{2x^2-3x-20}{x^2-x-12}$ hisoblang.

Yechilishi: ▶ $\lim\limits_{x\to 4}\frac{2x^2-3x-20}{x^2-x-12}=\lim\limits_{x\to 4}\frac{(x-4)(2x+5)}{(x+3)(x-4)}=\lim\limits_{x\to 4}\frac{2x+5}{x+3}=\frac{13}{7}.$ ◀

Har doim ham limitlarni to'g'ridan to'g'ri hisoblab bo'lmaydi. Masalan, talabalar 4- va 5- misollarga o'xshash misollarni yechganda kasr maxrajida 0 hosil bo'lganini ko'rib, limit mavjud emas degan noto'g'ri xulosaga kelishadi.

7-misol. $\lim\limits_{h\to 0}(4x^2+4xh+h^2)$ hisoblang.

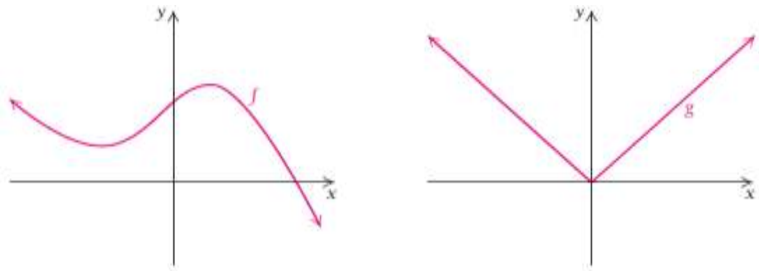
Yechilishi: ▶ Bu misolda x o'zgaruvchi emas, uni o'zgarmas son deb qaraymiz. O'zgaruvchi h ni o'rniga 0 ni qo'yib, limitni hisoblaymiz: $\lim\limits_{h\to 0}(4x^2+4xh+h^2)=\lim\limits_{h\to 0}(4x^2+4x\cdot 0+0^2)=4x^2$. ◀

2-vazifa. Limitni jadval va grafikdan foydalanib, hamda algebraik usulda xossalar yordamida hisoblang:

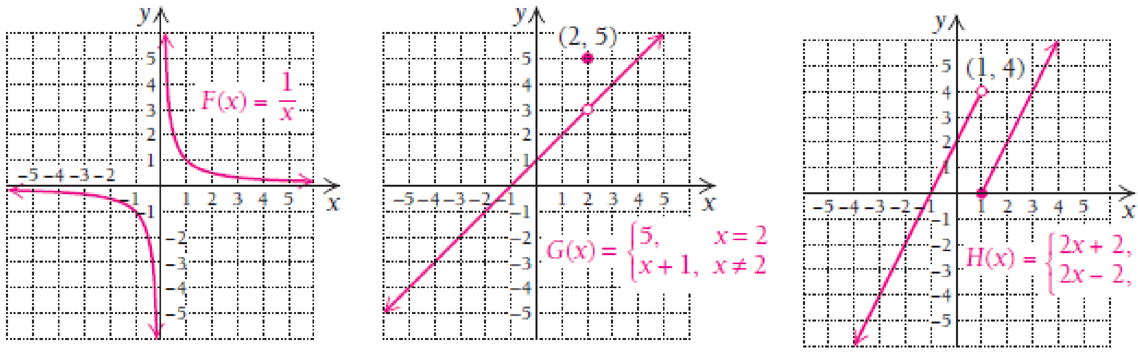
a) $\lim\limits_{x\to 2}\frac{x^2+4x-12}{x^2-4}$; b) $\lim\limits_{x\to 5}\frac{x^2-6x+5}{x^2-25}$; c) $\lim\limits_{x\to 0}\frac{x^4-2x^2}{x^2}$.

2.3.3. Funksiyaning uzluksizligi

Grafiklari keltirilgan funksiyalar butun haqiqiy sonlar o‘qida, ya’ni $(-\infty,\infty)$ da uzluksiz.



Grafiklarda hech qanday sakrash yoki bo‘yalmay qolgan joylari yo‘q. Shunga ko‘ra, uzluksizlikning sezgilarimizga asoslangan ta’rifini beramiz. Keyinchalik bu ta’rifni mukammalashtiramiz. Agar grafikning bir uchidan ushlab, uning ustidan qo‘limizni yurgizsak va qo‘limiz chizmadan uzilmasdan ikkinchi uchiga borsa, bu funksiya uzluksiz deymiz. Agar grafikning biror joyida qo‘limizni chizmadan olishga to‘g‘ri kelsa (sakratsak), funksiya uzilishga ega deymiz. $F(x)$, $G(x)$, $H(x)$ funksiyalar sonlar o‘qida uzlilishga ega ekanligini ko‘rish mumkin.



Bu funksiyalarning uchchalasi ham uzilishga ega, lekin ularning har biri o‘ziga xos ko‘rinishda.

● $F(x)$ funksiya $(-\infty,\infty)$ oraliqda uzluksiz emas, u $x=0$ nuqtada uziladi. Funksiya 2 bo‘lakdan iborat: $(-\infty,0)$ va $(0,\infty)$.

● $G(x)$ funksiya $x=2$ da uziladi va funksiya grafigi 5 ga sakraydi. Shuning uchun uni ham $(-\infty,2)$ va $(2,\infty)$ qismlarga ajratamiz.

● $H(x)$ funksiya ham uzilishga ega, bu funksiya 1 ga $x\to 1^-$ chapdan intilganda 4 ga, $x\to 1^+$ o‘ngdan intilganda 0 ga teng. Shuning uchun funksiyaning aniqlanish sohasi $(-\infty,1)$ va $(1,\infty)$ qismlardan iborat bo‘ladi.

$F(x),\ G(x),\ H(x)$ funksiyalarning barchasida **uzilish nuqtasi** mavjud.

$F(x)$ funksiyaning uzilish nuqtasi $x=0$, $G(x)$ funksiyaning uzilish nuqtasi $x=2$, $H(x)$ funksiya $x=1$ nuqtada uziladi.

Ta’rif. $f(x)$ funksiya $x=a$ **nuqtada uzluksiz deyiladi**, agarda

a) Funksiyaning $x=a$ nuqtadagi qiymati $f(a)$ mavjud bo‘lsa,

b) Funksiyaning $x\to a$ dagi limiti $\lim\limits_{x\to a} f(x)$ mavjud bo‘lsa,

c) Funksiyaning $x\to a$ dagi limiti funksiyaning shu nuqtadagi qiymatiga teng bo‘lsa: $\lim\limits_{x\to a} f(x)=f(a)$.

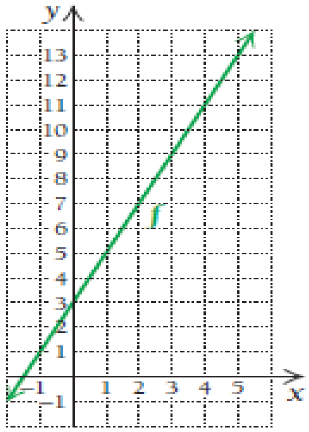
8-misol. $f(x)=2x+3$ funksiya $x=4$ nuqtada uzluksiz bo‘lishini isbotlang.

Yechilishi: ►

a) Funksiyaning $x=4$ nuqtadagi $f(4)=2x+3=2\cdot 4+3=11$ qiymati mavjud,

b) Funksiyaning $x\to 4$ dagi limiti $\lim\limits_{x\to 4} f(4)=11$ mavjud,

c) Funksiyaning $x \to 4$ dagi limiti funksiyaning shu nuqtadagi qiymatiga teng: $\lim_{x \to 4} f(4) = f(4) = 11$. Demak, funksiya $x = 4$ da uzluksiz ekan.

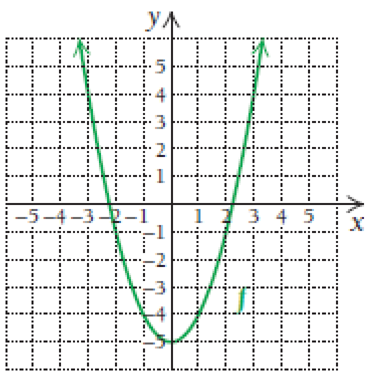


◀

9-misol. $f(x) = x^2 - 5$ funksiya $x = 3$ nuqtada uzluksizmi?

Yechilishi: ▶ Ratsional funksiyaning limiti haqidagi teoremaga ko‘ra, $\lim_{x \to 3} f(x) = \lim_{x \to 3} (3^2 - 5) = 4$.

Funksiyaning shu nuqtadagi qiymati esa $f(3) = 3^2 - 5 = 4$. Demak, $\lim_{x \to 3} f(x) = f(3)$. Ta’rifga ko‘ra, agar funksiyaning biror nuqtadagi limiti shu nuqtadagi qiymatiga teng bo‘lsa, funksuya bu nuqtada uzluksiz bo‘ladi. $f(x) = x^2 - 5$ funksiya $x = 3$ nuqtada uzluksiz ekan, shuningdek $f(x) = x^2 - 5$ funksiya barcha haqiqiy sonlarda uzluksizdir.



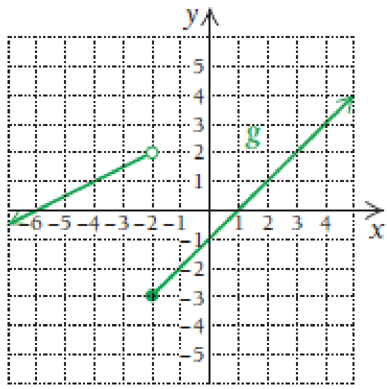
◀

10-misol. $g(x)=\begin{cases}\dfrac{x}{2}+3, & \text{agar} \quad x<-2\\ x-1, & \text{agar} \quad x\ge -2\end{cases}$ funksiyani $x=-2$ nuqtada uzluksizlikka tekshiring.

Yechilishi: ► $g(x)$ funksiyaning $x=-2$ nuqtada uzluksizligini bilish uchun $\lim\limits_{x\to -2} g(x)=g(-2)$ tenglikni tekshirish kerak. $\lim\limits_{x\to -2} g(x)$ limitni topish uchun chap va o‘ng limitlarni bir-biriga tengligini tekshiramiz:

$$\lim_{x\to -2^-} g(x)=\lim_{x\to -2^-}\left(\frac{x}{2}+3\right)=\lim_{x\to -2^-}\left(\frac{-2}{2}+3\right)=2;$$

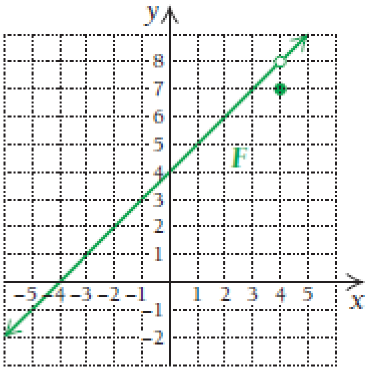
$$\lim_{x\to -2^+} g(x)=\lim_{x\to -2^+}(x-1)=\lim_{x\to -2^+}(-2-1)=-3.$$



$\lim\limits_{x\to -2^-} g(x)\neq \lim\limits_{x\to -2^+} g(x)$, ya’ni $\lim\limits_{x\to -2} g(x)$ limit mavjud emas. $g(x)$ funksiya $x=-2$ nuqtada uzilishga ega, boshqa barcha x qiymatlarda funksiya uzluksiz.◄

11-misol. $F(x)=\begin{cases}\dfrac{x^2-16}{x-4}, & \text{agar} \quad x\neq 4\\ 7, & \text{agar} \quad x=4\end{cases}$ funksiyani $x=4$ nuqtada uzluksizlikka tekshiring.

Yechilishi: ► $F(x)$ funksiya $x=4$ nuqtada uzluksiz bo‘lsa, $\lim\limits_{x\to 4} F(x)=F(4)$ tenglik o‘rinli bo‘lishi kerak. Funksiyaning formulasidan $F(4)=7$ ekanligini ko‘rish mumkin.



$$\lim_{x\to4} F(x)=\lim_{x\to4}\frac{x^2-16}{x-4}=\lim_{x\to4}\frac{(x-4)(x+4)}{x-4}=\lim_{x\to4}(x+4)=8.$$

Bundan $\lim\limits_{x\to4} F(x)\neq F(4)$ kelib chiqadi. Funksiya $x=4$ nuqtada uzilishga ega ekan. ◀

12-misol. $G(x)=\begin{cases}-x+3, & \text{agar } x\le 2\\ x^2-3, & \text{agar } x>2\end{cases}$ funksiyani barcha x larda uzluksizlikka tekshiring.

Yechilishi: ▶ $G(x)$ funksiya ikki xil funksiyaning birlashmasidan iborat. Ular $(-\infty,2]$ oraliqda aniqlangan $y=-x+3$ funksiya hamda $(2,\infty)$ oraliqda aniqlangan $y=x^2-3$ funksiyalardir.

$G(x)$ funksiya $x=2$ nuqtada uzluksiz bo‘lishi uchun $\lim\limits_{x\to2} G(x)=G(2)$ tenglik o‘rinli bo‘lishi kerak. Funksuya $x=2$ nuqtada

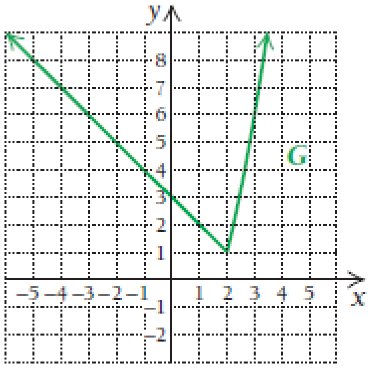
$$G(2)=-x+3=-2+3=1.$$

$\lim\limits_{x\to2} G(x)$ limitni topish uchun chap va o‘ng limitlarni bir-biriga tengligini tekshiramiz: $\lim\limits_{x\to2^-} G(x)=\lim\limits_{x\to2^-}(-x+3)=\lim\limits_{x\to2^-}(-2+3)=1;$

$$\lim_{x\to2^+} G(x)=\lim_{x\to2^+}(x^2-3)=\lim_{x\to2^+}(2^2-3)=1.$$

Demak, $\lim\limits_{x\to2^-} G(x)=\lim\limits_{x\to2^+} G(x)$, bundan $\lim\limits_{x\to2} G(x)=1$ ekanligi kelib chiqadi.

Natijada $\lim\limits_{x\to 2} G(x) = G(2)$ tenglik o‘rinli ekanligi isbotlandi. Funksiya $x = 2$ nuqtada va barcha x larda uzluksiz bo‘lar ekan.



◀

3-vazifa. Funksiyalarni uzluksizlikka tekshiring:

a) $G(x) = \begin{cases} 3x - 5, & \text{agar} \quad x < 2 \\ 2x + 1, & \text{agar} \quad x \geq 2 \end{cases}$ funksiyani $x = 2$ nuqtada;

b) $F(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & \text{agar} \quad x \neq 3 \\ 7, & \text{agar} \quad x = 3 \end{cases}$ funksiyani $x = 3$ nuqtada;

c) $p(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5}, & \text{agar} \quad x \neq 5 \\ c, & \text{agar} \quad x = 5 \end{cases}$ funksiyani $x = 5$ nuqtada.

11-misolda ko‘rdikki, bo‘lakli berilgan funksiya ham barcha haqiqiy sonlar to‘plamida uzluksiz bo‘lishi mumkin ekan. 12-misolda bo‘lakli funksiyalarning uzilishga ega bo‘lishini ko‘ramiz.

13-misol. Tadbirkorlik: narxdagi uzilishlar.

Rik Rok firmasi katta hajmli dekorativ landshaft toshlari bilan savdo qiladi. Hajmi 500 funtgacha bo‘lgan toshlarni har bir funtiga 2.50

$ dan, 500 funtdan kattalarini esa 2 $ dan sotadi. Narx funksiyasini bo‘lakli funksiya ko‘rinishida tasvirlash mumkin:

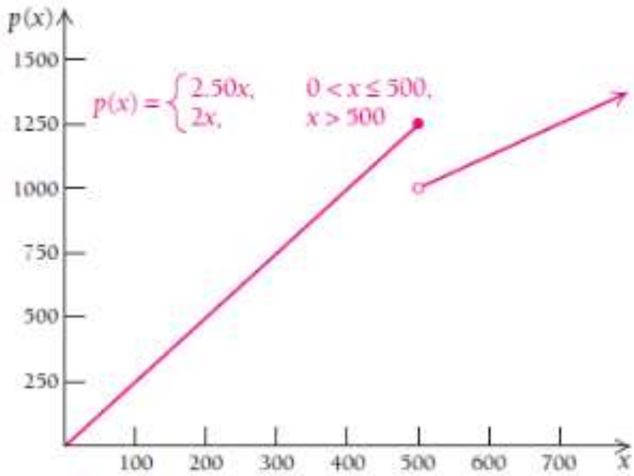
$$p(x)=\begin{cases} 2.50x, & \text{agar} \quad 0<x\le 500 \\ 2x, & \text{agar} \quad x>500 \end{cases}$$

Bunda x toshlar o‘lchami, $p(x)$ narxi. Funksiyaning $x=500$ da uzluksizlikka tekshiring.

Yechilishi: ► $\lim\limits_{x\to 500^-} p(x)=\lim\limits_{x\to 500^-} 2.50x=\lim\limits_{x\to 500^-} 2.50\cdot 500=1250$;

$$\lim_{x\to 500^+} p(x)=\lim_{x\to 500^+} 2x=\lim_{x\to 500^+} 2\cdot 500=1000.$$

O‘ng va chap limitlar o‘zaro teng emas, shuning uchun $x=500$ da funksiya uzilishga ega. Funksiya grafigini chizib ham uning uzilishga egaligini ko‘rish mumkin.



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14-misol. Tadbirkorlik: narxdagi uzilishlar. Rik Rok firmasining egasi agar 12-misoldagidek narxda savdo qilsa, zarar ko‘rar edi. Chunki 550 funtli toshlar 500 funtli toshlarga qaraganda arzonga sotilayotgan edi. Shuning uchun u 500 funtdan katta toshlar uchun qo‘shimcha to‘lov olishni o‘ylab topdi:

$$p(x) = \begin{cases} 2.50x, & \text{agar} \quad 0 < x \le 500 \\ 2x + k, & \text{agar} \quad x > 500 \end{cases}.$$

k qanday bo‘lganda $x = 500$ da uzluksiz bo‘ladi.

Yechilishi: ► Funksiya $x = 500$ da uzluksiz bo‘lishi uchun uning o‘ng va chap limitlari o‘zaro teng bo‘lishi kerak:

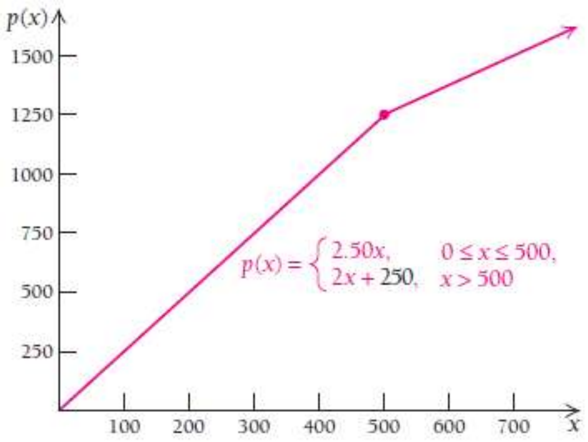
$$\lim_{x \to 500^-} p(x) = \lim_{x \to 500^+} p(x)$$

$$\lim_{x \to 500^-} p(x) = \lim_{x \to 500^-} 2.50x = \lim_{x \to 500^-} 2.50 \cdot 500 = 1250;$$

$$\lim_{x \to 500^+} p(x) = \lim_{x \to 500^+} (2x + k) = \lim_{x \to 500^+} (2 \cdot 500 + k) = 1250.$$

$$1000 + k = 1250,$$

$k = 250$ bo‘lganda $x = 500$ da uzluksiz bo‘ladi.



◄

Ta’rif. Agar $y = f(x)$ funksiya x_0 nuqtada va uning atrofida aniqlangan bo‘lib, argumentning cheksiz kichik orttirmasiga funksiyaning cheksiz kichik orttirmasi mos kelsa, ya’ni

$$\lim_{\Delta x \to 0} \Delta y = 0$$

bo‘lsa, funksiya x_0 nuqtada uzluksiz deyiladi.

15-misol. $y = x^2$ funksiya $x_0 = 1$ nuqtada uzluksizligini ko‘rsating.

Yechilishi: ► Bu funksiya barcha haqiqiy sonlar uchun aniqlangan.

Δy ni tuzamiz:

$$\Delta y = f(x_0 + \Delta x) - f(x_0) = (1 + \Delta x)^2 - 1^2 = 1 + 2\Delta x + (\Delta x)^2 - 1 = 2\Delta x + (\Delta x)^2 .$$

Bundan $\lim\limits_{\Delta x \to 0} \Delta y = \lim\limits_{\Delta x \to 0} (2\Delta x) + (\Delta x)^2) = 0$. Shunday qilib, $\lim\limits_{\Delta x \to 0} \Delta y = 0$, demak, $y = x^2$ funksiya $x_0 = 1$ nuqtada uzluksiz. ◄

2.3.4. Uzilish nuqtalari va ularning turlari

Ta'rif. Agar x_0 nuqtada $y = f(x)$ funksiya uchun quyidagi shartlardan kamida bittasi bajarilsa, x_0 nuqta $f(x)$ funksiyaning **uzilish nuqtasi**, funksiyaning o'zi esa **uzlukli funksiya** deyiladi:

1) funksiya x_0 nuqtada aniqlanmagan;
2) funksiya x_0 nuqtada aniqlangan, lekin $f(x_0 - 0)$ va $f(x_0 + 0)$ bir tomonlama limitlardan kamida biri mavjud emas;
3) funksiya x_0 nuqtada aniqlangan, bir tomonlama limitlar mavjud, lekin o'zaro teng emas;
4) funksiya x_0 nuqtada aniqlangan, bir tomonlama limitlar mavjud, o'zaro teng, lekin ular funksiyaning bu nuqtadagi qiymatiga teng emas: $f(x_0 - 0) = f(x_0 + 0) \neq f(x_0)$.

Uch turdagi uzilish nuqtalari farq qilinadi.

1. Yo'qotiladigan (bartaraf qilinadigan) uzilish

Ta'rif. x_0 nuqtada $y = f(x)$ funksiya aniqlanmagan, biroq bir tomonlama limitlar mavjud va o'zaro teng, ya'ni $f(x_0 - 0) = f(x_0 + 0)$ bo'lsa, x_0 nuqta **yo'qotiladigan uzilish nuqtasi** deyiladi.

16-misol. ► $x_0 = 0$ nuqta $f(x) = \dfrac{\sin x}{x}$ funksiyaning uzilish nuqtasidir.

Biroq, $\lim\limits_{x\to+0}\frac{\sin x}{x}=1$ va $\lim\limits_{x\to-0}\frac{\sin x}{x}=1$, ya'ni $f(-0)=f(+0)$ bir tomonlama limitlar mavjud va o'zaro teng, ammo $f(x)$ mavjud emas, demak, x_0 yo'qotiladigan uzilish nuqtasi, $f(0)=f(-0)=f(+0)=1$ deb olamiz. Shu bilan uzilish nuqtasini yo'qotamiz. ◄

2. Birinchi tur uzilish nuqtasi

Ta'rif. Agar funksiya x_0 nuqtada aniqlangan yoki aniqlanmagan, lekin bir tomonlama limitlar mavjud va o'zaro teng bo'lmasa, ya'ni $f(x_0-0)\neq f(x_0+0)$ bo'lsa, unga **birinchi tur uzilish nuqtasi** deyiladi. $h=f(x_0-0)-f(x_0+0)$ soni funksiyaning x_0 **nuqtadagi sakrashi** deyiladi.

17-misol. ► $f(x)=\frac{|x|}{x}$ funksiya $x=0$ nuqtada aniqlanmagan.

$$f(+0)=\lim_{x\to+0}\frac{|x|}{x}=\lim_{x\to+0}\frac{+x}{x}=1,\qquad f(-0)=\lim_{x\to-0}\frac{|x|}{x}=\lim_{x\to+0}\frac{-x}{x}=-1,$$

ya'ni $f(+0)\neq f(-0)$ va $h=1-(-1)=2$.

Demak, $x=0$ – birinchi tur uzilish nuqtasi. ◄

3. Ikkinchi tur uzilish nuqtasi

Ta'rif. Agar x_0 nuqtada bir tomonlama limitlardan kamida biri mavjud emas yoki cheksizlikka teng bo'lsa, x_0 nuqta **ikkinchi tur uzilish nuqtasi** deyiladi.

18-misol. ► $f(x)=2^{\frac{1}{x-1}}$ funksiya $x=1$ nuqtada mavjud emas.

$$f(1-0)=\lim_{x\to 1-0} 2^{\frac{1}{x-1}}=2^{-\infty}=0,$$

$$f(1+0)=\lim_{x\to 1+0} 2^{\frac{1}{x-1}}=2^{\infty}=\infty$$

Demak, $x=1$ ikkinchi tur uzilish nuqtasi. ◄

19-misol. ► $f(x)=\sin\frac{1}{x}$ funksiya $x=0$ nuqtada aniqlanmagan.

$$f(\pm 0)=\lim_{x\to\pm 0} Sin\frac{1}{x}$$

tayin limitga ega emas, demak, $x=0$ ikkinchi tur uzilish nuqtasi. ◄

20-misol. Berilgan funksiyani uzluksizlikka tekshiring va uning grafigini chizing:

$$f(x)=\begin{cases} x^2, & -\infty<x\le 0,\\ (x-1)^2, & 0<x\le 2,\\ 5-x, & 2<x<+\infty. \end{cases}$$

Yechilishi: ► $f(x)$ funksiya $(-\infty;2],(0;2]$ va $(2;+\infty)$ oraliqda aniqlangan va bu oraliqlarda uzluksiz bo`lgan elementar funksiyalar bilan berilgan. Shunday ekan, funksiya faqat va $x_2=2$ nuqtada uzilishi mumkin. $x_1=0$ nuqta uchun:

$$\lim_{x\to 0-0} f(x)=\lim_{x\to 0-0} x^2=0,$$

$$\lim_{x\to 0+0} f(x)=\lim_{x\to 0+0} (x-1)^2=1,$$

$$f(0)=x^2\Big|_{x=0}=0,$$

ya'ni, $f(x)$ funksiya $x_1=0$ nuqtada I tur uzilishga ega.

$x_2=2$ nuqta uchun quyidagilarni topamiz:

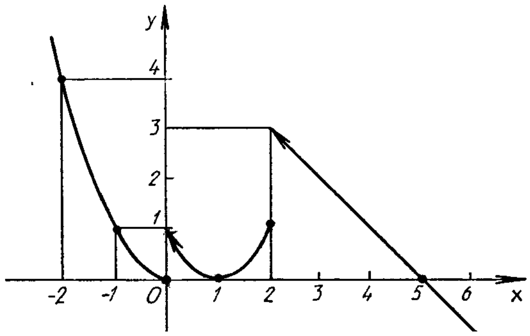
$$\lim_{x\to 2-0} f(x)=\lim_{x\to 2-0}(x-1)^2=1,$$

$$\lim_{x\to 2+0} f(x)=\lim_{x\to 2+0}(5-x)=3,$$

$$f(0)=(x-1)^2\Big|_{x=2}=1,$$

ya'ni, $f(x)$ funksiya $x_2=2$ nuqtada I tur uzilishga ega.

Berilgan funksiyaning grafigi quyidagicha bo'ladi:



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21-misol. $f(x)=8^{1/(x-3)}+1$ funksiyani $x_1=3$ va $x_2=4$ nuqtalarda uzluksizlikka tekshiring.

Yechilishi: ▶ $x_1=3$ nuqtada tekshiramiz.

$$\lim_{x\to 3-0} f(x)=\lim_{x\to 3-0}\left(8^{1/(x-3)}+1\right)=8^{-\infty}+1=1,$$

$$\lim_{x\to 3+0} f(x)=\lim_{x\to 3+0}\left(8^{1/(x-3)}+1\right)=8^{\infty}+1=\infty,$$

$x_1=3$ nuqtada $f(x)$ funksiya II tur uzulishga ega.

$x_2=4$ nuqtada funksiyani uzluksizlikka tekshiramiz:

$$\lim_{x\to 4-0} f(x) = \lim_{x\to 4-0}\left(8^{1/(x-3)}+1\right) = 9,$$

$$\lim_{x\to 4+0} f(x) = \lim_{x\to 4+0}\left(8^{1/(x-3)}+1\right) = 9,$$

$$f(4) = 8^{1/(4-3)}+1 = 9.$$

Funksiya uzluksizligining ta'rifiga ko'ra, $x_2 = 4$ nuqtada funksiya uzluksiz. ◀

2.3 bo'limda nimalarni o'rgandik?

● Aniqlanish sohasiga tegishli a nuqta uchun x a ga intilgandagi f **ratsional funksiyaning limitini** limit xossalaridan foydalanib, algebraik usulda hisoblash mumkin.

● Agar algebraik usulda $\frac{0}{0}$ aniqmaslik kelib chiqsa ham limit mavjud bo'lishi mumkin. Bunda limitni hisoblash uchun algebraik usuldan yoki jadval, grafik usuldan foydalaniladi.

● Agar funksiya uzluksiz bo'lsa, uning grafigini chizganda qalamni qog'ozdan uzmasdan chiziladi.

● $f(x)$ funksiya $x = a$ **nuqtada uzluksiz** bo'ladi, agarda

1) Funksiyaning $x = a$ nuqtadagi qiymati $f(a)$ mavjud,

2) Funksiyaning $x \to a$ dagi limiti $\lim_{x\to a} f(x)$ mavjud,

3) Funksiyaning $x \to a$ dagi limiti funksiyaning shu nuqtadagi qiymatiga teng bo'lsa: $\lim_{x\to a} f(x) = f(a)$.

- Agar bu shartlardan birortasi bajarilmasa, funksiya $x = a$ nuqtada uzilishga ega bo‘ladi.

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-8 misollarda limitlarning to‘g‘ri yoki noto‘g‘riligini aniqlang:

1. $\lim\limits_{x\to 5} 14 = 14$.

2. Agar $\lim\limits_{x\to 2} f(x) = 9$ bo‘lsa, u holda $\lim\limits_{x\to 2}\sqrt{f(x)} = 3$ bo‘ladi.

3. Agar $\lim\limits_{x\to 2} f(x) = 5$ bo‘lsa, u holda $\lim\limits_{x\to 2}[f(x)]^2 = 10$ bo‘ladi.

4. Agar $\lim\limits_{x\to 1} g(x) = 7$ bo‘lsa, u holda $\lim\limits_{x\to 1}[c\cdot g(x)] = 7c$ bo‘ladi.

5. Agar $f(x)$ funksiya $x = 2$ nuqtada uzluksiz bo‘lsa, $f(2)$ mavjud bo‘lishi kerak.

6. Agar $g(x)$ funksiya $x = -3$ nuqtada uzilishga ega bo‘lsa, $g(-3)$ mavjud bo‘lmasligi kerak.

7. Agar $\lim\limits_{x\to 1} g(x)$ limit mavjud bo‘lsa, $g(x)$ funksiya $x = 1$ da uzluksiz bo‘ladi.

8. Agar $\lim\limits_{x\to -1} F(x)$ limit $F(-1)$ funksiya qiymatiga teng bo‘lsa, $F(x)$ funksiya $x = -1$ da uzluksiz bo‘ladi.

9-20 misollarda ratsional funksiyaning limiti haqidagi teoremadan foydalanib, hisoblang:

9. $\lim\limits_{x\to -1}(4x+9)$

10. $\lim\limits_{x\to 5}(2x-7)$

11. $\lim_{x\to -1}(x^2-36)$

12. $\lim_{x\to -3}(x^3+39)$

13. $\lim_{x\to 3}(x^2-5x+22)$

14. $\lim_{x\to 6}(x^2-4x+16)$

15. $\lim_{x\to -1}(x^5+x^4-3x+14)$

16. $\lim_{x\to 3}(x^4+2x^3-3x^2+1)$

17. $\lim_{x\to 4}\frac{x^2-36}{x-2}$

18. $\lim_{x\to -2}\frac{x^3-27}{x-3}$

19. $\lim_{x\to 1}\frac{x^2-8}{x-2}$

20. $\lim_{x\to 9}\frac{x^2-67}{x-2}$

21-34 misollarda limitni hisoblashda $\frac{0}{0}$ aniqmaslik kelib chiqsa algebraik, jadval va grafik usullardan foydalaning:

21. $\lim_{x\to 2}\frac{2x^2-8}{x-2}$

22. $\lim_{x\to -8}\frac{x^2-64}{x+8}$

23. $\lim_{x\to 1}\frac{x^2+5x-6}{x^2-1}$

24. $\lim_{x\to -2}\frac{x^2-2x-8}{x^2-4}$

25. $\lim_{x\to 2}\frac{3x^2+x-14}{x^2-4}$

26. $\lim_{x\to -3}\frac{2x^2-x-21}{9-x^2}$

27. $\lim_{x\to 1}\frac{x^3-1}{x-1}$

28. $\lim_{x\to 2}\frac{x^3-8}{2-x}$

29. $\lim_{x\to 16}\frac{\sqrt{x}-4}{x-16}$

30. $\lim_{x\to 9}\frac{3-\sqrt{x}}{x-9}$

31. $\lim_{x\to 25}\frac{25-x}{\sqrt{x}-5}$

32. $\lim_{x\to -1}\frac{x^2+5x+4}{x^2+2x+1}$

33. $\lim_{x\to 2}\frac{x^2+3x-10}{x^2-4x+4}$

34. $\lim_{x\to -1}\frac{x^3+3x^2+3x+1}{(x+1)^2}$

35-40 misollarda limit xossalaridan foydalanib hisoblang:

35. $\lim_{x\to 5}\sqrt{x^2-16}$

36. $\lim_{x\to 3}\sqrt[4]{x^3+2x-17}$

37. $\lim\limits_{x\to 3}\sqrt{x^2-16}$

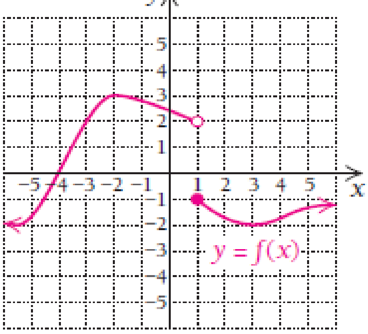
38. $\lim\limits_{x\to 4^-}\sqrt{x^2-16}$

39. $\lim\limits_{x\to -4^+}\sqrt{x^2-16}$

40. $\lim\limits_{x\to 3^+}\sqrt{x^2-9}$.

41-45 misollarda grafigi keltirilgan funksiyalarning qaysilari (-6,6) oraliqda uzluksiz:

41.



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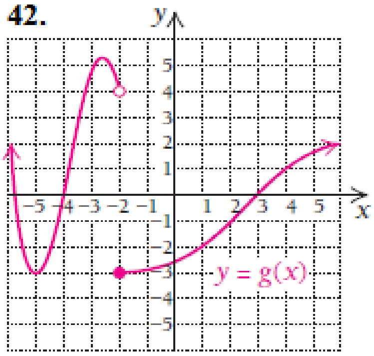

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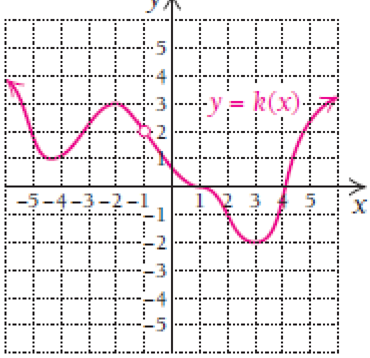

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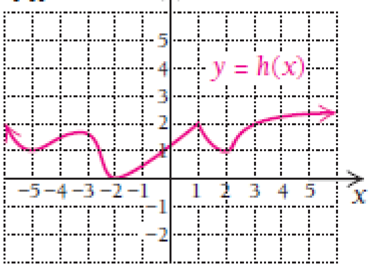

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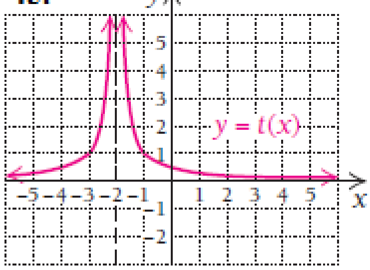


46-50 misollarda 41-45 misollardagi $f(x)$, $g(x)$, $k(x)$, $h(x)$, $t(x)$ funksiyalarning grafiklaridan foydalanib, savollarga javob bering:

46. 1) $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1^-} f(x)$ va $\lim_{x\to 1} f(x)$ hisoblang.

2) $f(1)$ ni toping.

3) $x=1$ da $f(x)$ funksiya uzluksizmi? Tushuntiring.

4) $\lim_{x\to -2} f(x)$ hisoblang.

5) $f(-2)$ ni toping.

6) $x=-2$ da $f(x)$ funksiya uzluksizmi? Tushuntiring.

47. 1) $\lim_{x\to 1^+} g(x)$, $\lim_{x\to 1^-} g(x)$ va $\lim_{x\to 1} g(x)$ hisoblang.

2) $g(1)$ ni toping.

3) $x=1$ da $g(x)$ funksiya uzluksizmi? Tushuntiring.

4) $\lim_{x\to -2} g(x)$ hisoblang.

5) $g(-2)$ ni toping.

6) $x=-2$ da $g(x)$ funksiya uzluksizmi? Tushuntiring.

48. 1) $\lim_{x\to -1} k(x)$ hisoblang.

2) $k(-1)$ ni toping.

3) $x=-1$ da $k(x)$ funksiya uzluksizmi? Tushuntiring.

4) $\lim_{x\to 3} k(x)$ hisoblang.

5) $k(3)$ ni toping.

6) $x=3$ da $k(x)$ funksiya uzluksizmi? Tushuntiring.

49. 1) $\lim_{x\to 1} h(x)$ hisoblang.

2) $h(1)$ ni toping.

3) $x=1$ da $h(x)$ funksiya uzluksizmi? Tushuntiring.

4) $\lim_{x\to -2} h(x)$ hisoblang.

5) $h(-2)$ ni toping.

6) $x=-2$ da $h(x)$ funksiya uzluksizmi? Tushuntiring.

50. 1) $\lim_{x\to 1} t(x)$ hisoblang.

2) $t(1)$ ni toping.

3) $x=1$ da $t(x)$ funksiya uzluksizmi? Tushuntiring.

4) $\lim_{x\to -2} t(x)$ hisoblang.

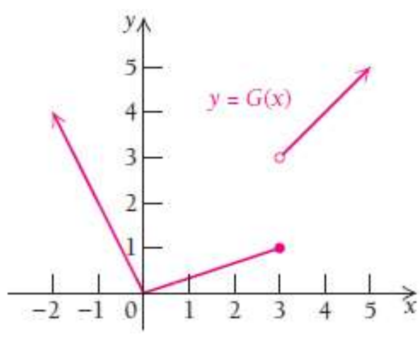
5) $t(-2)$ ni toping.

6) $x=-2$ da $t(x)$ funksiya uzluksizmi? Tushuntiring.

51-52 misollarda $G(x)$ va $C(x)$ funksiyalarning grafiklaridan foydalanib, savollarga javob bering:

51. 1) $\lim_{x\to 3^+} G(x)$, $\lim_{x\to 3^-} G(x)$, $\lim_{x\to 3} G(x)$ hisoblang.

2) $G(3)$ ni toping.



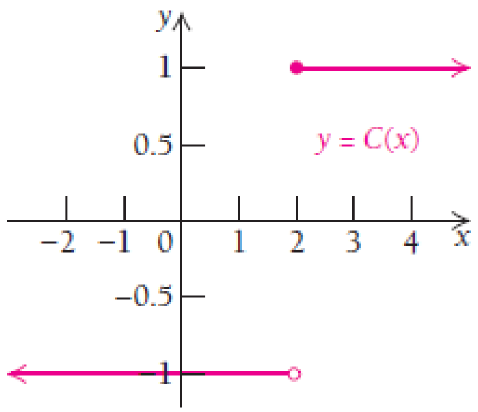
3) $x=3$ da $G(x)$ funksiya uzluksizmi? Tushuntiring.

4) $x=0$ da $G(x)$ funksiya uzluksizmi? Tushuntiring.

5) $x=2.9$ da $G(x)$ funksiya uzluksizmi? Tushuntiring.

52. Agar funksiya $C(x)=\begin{cases}-1, & \text{agar} \quad x<2\\ 1, & \text{agar} \quad x\geq 2\end{cases}$ bo‘lsa,

1) $\lim\limits_{x\to 2^+} C(x)$, $\lim\limits_{x\to 2^-} C(x)$, $\lim\limits_{x\to 2} C(x)$ hisoblang.



2) $C(2)$ ni toping.

3) $x=2$ da $C(x)$ funksiya uzluksizmi? Javobingizni tushuntiring.

4) $x=1.95$ da $C(x)$ funksiya uzluksizmi? Javobingizni tushuntiring.

53. $f(x)=3x-2$ funksiya $x=5$ nuqtada uzluksizmi?

54. $f(x)=x^2-3x$ funksiya $x=4$ nuqtada uzluksizmi?

55. $G(x)=\dfrac{1}{x}$ funksiya $x=0$ nuqtada uzluksizmi?

56. $C(x)=\sqrt{x}$ funksiya $x=-1$ nuqtada uzluksizmi?

57-75 misollarda javobingizni tushuntirib bering:

57. $C(x)=\begin{cases}\frac{x}{3}+4, & \text{agar } x\le 3\\ 2x-1, & \text{agar } x>3\end{cases}$ funksiya $x=3$ da uzluksizmi?

58. $f(x)=\begin{cases}\frac{x}{2}+1, & \text{agar } x<4\\ 7-x, & \text{agar } x\ge 4\end{cases}$ funksiya $x=4$ da uzluksizmi?

59. $g(x)=\begin{cases}\frac{x}{3}+4, & \text{agar } x\le 3\\ 2x-5, & \text{agar } x>3\end{cases}$ funksiya $x=3$ da uzluksizmi?

60. $f(x)=\begin{cases}\frac{x}{2}+1, & \text{agar } x<4\\ -x+5, & \text{agar } x>4\end{cases}$ funksiya $x=4$ da uzluksizmi?

61. $C(x)=\begin{cases}\frac{x}{3}+4, & \text{agar } x\le 3\\ x-1, & \text{agar } x>3\end{cases}$ funksiya $x=3$ da uzluksizmi?

62. $H(x)=\begin{cases}\frac{x^2-4}{x-2}, & \text{agar } x\ne 2\\ 5, & \text{agar } x=2\end{cases}$ funksiya $x=2$ da uzluksizmi?

63. $H(x)=\begin{cases}\frac{x^2-1}{x-1}, & \text{agar } x\ne 1\\ 4, & \text{agar } x=1\end{cases}$ funksiya $x=1$ da uzluksizmi?

64. $f(x)=\begin{cases}\frac{x^2-4x-5}{x-5}, & \text{agar } x<5\\ x+1, & \text{agar } x\ge 5\end{cases}$ funksiya $x=5$ da uzluksizmi?

65. $f(x)=\begin{cases}\frac{x^2-3x-4}{x-4}, & \text{agar } x<4\\ 2x-3, & \text{agar } x\ge 4\end{cases}$ funksiya $x=4$ da uzluksizmi?

66. $k(x)=\frac{1}{x^2-7x+10}$ funksiya $x=5$ da uzluksizmi?

67. $k(x)=\frac{1}{x^2-6x+8}$ funksiya $x=3$ da uzluksizmi?

68. $f(x)=\dfrac{1}{x^2-7x+10}$ funksiya $x=4$ da uzluksizmi?

69. $k(x)=\dfrac{1}{x^2-6x+8}$ funksiya $x=2$ da uzluksizmi?

70. $g(x)=x^2-3x+2$ funksiya (-4,4) oraliqda uzluksizmi?

71. $f(x)=x^2-5x+6$ funksiya (-5,5) oraliqda uzluksizmi?

72. $f(x)=\dfrac{1}{x}+3$ funksiya (-7,7) oraliqda uzluksizmi?

73. $f(x)=\dfrac{1}{x-1}$ funksiya $(0,\infty)$ oraliqda uzluksizmi?

74. $f(x)=4x^3-6x$ funksiya sonlar o‘qida uzluksizmi?

75. $f(x)=\dfrac{3}{x-5}$ funksiya sonlar o‘qida uzluksizmi?

76- 85 misollarda agar funksiyaning uzilish nuqtasi mavjud bo‘lsa, uni toping va funksiya grafigini chizing:

76. $f(x)=\begin{cases} x+4, & x<-1; \\ x^2+2, & -1\le x<1; \\ 2x, & x\ge 1. \end{cases}$

77. $f(x)=\begin{cases} x+2, & x\le -1; \\ x^2+1, & -1<x\le 1; \\ -x+3, & x\ge 1. \end{cases}$

78. $f(x)=\begin{cases} -x, & x\le 0; \\ -(x-1)^2, & 0<x<2; \\ x-3, & x\ge 2. \end{cases}$

81. $f(x)=\begin{cases} -x, & x\le 0; \\ \sin x, & 0<x\le \pi; \\ x-2, & x>\pi. \end{cases}$

82. $f(x)=\begin{cases} -(x+1), & x\le -1; \\ (x+1)^2, & -1<x\le 0; \\ x, & x>0. \end{cases}$

83. $f(x)=\begin{cases} -x^2, & x\le 0; \\ tg\, x, & 0<x\le \pi/4; \\ 2, & x>\pi/4. \end{cases}$

79. $f(x)=\begin{cases}\cos x, & x\le 0;\\ x^2+1, & 0<x<1;\\ x, & x\ge 1.\end{cases}$

84. $f(x)=\begin{cases}-2x, & x\le 0;\\ x^2+1, & 0<x\le 1;\\ 2, & x>1.\end{cases}$

80. $f(x)=\begin{cases}-x, & x\le 0;\\ x^2, & 0<x<2;\\ x+1, & x\ge 2.\end{cases}$

85. $f(x)=\begin{cases}-2x, & x\le 0;\\ \sqrt{x}, & 0<x<4;\\ 1, & x\ge 4.\end{cases}$

86- 95 misollarda funksiyaning 2 ta argumenti berilgan. Quyidagilarni tekshiring va hisoblang:

a) Berilgan argument qiymatlarida funksiyalar uzluksizmi?

b) Agar uzilishga ega bo‘lsa, funksiyaning shu qiymatlardagi chap va o‘ng limitlarini toping.

c) Sxematik grafigini chizing.

86. $f(x)=9^{1/(2-x)}$, $x_1=0$, $x_2=2$.

87. $f(x)=10^{1/(7-x)}$, $x_1=5$, $x_2=7$.

88. $f(x)=4^{1/(3-x)}$, $x_1=1$, $x_2=3$.

89. $f(x)=14^{1/(6-x)}$, $x_1=4$, $x_2=6$.

90. $f(x)=12^{1/x}$, $x_1=0$, $x_2=2$.

91. $f(x)=15^{1/(8-x)}$, $x_1=6$, $x_2=8$.

92. $f(x)=3^{1/(4-x)}$, $x_1=2$, $x_2=4$.

93. $f(x)=11^{1/(4+x)}$, $x_1=-4$, $x_2=-2$.

94. $f(x)=8^{1/(5-x)}$, $x_1=3$, $x_2=5$.

95. $f(x)=13^{1/(5+x)}$, $x_1=-5$, $x_2=-3$.

Funksiya limitining tatbiqlariga oid masalalar

96. Tadbirkorlik. Konditer fabrikasi konfetni 1 kg dan 20 kg gacha 1.5 dollardan sotadi. 20 kg dan ko‘p bo‘lsa, 1.25 dollar va yana k dollar qo‘shimcha narx qo‘shiladi. Agar x konfet massasini ifodalasa, u holda

$$p(x)=\begin{cases}1.50x, & \text{agar } x\le 20\\ 1.25x+k, & \text{agar } x>20\end{cases}$$

konfet narxini bildiradi.

1) Agar $p(x)$ funksiya $x=20$ da uzluksiz bo‘lsa, k ni toping.

2) Nima uchun $p(x)$ funksiya $x=20$ da uzluksiz bo‘ladi?

97. Fizika. Laborant pech ichidagi t haroratni o‘lchab turadi. Harorat 60 daqiqa ichida 0^0 C dan boshlab har daqiqada 2^0 C ga ko‘tarilishi kerak. 60-daqiqada laborant pechni har daqiqada 3^0 C dan sovutishi kerak. $T(t)=\begin{cases}2t, & \text{agar } t\le 60\\ k-3t, & \text{agar } t>60\end{cases}$ funksiya pech ichidagi haroratni bildiradi.

3) Agar $T(t)$ funksiya $t=60$ da uzluksiz bo‘lsa, k ni toping.

4) Nima uchun $T(t)$ funksiya $t=60$ da uzluksiz bo‘ladi?

98. Quyidagi limitlar mavjud bo‘lsa, ularni hisoblang. Agar mavjud bo‘lmasa, sababini tushuntiring.

a) $\lim\limits_{x\to 0}\frac{|x|}{x}$; b) $\lim\limits_{x\to 2}\frac{x^3+8}{x^2-4}$;

c) $\lim\limits_{x\to -1}\frac{|x+1|}{x+1}$; d) $\lim\limits_{x\to -3}\frac{x^3-27}{x^2-3}$.

2.4. O‘rta qiymat

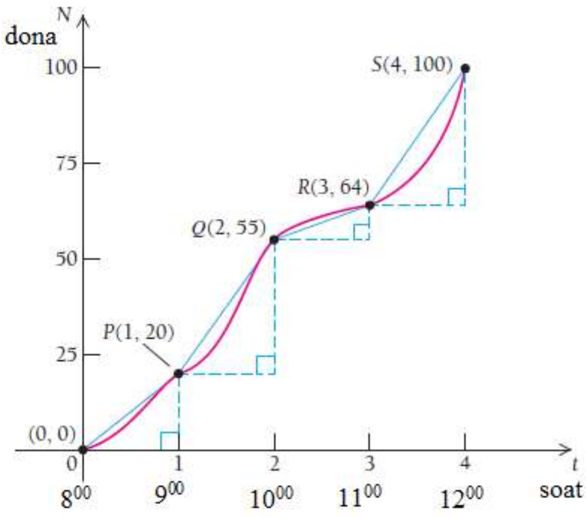
2.4.1 Orttirmaning o‘rta qiymati

Aytaylik, avtomobil 2 soatda 110 km yo’lni bosib o‘tdi. U holda uning o‘rtacha tezligi $\frac{110\,\text{km}}{2\,\text{soat}} = 55\,\text{km/soat}$ ga teng bo‘ladi. Faraz qilaylik, siz trassada avtomobilda harakatlanib ketayotganda tezlikni oshirdingiz. Spidometrga qaraganda ko‘rdingizki, oniy tezligingiz 55 km/s. Bular ikki xildagi juda katta farq qiluvchi tushunchalardir. Birinchi holat bilan siz tanish bo‘lsangiz kerak. Ikkinchi holat limit yordamida tushuntiriladi. Oniy tezlikni tushunish uchun avvalo o‘rtacha tezlikni aniq tassavvur qilishimiz kerak.

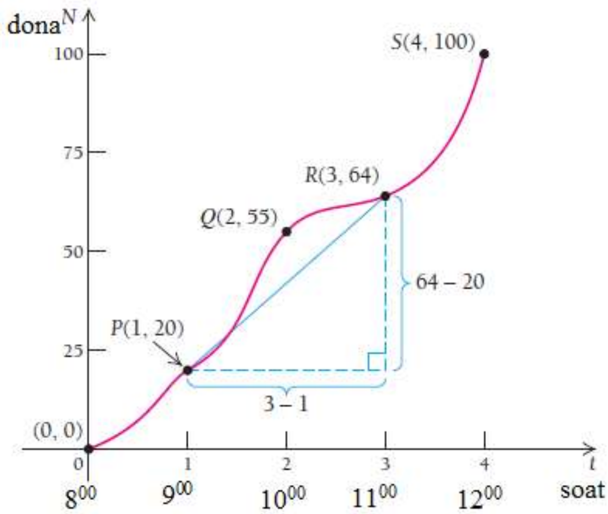
1-misol. Sanoat psixologlari ishlab chiqarish korxonalarining ishchilari uchun xarakterli bo‘lgan egri chiziqni yasashdi. Grafikda Lazurri firmasining 1 kunda tushlikkacha tayyorlagan kostyum-shimlari soni keltirilgan.

►1) Soat 9^{00} dan 10^{00} gacha Lazurri firmasida nechta dona kostyum-shim tikilgan?

Soat 9^{00} gacha 20 dona kostyum, soat 10^{00} gacha 55 dona kostyum-shim tikilgan. Shunda soat 9^{00} dan 10^{00} gacha 55-20=35 dona kostyum-shim tikilgan bo‘ladi. Soat 9^{00} dan 11^{00} gacha Lazurri firmasida o‘rtacha norma qancha bo‘lgan?



Bunda soat 9^{00} dan 11^{00} gacha har soatda nechta kostyum-shim tikilganini topish kerak:



$$\frac{64\,\text{dona - }20\,\text{dona}}{\text{soat }11-\text{soat }9}=\frac{44\,\text{dona}}{2\ \text{soat}}=22\,\text{dona/soat}$$

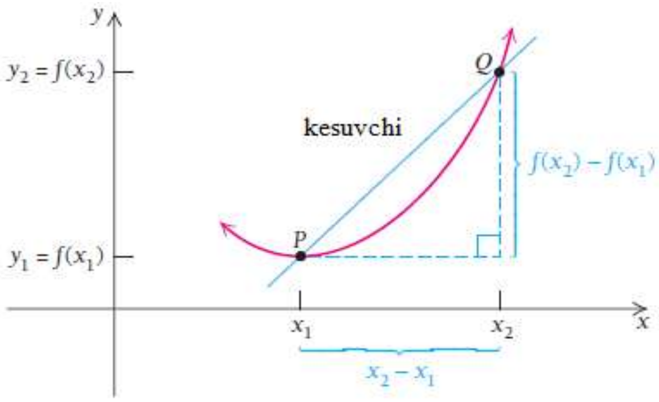
Soat 9^{00} dan 11^{00} gacha har soatda o‘rtacha 22 dona kostyum-shim tikilgan. ◄

$y=f(x)$ funksiya va uning ikkita x_1 va x_2 kirish qiymatlari berilgan bo‘lsin, u holda kirishdagi o‘zgarish yoki x ning o‘zgarishi x_2-x_1, chiqishdagi o‘zgarish yoki y ning o‘zgarishi y_2-y_1 bo‘ladi, bunda

$$y_1 = f(x_1) \text{ va } y_2 = f(x_2).$$

Ta'rif. x ning x_1 dan x_2 ga o'zgargandagi y **ning** x **ga nisbatan o'rtacha o'zgarishi deb,** funksiya orttirmasining argument (kirish) orttirmasiga nisbatiga aytiladi: $\frac{y_2 - y_1}{x_2 - x_1}$, bunda $x_2 \neq x_1$.

Agar funksiya grafigiga qarasak, $\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ ekanini ko'ramiz. Bunda $P(x_1, y_1)$ va $Q(x_2, y_2)$ nuqtalardan o'tgan og'ma yoki o'zgarishning o'rtacha qiymati deyiladi. $\vec{PQ}$ ga kesuvchi ham deyiladi. Kesuvchining og'ish burchagini f ning o'rtacha o'zgarishi sifatida tasvirlash mumkin ekan.



◄

1-vazifa. Quyidagi holatlar uchun o'zgarishning o'rtacha tezligini toping. Albatta o'lchov birligini ko'rsating:

a) 8 soat diametri 4 mm bo'lgan yomg'ir yog'di;

b) Avtomobilingiz 40 gallon gazda 250 km yuradi;

c) Soat 14^{00} da harorat 50^0 C edi, soat 17^{00} da 44^0 C bo'ldi.

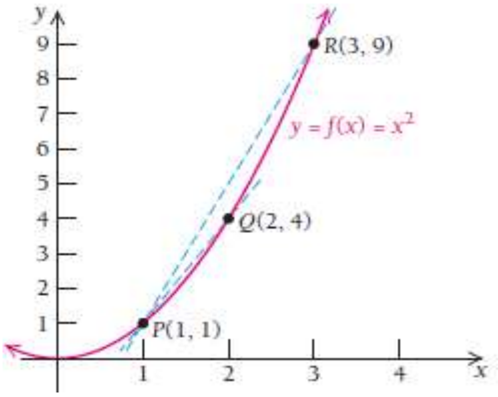
2-misol. Quyidagi holatlar uchun $y = f(x) = x^2$ funksiyaning o'rtacha o'zgarishini toping:

a) x ning qiymati 1 dan 3 gacha o'zgarganda;

b) x ning qiymati 1 dan 2 gacha o'zgarganda;

c) x ning qiymati 2 dan 3 gacha o'zgarganda.

Yechilishi: ► Bu diagramma hisoblashlarda bizga unchalik muhim emas, lekin ikkita kesuvchining qanday joylashganligini ko'rishimiz uchun chizdik.



a) $x_1 = 1$ da $y_1 = f(x_1) = f(1) = 1^2 = 1$ va $x_2 = 3$ da $y_2 = f(x_2) = f(3) = 3^2 = 9$.

Funksiya o'zgarishining o'rtacha qiymati

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{9-1}{3-1} = 4.$$

b) $x_1 = 1$ da $y_1 = f(x_1) = f(1) = 1^2 = 1$ va $x_2 = 2$ da $y_2 = f(x_2) = f(2) = 2^2 = 4$.

Funksiya o'zgarishining o'rtacha qiymati

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{4-1}{2-1} = 3.$$

c) $x_1 = 2$ da $y_1 = f(x_1) = f(2) = 2^2 = 4$ va $x_2 = 3$ da $y_2 = f(x_2) = f(3) = 3^2 = 9$.

Funksiya o'zgarishining o'rtacha qiymati

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{9-4}{3-2} = 5. \quad ◀$$

2-vazifa. Quyidagi holatlar uchun $f(x) = x^3$ funksiyaning o‘rtacha o‘zgarishini toping:

a) x ning qiymati 1 dan 4 gacha o‘zgarganda;

b) x ning qiymati 1 dan 2 gacha o‘zgarganda;

c) x ning qiymati 1 dan 1.2 gacha o‘zgarganda.

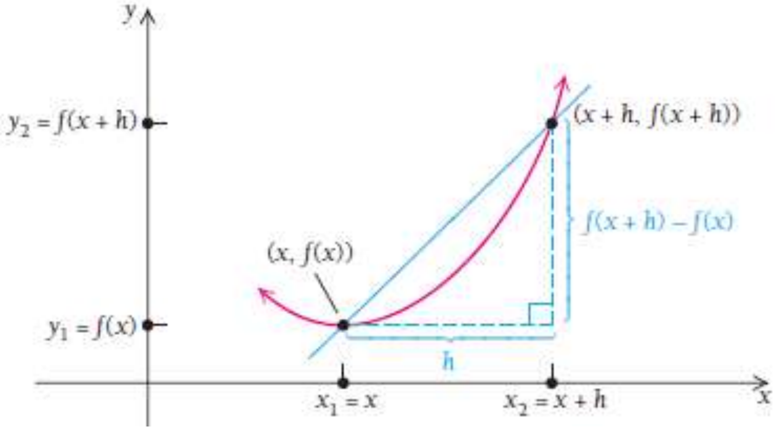
Chiziqli funksiya uchun o‘zgarishning o‘rtacha qiymati – ixtiyoriy nuqtadagi qiymatiga teng, ya’ni hamma joyida bir xil bo‘ladi. Chunki og‘ma (kesuvchi) to‘g‘ri chiziqning o‘zi bilan ustma-ust tushadi. 2-misoldagi nochiziqli funksiyada esa o‘zgarishning o‘rtacha qiymati mavjud. x ning x_1 dan x_2 gacha o‘zgarishiga qarab, har xil oraliqda funksiya ham har xil o‘zgaradi. Endi funksiya o‘zgarishining o‘rtacha qiymatini topish uchun formula keltirib chiqaramiz.

2.4.2. Orttirmalarning nisbati - o‘zgarishning o‘rtacha qiymati sifatida

O‘zgarishning o‘rta qiymati uchun formula yozamiz, bunda x_1 o‘rniga x deb, x_2 o‘rniga $x+h$ deb belgilashlar kiritamiz. Bunda h kattalik x_1 dan x_2 gacha bo‘lgan gorizontal masofa. Shunday qilib, x_1 yoki x dan x_2 ga boorish uchun biz h masofani o‘tishimiz kerak. Shunda

$x_2 = x + h$ ga teng bo‘ladi. U holda o‘zgarishning o‘rta qiymati yoki orttirmalar nisbati quyidagiga teng bo‘ladi:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$



Ta’rif. x ga nisbatan $f(x)$ ning o‘rtacha o‘zgarishiga **orttirmalar nisbati deb ham yuritiladi**: $\frac{f(x+h) - f(x)}{h}$, bunda $h \neq 0$. Orttirmalar nisbati $(x, f(x))$ nuqtadan $(x+h, f(x+h))$ nuqtaga o‘tkazilgan kesuvchining og‘ish burchagiga teng.

Yodda saqlang: $f(x+h) \neq f(x) + f(h)$.

3-misol. $f(x) = x^2$ funksiya uchun orttirmalar nisbatini toping:

a) $x = 7$ va $h = 4$;

b) $x = 7$ va $h = 0.1$;

c) $x = 7$ va $h = 0.01$.

Yechilishi: ►

$x=7$ va $h=4$ bo‘lgan hol uchun orttirmalar nisbatini topamiz:

$$\frac{f(x+h)-f(x)}{h}=\frac{f(7+4)-f(7)}{4}=\frac{11^2-7^2}{4}=\frac{121-49}{4}=\frac{72}{4}=18.$$

Bu yerda $f(11)=11^2=121$, $f(7)=7^2=49$. Orttirmalar nisbati 18 ga teng, bu $(7,49)$ va $(11,121)$ nuqtalardan o‘tkazilgan kesuvchining absissalar o‘qiga og‘ish burchagiga teng.

a) $x=7$ va $h=0.1$ bo‘lganda orttirmalar nisbatini hisoblaymiz:

$$\frac{f(x+h)-f(x)}{h}=\frac{f(7+0.1)-f(7)}{0.1}=\frac{7.1^2-7^2}{0.1}=\frac{50.41-49}{0.1}=\frac{1.41}{0.1}=14.1$$

c) $x=7$ va $h=0.01$ bo‘lganda orttirmalar nisbati quyidagicha:

$$\frac{f(x+h)-f(x)}{h}=\frac{f(7+0.01)-f(7)}{0.01}=\frac{7.01^2-7^2}{0.01}=\frac{49.1401-49}{0.01}=\frac{0.1401}{0.01}=14.01$$ ◀

Hisoblash ishlarini osonlashtirish maqsadida 3-misoldagi funksiya uchun formula keltirib chiqaramiz.

4-misol. $f(x)=x^2$ funksiyaning $x=7$ va $h=0.1$ hamda $x=7$ va $h=0.01$ qiymatlardagi orttirmalar nisbatini toping.

Yechilishi: ▶ Bizda $f(x)=x^2$ funksiya bor. Bundan

$$f(x+h)=(x+h)^2=x^2+2xh+h^2;$$

$$f(x+h)-f(x)=(x+h)^2-x^2=(x^2+2xh+h^2)-x^2=2xh+h^2.$$

U holda $\frac{f(x+h)-f(x)}{h}=\frac{2xh+h^2}{h}=\frac{h(2x+h)}{h}=2x+h$ formulani hosil qilamiz. Bu ifoda orttirmalar nisbatining soddalashgan ko‘rinishi, u faqat $h\neq 0$ bo‘lganda o‘rinli. Endi formulani misolga qo‘llaymiz:

$x=7$ va $h=0.1$ uchun $\frac{f(x+h)-f(x)}{h}=2x+h=2\cdot 7+0.1=14.1$

$x = 7$ va $h = 0.01$ uchun $\dfrac{f(x+h) - f(x)}{h} = 2x + h = 2 \cdot 7 + 0.01 = 14.01$.

Formuladan foydalanib, h ning 0 ga yanada yaqin borgandagi, ya'ni $h \to 0$ dagi orttirmalar nisbatini hisoblash mumkin.◄

5-misol. $f(x) = x^3$ funksiyaning orttirmalar nisbati formulasini yozing.

Yechilishi: ► Bizda $f(x) = x^3$ funksiya bor. Bundan

$$f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3;$$

$$f(x+h) - f(x) = (x+h)^3 - x^3 = (x^3 + 3x^2h + 3xh^2 + h^3) - x^3 = 3x^2h + 3xh^2 + h^3;$$

Bundan

$$\frac{f(x+h) - f(x)}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} = \frac{h(3x^2 + 3xh + h^2)}{h} = 3x^2 + 3xh + h^2, \; h \neq 0.$$◄

Hisob fanida ko‘p qo‘llaniladigan ratsional va kvadrat ildizli funksiyalarni ko‘rib chiqamiz. Ularning orttirmalar nisbati uchun formulalar keltirib chiqaramiz.

6-misol. $f(x) = \dfrac{1}{x}$ funksiyaning orttirmalar nisbati formulasini yozing.

Yechilishi: ► Argument (kirish) orttirmasi $f(x+h) = \dfrac{1}{x+h}$ ga teng.

Funksiya orttirmasi esa

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{x(x+h)} = \frac{-h}{x(x+h)};$$

Orttirmalar nisbati

$$\frac{f(x+h)-f(x)}{h}=\frac{\dfrac{-h}{x(x+h)}}{h}=\frac{-h}{x(x+h)}\cdot\frac{1}{h}=-\frac{1}{x(x+h)}$$

ga teng, bunda $h\neq 0$. ◀

7-misol. $f(x)=\sqrt{x}$ funksiyaning orttirmalar nisbati formulasini yozing.

Yechilishi: ▶ Argumentga orttirma beramiz: $f(x+h)=\sqrt{x+h}$. Endi funksiyaga orttirma beramiz: $f(x+h)-f(x)=\sqrt{x+h}-\sqrt{x}$. Ortirmalar nisbatini hisoblashda algebraik soddalashtirishlar bajaramiz:

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}=$$

$$=\frac{x+h-x}{h(\sqrt{x+h}-\sqrt{x})}=\frac{1}{\sqrt{x+h}-\sqrt{x}},\quad h\neq 0$$ ◀

Yuqorida keltiirilgan 4-7 misollarda orttirmalar nisbatini hisoblash uchun algebraik soddalashtirishlar bajarganimizdan keyin ikkita o‘zgaruvchi qoldi: x va h bunda $h\neq 0$ bo‘lishi kerak. h o‘zgaruvchi 0 ga teng bo‘lmasa-da, 0 ga yaqin borishi mumkin. Keyingi bo‘limda $h\to 0$ dagi limitni o‘rganamiz.

2.4 bo‘limda nimalarni o‘rgandik?

- **Orttirmaning o‘rtacha qiymati** – ikki nuqtadan o‘tuvchi to‘g‘ri chiziqning absissalar o‘qiga og‘ish burchagiga teng. Agar ikkita nuqta (x_1, y_1) va (x_2, y_2) bo‘lsa, u holda orttirmaning o‘rta qiymati $\frac{y_2 - y_1}{x_2 - x_1}$ ga teng.

- Agar ikkita nuqta berilgan funksiyaning yechimlari bo‘lsa, u holda **orttirmaning o‘rtacha qiymati** yoki og‘ish burchagi $\frac{f(x+h) - f(x)}{h}$ ga teng, bunda h gorizontal o‘qdagi x qiymatlarning farqi. Bu nisbatga **orttirmalar nisbati** deyiladi. Ikki nuqtani tutashtiruvchi to‘g‘ri chiziq esa **kesuvchi** deyiladi.

- Orttirmalar nisbati – og‘ma to‘g‘ri chiziqning formulasi. Ikkala formula ham ikki nuqtadan o‘tuvchi to‘g‘ri chiziqning absissalar o‘qiga og‘ish burchagiga teng.

- Orttirmalar nisbatini hisoblashdan oldin x va h o‘zgaruvchilar ustida algebraik soddalashtirishlar bajarish kerak.

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-15 misollarda dastlab a) orttirmalar nisbatini hisoblang va b) jadvalni to'ldiring:

x	h	$\frac{f(x+h)-f(x)}{h}$
5	2	
5	1	
5	0.1	
5	0.01	

1. $y=3x^2$; **2.** $y=3(x+1)^2$; **3.** $y=-3x^2$

4. $y=x^2+x$; **5.** $y=x^2-x$; **6.** $y=5x^2$

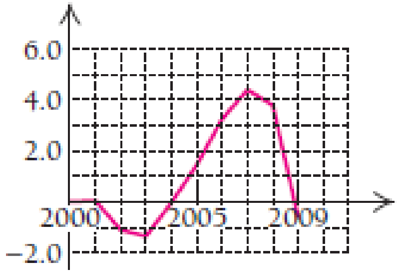
7. $y=\frac{2}{x}$; **8.** $y=\frac{9}{x}$; **9.** $y=-2x+5$

10. $y=1-x^3$; **11.** $y=11x^3$; **12.** $y=2x+3$

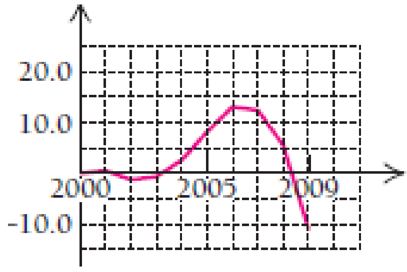
13. $y=x^2-3x$; **14.** $y=x^2-4x$; **15.** $y=x^2+4x-3$

16-23 misollarda berilgan diagrammalardan foydalanib, 2000-2005, 2005-2009, 2000-2009 yillardagi sohalardagi bandlikning o'rta qiymatini toping:

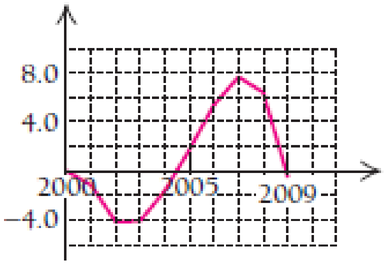
16. Yangi xodimlar



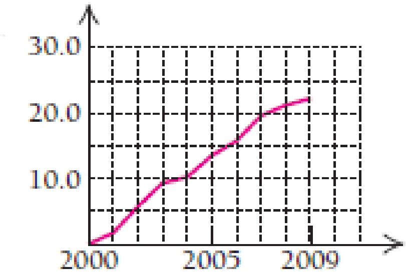
17. Qurilish



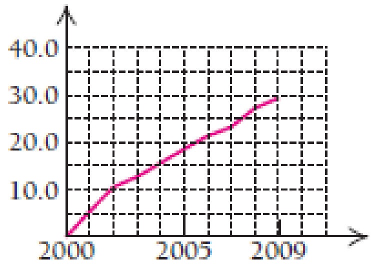
18. Mutaxassis xizmatlari



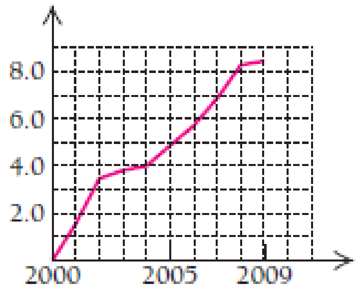
19. Sog‘liqni saqlash



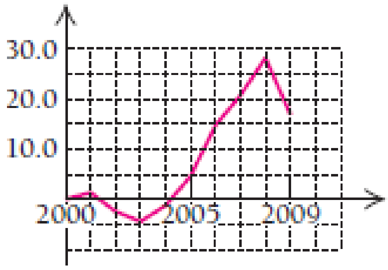
20. Ta’lim



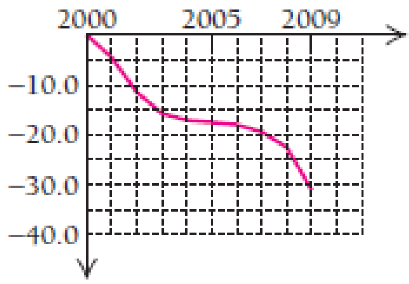
21. Hukumat



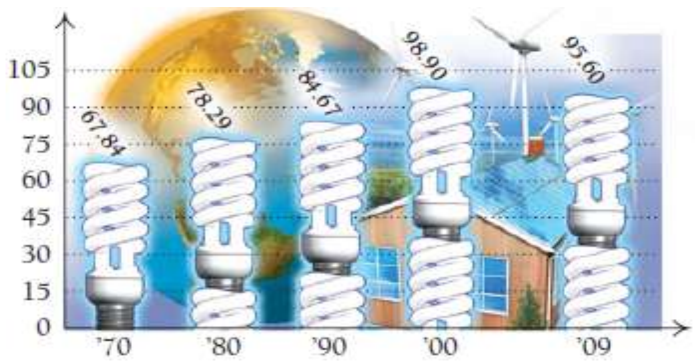
22. Tog‘ qazilmalari



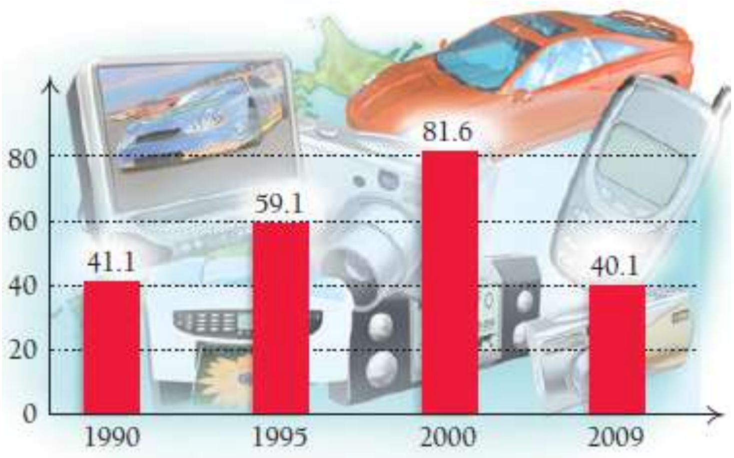
23. Ishlab chiqarish



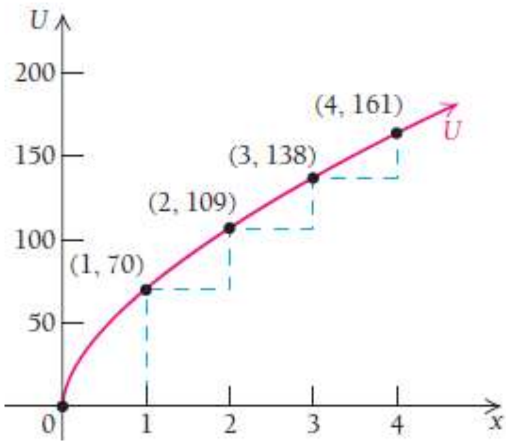
24. Diagramma asosida 1970-1980, 1980-1990, 2000-2009 yillardagi O‘zbekistondagi energiya iste’moli o‘zgarishining o‘rta qiymatini toping.



25. Diagrammadan foydalanib, 1990-1995, 1995-2000, 2000-2009 yillardagi O‘zbekiston – Rossiya iqtisodiy aloqalari o‘zgarishining o‘rta qiymatini toping.



26. Foyda. Quyida U iqtisodiy kattalik bo‘lgan foyda funksiyasining diagrammasi.



a) x 0 dan 1 gacha,

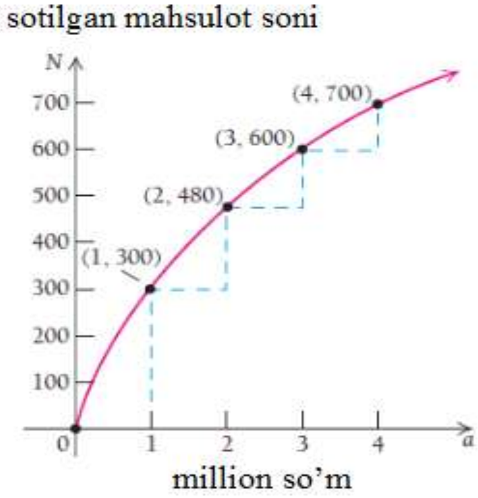
1 dan 2 gacha,

2 dan 3 gacha,

3 dan 4 gacha o‘zgarganda funksiya o‘zgarishining o‘rta qiymatini toping.

b) Nima uchun x kattalashgani sari orttirmalar nisbati kichrayib boradi?

27. Reklamaga a so‘m sarflangandan so‘ng $N(a)$ dona mahsulot sotiladi.



a) a 0 dan 1 gacha,

1 dan 2 gacha,

2 dan 3 gacha,

3 dan 4 gacha o‘zgarganda funksiya o‘zgarishining o‘rta qiymatini toping.

b) Nima uchun a kattalashgani sari orttirmalar nisbati kichrayib boradi?

28. Bilet narxi. Futbol oliy liga o‘yinlaridagi biletning o‘rtacha narxlari

$$p(x) = 0.03x^2 + 0.56x + 8.63$$

tenglama bilan approksimatsiyalanadi. Bu yerda x yillar orasidagi farq.

a) $p(4)$ ni hisoblang; b) $p(17)$ ni hisoblang;

c) $p(17) - p(4)$ ni hisoblang; d) $\frac{p(17) - p(4)}{17 - 4}$ ni hisoblang.

29. Murakkab foiz. Kvartaliga 6 % li omonatga 2000 $ bir necha yilga qo‘yilgan. Agar t vaqtda $A(t)$ pul miqdori quyidagi tenglama bilan hisoblanadi:

$$A(t) = 2000(1.015)^t$$

a) $A(3)$ ni hisoblang; b) $A(5)$ ni hisoblang;

c) $A(5) - A(3)$ ni hisoblang; d) $\frac{A(5) - A(3)}{5-3}$ ni hisoblang.

30. Kredit karta bo‘yicha qarzdorlik. Yiliga 14% ustama bo‘yicha 5000$ uzoq muddatli kredit olindi. t yildan keyin umumiy summa $A(t)$ bilan ifodalanadi:

$$A(t) = 5000 \cdot (1.14)^t.$$

$\frac{A(5) - A(3)}{5-3}$ ni hisoblang va bu qiymat nimani bildirishini tushuntiring.

31. Kredit karta bo‘yicha qarzdorlik. Yiliga 17% ustama bo‘yicha 3000$ uzoq muddatli kredit olindi. t yildan keyin umumiy summa $A(t)$ bilan ifodalanadi:

$$A(t) = 3000 \cdot (1.17)^t.$$

$\frac{A(4) - A(3)}{4-3}$ ni hisoblang va bu qiymat nimani bildirishini tushuntiring.

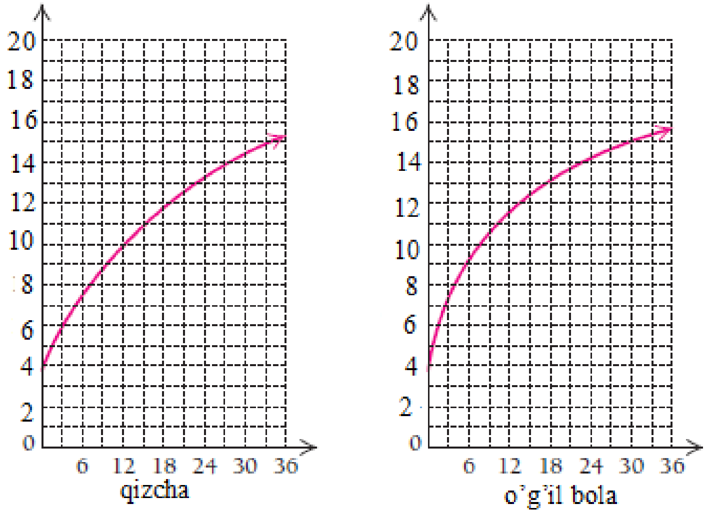
32. Umumiy talab. Faraz qilaylik, Spotr Stylz.Inc mobil telefonlar uchun ishlab chiqarilgan x (soni) qora ko‘zoynaklar narxini belgiladi:

$$C(x) = -0.05x^2 + 50x$$

ga teng bo‘lsa, $\dfrac{C(301)-A(300)}{301-300}$ ni hisoblang va bu natijaning kompaniyaga qanday ta’siri borligini tushuntiring.

33. Umumiy daromad. Faraz qilaylik, Spotr Stylz.Inc mobil telefonlar uchun ishlab chiqarilgan x (soni) qora ko‘zoynaklarni sotishdan tushgan daromad: $C(x)=-0.01x^2+1000x$ ga teng. $\dfrac{C(301)-A(300)}{301-300}$ ni hisoblang va bu natijaning kompaniyaga qanday ta’siri borligini tushuntiring.

34. Chaqaloqning o‘sishi. Chaqaloqning yoshiga nisbatan o‘rtacha vazni grafikda keltirilgan.



a) Birinchi 12 oyda qizchaning o‘rtacha o‘sishini toping (kg/oy);

b) Ikkinchi 12 oyda qizchaning o‘rtacha o‘sishini toping (kg/oy);

c) Birinchi 24 oyda qizchaning o‘rtacha o‘sishini toping (kg/oy);

d) (a)-(c) javoblar asosida 12 oylik qizchaning o‘sish tezligini baholang.

e) Qachon diagramma maksimal o‘sish tezligini ko‘rsatadi?

35. Anatomiya. Chaqaloq vaznining ortishi. Chaqaloqning yoshiga nisbatan o‘rtacha vazni grafikda keltirilgan (34-misolda).

a) Birinchi 15 oyda o‘g‘il bolaning o‘rtacha o‘sishini toping (kg/oy);

b) Ikkinchi 15 oyda o‘g‘il bolaning o‘rtacha o‘sishini toping (kg/oy);

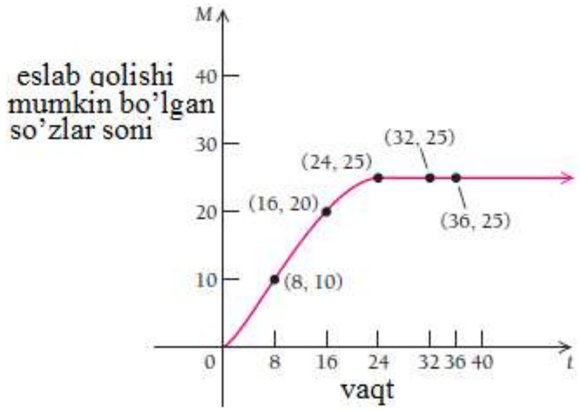
c) Birinchi 30 oyda o‘g‘il bolaning o‘rtacha o‘sishini toping (kg/oy);

d) (a)-(c) javoblar asosida 15 oylik o‘g‘il bolaning o‘sish tezligini baholang.

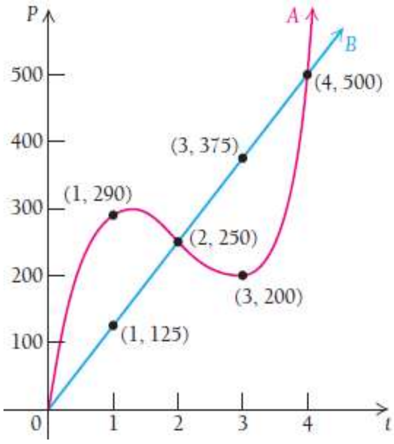
36. Xotira. Insonning 1 daqiqada eslab qolishi mumkin bo‘lgan so‘zlar soni quyidagi diagrammada keltirilgan:

1) $0 \le t \le 8$, $8 \le t \le 16$, $16 \le t \le 24$, $24 \le t \le 32$, $32 \le t \le 36$ daqiqa oralig‘ida o‘zgarganda M ning o‘rtacha o‘zgarishini toping

2) Nima uchun o‘rtacha o‘zgarishni $0 \le t \le 24$ daqiqa oralig‘ida hisoblash kerak?



37. Aholi sonining o‘sishi. Quyidagi egri chiziqlar 4 yil davomida A va B davlatlar aholisining o‘sishini ko‘rsatadi.



a) $0 \le t \le 4$ yil davomida aholi soni o‘sishini o‘rtacha qiymatini toping.

b) Agar hisoblashni har bir yil uchun alohida bajarib, so‘ngra yig‘indisi olinsa, o‘sish ko‘rsatkichi o‘zgaradimi?

c) Agar t ning qiymati $0 \le t \le 1$ gacha, $1 \le t \le 2$ gacha, $2 \le t \le 3$, $3 \le t \le 4$

38- 45 misollar 1-ajoyib limitni hisoblashga doir:

38. $\lim\limits_{x \to 0} \frac{\text{tg} mx}{\sin nx}$

39. $\lim\limits_{h \to 0} \frac{\sin(a+2h) - 2\sin(a+h) + \sin a}{h^2}$

40. $\lim\limits_{x \to x_0} \frac{\text{tg}\, x - \text{tg}\, x_0}{x - x_0}$

41. $\lim\limits_{x \to \pi/4} \frac{\sin x - \cos x}{\pi - 4x}$

42. $\lim\limits_{x \to \pi/2} \frac{\cos x}{\pi - 2x}$

43. $\lim\limits_{x \to 0} \frac{\text{tg} x - \sin x}{x^3}$

44. $\lim\limits_{x \to 0} \frac{\sin 3x - \sin x}{\ln(x+1)}$

45. $\lim\limits_{x \to 0} \frac{\sin 2x}{\ln(x+1)}$

2.5. Funksiya hosilasini hisoblash

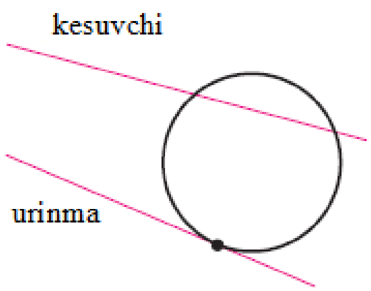
$y=f(x)$ funksiya grafigidagi $(x, f(x))$ va $(x+h, f(x+h))$ nuqtalarni tutashtiruvchi kesuvchining og‘ish burchagi $[x, x+h]$ kesmada $f(x)$ ni o‘zgarishining o‘rtacha qiymatini ifodalaydi. Bu orttirmalar nisbatiga teng bo‘ladi:

$$\frac{f(x+h)-f(x)}{h}, \quad h \neq 0 .$$

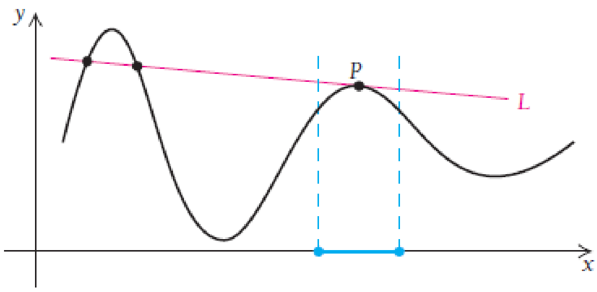
1.3-bo‘limda bir nechta funksiya uchun orttirmalar nisbatini hisobladik, ularni maksimal darajada soddalashtirdik. Soddalashgan orttirmalar nisbati x va h o‘zgaruvchilardan iborat bo‘ladi. Esingizda bo‘lsa, h qiymat x va $x+h$ orasidagi gorizontal masofaga teng. h nolga teng bo‘lishi mumkin emas. Lekin limitga o‘xshab, h nolga intiladi deb hisoblashimiz mumkin. Ushbu bo‘limda shu holatni tadqiq qilamiz.

2.5.1. Urinma

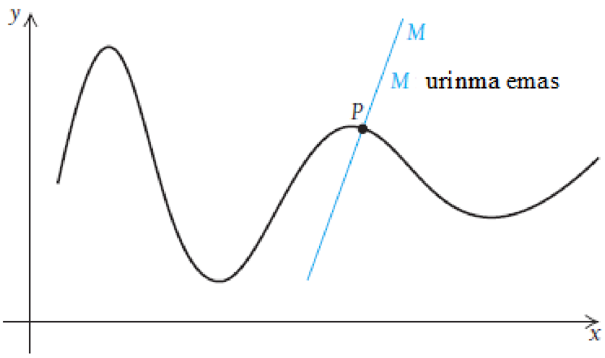
Aylana bilan bitta umumiy nuqtaga ega bo‘lgan to‘g‘ri chiziqqa **urinma** deyiladi.



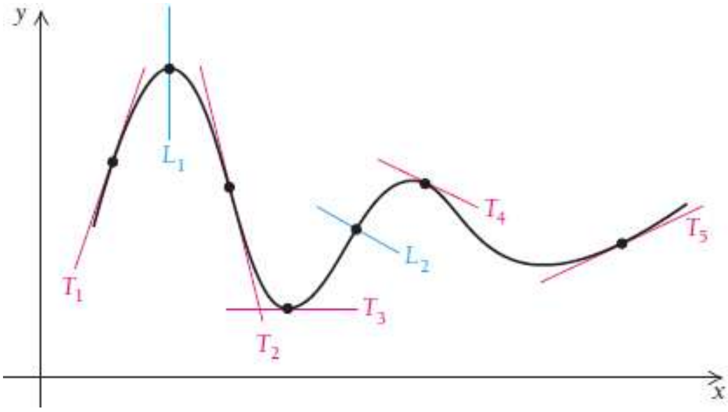
Egri chiziqqa o‘tkazilgan urinma ham u bilan bitta umumiy nuqtaga ega bo‘ladi. L to‘g‘ri chiziq egri chiziqqa P nuqtada o‘tkazilgan urinma bo‘ladi.



M to‘g‘ri chiziq P nuqtada egri chiziqqa o‘tkazilgan kesuvchi bo‘ladi.



L_1 va L_2 kesuvchilar, T_1, T_2, T_3, T_4, T_5 to‘g‘ri chiziqlar urinmalardir.



Agar egri chiziq silliq bo‘lsa (hech qanday uchlari bo‘lmasa), u holda egri chiziqning har bir nuqtasidan urinma o‘tkazish mumkin.

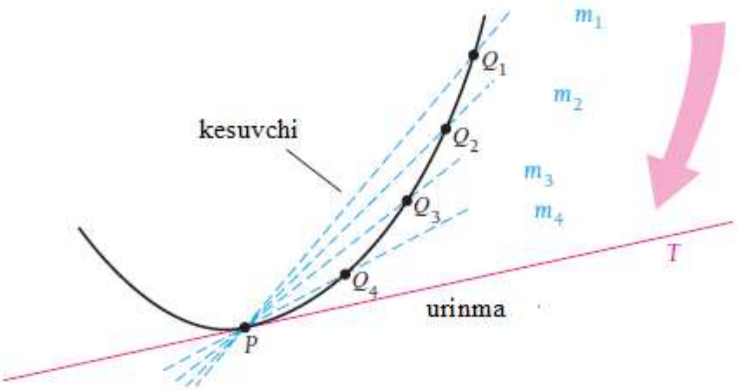
2.5.2. Limit yordamida hosilani hisoblash

Endi urinma ixtiyoriy egri chiziq uchun qanday ahamiyatga ega ekanini aniqlaymiz. Buning uchun limit tushunchasidan foydalanamiz.

Egri chiziqqa P nuqtada o‘tkazilgan urinmani topish uchun P va Q_1, Q_2, Q_3, Q_4 nuqtalardan o‘tkazilgan kesuvchilarni qaraymiz.

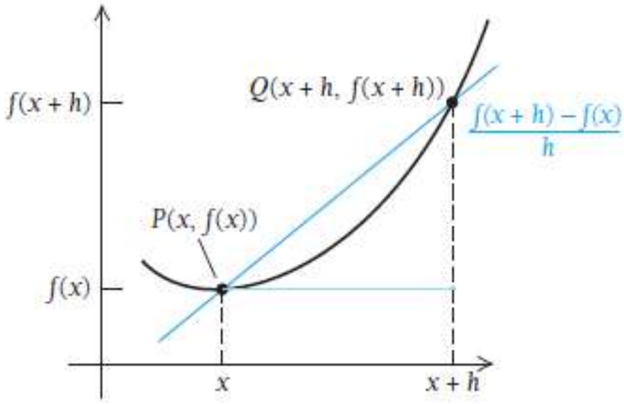
Q nuqtalar P ga yaqinlashgani sari kesuvchilar ham T chiziqqa yaqinlashib boradi. Har bir kesuvchi og‘ish koeffitsiyentiga ega. Bular m_1, m_2, m_3, m_4 koeffitsiyentlar bo‘lib, kesuvchilar T chiziqqa yaqinlashganda, ular m ga yaqinlashadi. Biz T chiziqni urinma sifatida aniqlaymiz, bu chiziq P nuqtadan o‘tadi va Q nuqtalar P ga yaqinlashganda ularning og‘ish koeffitsiyentlarining limiti m og‘ish koeffitsiyentiga intiladi.

Multiplikatsiya kadrlarini esga soling, daftar varaqlarini o‘tkazganingizda Q nuqtalar P qo‘zg‘almas nuqtaga yaqinlashib borib, barcha kesuvchilar urinma “ustiga” yotadi.

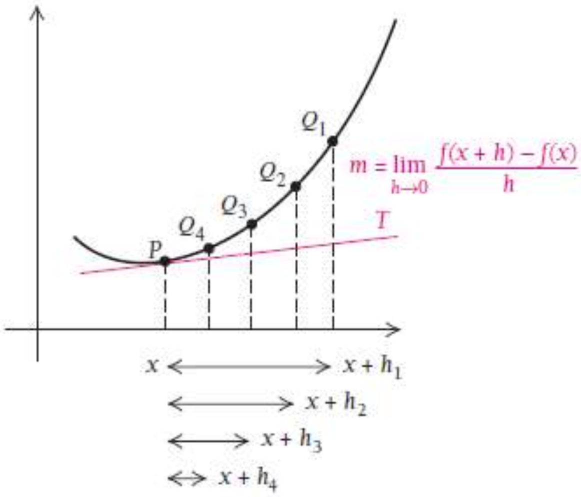


Limit m ni qanday hisoblaymiz?

Faraz qilaylik, P nuqtaning koordinatalari $(x, f(x))$. U holda Q nuqtaning absissasi $x+h$, ya'ni absissaning ustiga biror son qo'shilganiga teng yoki Q ning koordinatalari $(x+h, f(x+h))$ dan iborat bo'ladi.



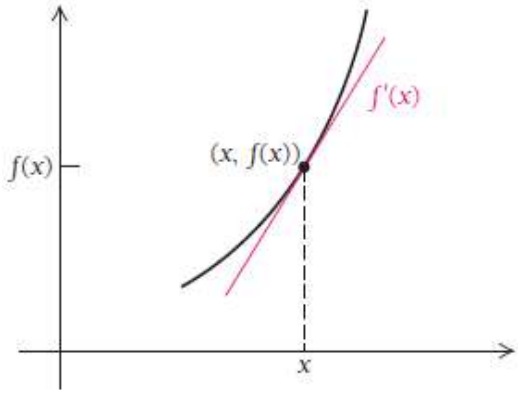
2.3 - bo'limdan bilamizki, PQ kesuvchining og'ish koeffitsiyenti $\dfrac{f(x+h)-f(x)}{h}$ ga teng. Bizning holatda Q nuqtalar P ga yaqinlashganda



$x+h$ qiymatlar ham x ga yaqinlashib boradi. Bunda h ham nolga yaqinlashadi, ya'ni $h \to 0$ deb olish mumkin. Shunda quyidagi tasdiq o'rinli bo'ladi:

Urinmaning og‘ish koeffitsiyenti $m=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ ga teng. Bu limit $f(x)$ ning x dagi oniy hosilasiga teng bo‘ladi.

Endi $f(x)$ funksiya hosilasiga ta’rif berish mumkin. Funksiyaning x dagi hosilasini m ning o‘rniga $f'(x)$ deb belgilaymiz. $f'(x)$ ni “**funksiyaning nuqtadagi hosilasi**” yoki “f shtrix x” deb o‘qish mumkin.



Ta’rif. $y=f(x)$ **funksiyaning nuqtadagi hosilasi** deb, quyidagi tenglik bilan aniqlanadigan $f'(x)$ funksiyaga aytiladi:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Agar $f'(x)$ mavjud bo‘lsa, u holda $f(x)$ funksiyani **nuqtada differensiallanuvchi** deyiladi.

Funksiya hosilasini hisoblash 3 ta qadamdan iborat:

1. Orttirmalar nisbatini yozamiz: $\frac{f(x+h)-f(x)}{h}$;

2. Bu nisbatni soddalashtiramiz;

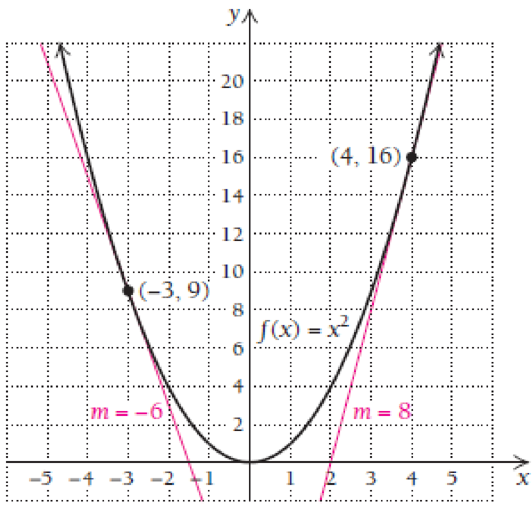
3. Soddalashgan ifodani $h\to 0$ da limitini hisoblaymiz.

1-misol. $f(x)=x^2$ funksiyaning $f'(x)$ hosilasini toping.

Yechilishi: ▶ 1) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{(x+h)^2-x^2}{h}$, $h\neq 0$;

1) $$\frac{f(x+h)-f(x)}{h}=\frac{(x^2+2xh+h^2)-x^2}{h}=\frac{2xh+h^2}{h}=\frac{h(2x+h)}{h}=2x+h;$$

2) $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}(2x+h)=2x,$$



Bulardan $f(x)=x^2$ funksiyaning hosilasi $f'(x)=2x$ ekanligi kelib chiqadi. $f'(x)=2x$ formuladan $f'(-3)=2\cdot(-3)=-6$ va $f'(4)=2\cdot 4=8$ ni hosil qilamiz. Bu tengliklar $x=-3$ nuqtada funksiyaga o‘tkazilgan urinmaning og‘ish koeffitsiyenti -6 ga, $x=4$ nuqtada funksiyaga o‘tkazilgan urinmaning og‘ish koeffitsiyenti 8 ga teng ekanligini bildiradi.

Shuningdek,

- Egri chiziqqa $(-3,9)$ nuqtada o‘tkazilgan urinmaning og‘ish koeffitsiyenti -6;
- Egri chiziqqa $(4,16)$ nuqtada o‘tkazilgan urinmaning og‘ish koeffitsiyenti 8;

● Funksiyaning $x=-3$ nuqtadagi hosilasi -6;

● Funksiyaning $x=4$ nuqtadagi hosilasi 8 deb aytishimiz mumkin. ◄

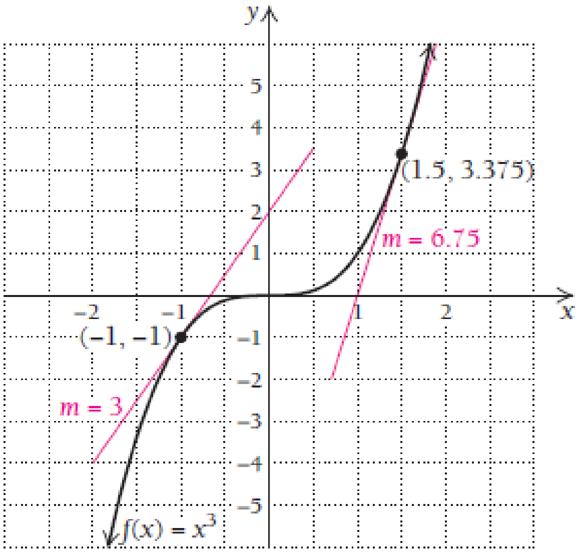
2-misol. $f(x)=x^3$ funksiyaning hosilasini toping.

Yechilishi: ► 1) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{(x+h)^3-x^3}{h}$, $h\neq 0$;

2) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}=\dfrac{3x^2h+3xh^2+h^3}{h}=3x^2+3xh+h^2$;

3) $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}(3x^2+3xh+h^2)=3x^2$.

Demak, $f'(x)=3x^2$.



$f'(x)=3x^2$ formuladan $f'(-1)=3\cdot(-1)^2=3$ va $f'(1.5)=3\cdot 1.5^2=6.75$ nuqtalardagi urinmalarning og‘ish koeffitsiyentlarini topishimiz mumkin.◄

1-vazifa. 1- va 2-misollardan foydalanib, $f(x)=x^3+x^2$ funksiya hosilasini toping, so'ngra $f'(-2)$ hamda $f'(4)$ nuqtalardagi hosila qiymatlarini aniqlang.

Orttirmalar nisbatini hisoblaganda ko'p uchraydigan xato funksiya orttirmasini $f(x+h)-f(x)$ deb yozish o'rniga $f(x)+h$ yozishdir:

$$f(x+h)-f(x)\neq f(x)+h.$$

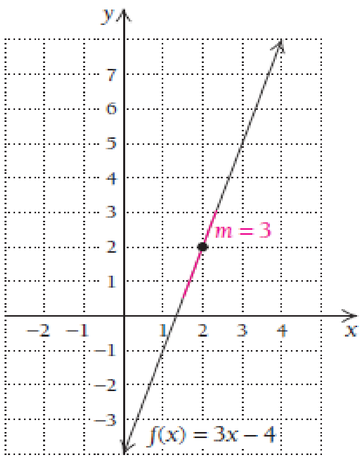
3-misol. $f(x)=3x-4$ funksiyaning hosilasini toping.

Yechilishi: ► 1) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{[3(x+h)-4]-[3x-4]}{h}$, $h\neq 0$;

2) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{3x+3h-4-3x+4}{h}=\dfrac{3h}{h}=3$;

3) $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}3=3$.

$f(x)=3x-4$ funksiyaning hosilasi $f'(x)=3$ ga teng. $f'(2)=3$, ya'ni



◄

3-misoldan ko'rinadiki, to'g'ri chiziqqa ixtiyoriy nuqtada o'tkazilgan urinma to'g'ri chiziqning o'zi bilan ustma-ust tushadi, ya'ni to'g'ri chiziq bilan urinmalarning og'ish burchaklari bir xilda 3 ga teng.

Xulosa: $f(x) = mx + b$ **funksiyaning hosilasi** $f'(x) = m$ **ga teng.**

2.5-bo'limda hosilani hisoblashning oson usulini o'rganamiz. Hozirda limit usulida hosilani hisoblash bizga hosilaning mazmun mohiyatini chuqurroq tushunishga yordam beradi.

4-misol. $y = \dfrac{2x}{3x+1}$ funksiyaning hosilasini ta'rif asosida hisoblang.

Yechilishi: ► Funksiyaning hosilasini hisoblash uchun

1) Orttirmalar nisbatini yozamiz: $\dfrac{f(x+h) - f(x)}{h}$;

$$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{2(x+h)}{3(x+h)+1} - \dfrac{2x}{3x+1}}{h} = \frac{6x^2 + 6xh + 2x + 2h - 6x^2 - 6xh - 2x}{(3(x+h)+1)(3x+1)h}$$

2) Bu nisbatni soddalashtiramiz:

$$\frac{f(x+h)-f(x)}{h} = \frac{2h}{(3(x+h)+1)(3x+1)h} = \frac{2}{(3(x+h)+1)(3x+1)}$$

3) Soddalashgan ifodani $h \to 0$ da limitini hisoblaymiz.

$$y' = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \frac{2}{(3x+1)^2}.$$ ◄

5-misol. $f(x) = \dfrac{1}{x}$ funksiya berilgan.

a) $f'(x)$ hosilani toping;

b) $f'(2)$ ni hisoblang;

c) Funksiya grafigiga $x = 2$ nuqtada o'tkazilgan urinma tenglamasini yozing.

Yechilishi: ► a) $f'(x)$ hosilani 3 ta qadam bilan topamiz:

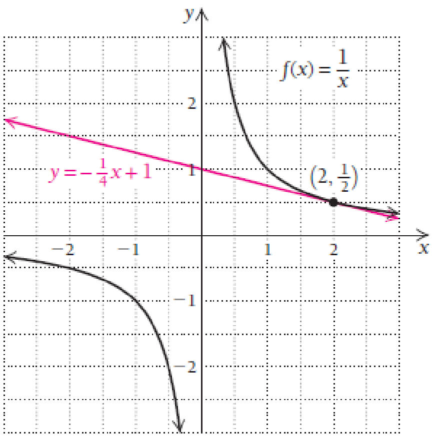
1) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}, \quad h\neq 0;$

2) $\dfrac{f(x+h)-f(x)}{h}=\dfrac{\dfrac{1}{x+h}\cdot\dfrac{x}{x}-\dfrac{1}{x}\cdot\dfrac{x+h}{x+h}}{h}=\dfrac{x-(x+h)}{h\cdot(x+h)\cdot x}=-\dfrac{1}{x(x+h)};$

3) $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\left(-\dfrac{1}{x(x+h)}\right)=-\dfrac{1}{x^2}.$

b) $f'(2)$ ni hisoblaymiz: $f'(2)=-\dfrac{1}{2^2}=-\dfrac{1}{4}$

c) Funksiya grafigiga $x=2$ nuqtada o‘tkazilgan urinma tenglamasini yozish uchun dastlab, $f(x)=\dfrac{1}{x}$ funksiyaga urinma o‘tkaziladigan nuqtani aniqlaymiz: $f(2)=\dfrac{1}{2}$, bu nuqta $\left(2,\dfrac{1}{2}\right)$ ekan. Og‘ish koeffitsiyenti esa $m=-\dfrac{1}{4}$ ga teng.



Urinma tenglamasiga asosan $y-y_1=m(x-x_1)$

$y-\dfrac{1}{2}=-\dfrac{1}{4}(x-2)$, $\qquad y-\dfrac{1}{2}=-\dfrac{1}{4}x+\dfrac{1}{2}$, $\qquad y=-\dfrac{1}{4}x+1$.

Shunday qilib, funksiya grafigiga $x = 2$ nuqtada o‘tkazilgan urinma tenglamasi $y = -\frac{1}{4}x + 1$ ko‘rinishda bo‘ladi.◄

2-vazifa. $f(x) = -\frac{2}{x}$ **funksiya hosilasini toping.**

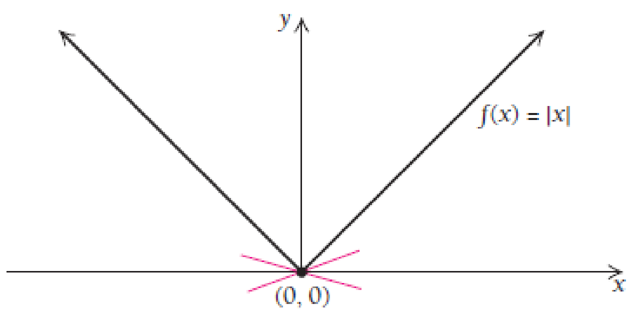
$f(x) = \frac{1}{x}$ funksiyaning $f(0)$ nuqtadagi qiymati mavjud emas. Shuning uchun bu funksiyaning $f'(0)$ hosilasini ham hisoblab bo‘lmaydi, mavjud emas.

Xulosa. Agar funksiya biror nuqtada aniqlanmagan bo‘lsa, bu funksiyaning shu nuqtada hosilasi mavjud bo‘lmaydi. Yoki agar funksiya biror nuqtada uzilishga ega bo‘lsa, funksiyaning uzilish nuqtasida hosilasi mavjud emas.

Shunday $f(x)$ funksiyalar ham borki, funksiya biror nuqtada aniqlangan, lekin bu nuqtada uning $f'(x)$ hosilasini hisoblab bo‘lmaydi.

6-misol. $f(x) = |x|$ funksiyaning $x = 0$ nuqtada hosilasini toping.

Yechilishi: ► $f(x) = |x|$ funksiya $x = 0$ nuqtada uzluksizlikning barcha shartlarini qanoatlantiradi, lekin bu nuqtada funksiyaning hosilasi mavjud emas. Bu nuqtada funksiyaga o‘tkazilgan urinma qanday?



Faraz qiling, (0, 0) nuqtada funksiyaga urinma o‘tkazmoqchimiz. Funksiya bu nuqtada uchli (silliq emas), bu nuqtada funksiyaga cheksiz ko‘p urinmalar o‘tadi, ularning og‘ish koeffitsiyentlari ham cheksiz bo‘lishi kerak. Keling $x=0$ nuqtada funksiya hosilasini topishga harakat qilamiz.

$$f(x)=|x|=\begin{cases} x, & \text{agar } x\geq 0 \\ -x, & \text{agar } x<0 \end{cases}$$

funksiya hosilasi

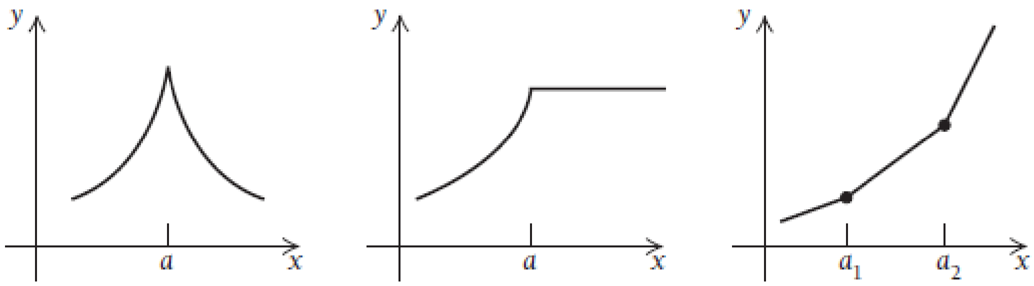
$$f'(x)=\begin{cases} 1, & \text{agar } x>0 \\ -1, & \text{agar } x<0 \end{cases}.$$

O‘ng va chap limitlar bir-biriga teng emas: $\lim_{x\to 0^+} f'(x) \neq \lim_{x\to 0^-} f'(x)$.

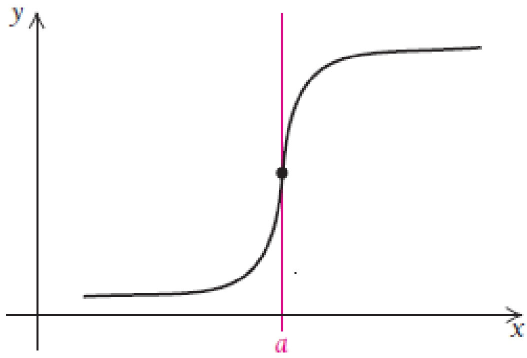
Shuning uchun $f(x)=|x|$ funksiyaning $x=0$ nuqtada $f'(0)$ hosilasi mavjud emas.◀

Xulosa. Agar funksiya grafigi “uchli” bo‘lsa, bu funksiyaning shu nuqtada hosilasi mavjud bo‘lmaydi.

Quyidagi chizmalarda funksiyada qanday “uchlar” bo‘lishi mumkinligi ko‘rsatilgan.



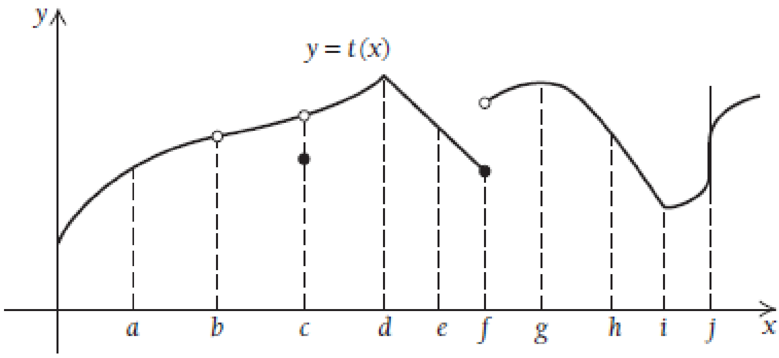
Xuddi shuningdek, agar funksiyaga uning biror nuqtasida vertikal urinma o‘tkazish mumkin bo‘lsa, funksiya bu nuqtada hosilaga ega emas. Misol uchun pastdagi rasmda $x = a$ nuqtada o‘tkazilgan urinma vertikal to‘g‘ri chiziqdan iborat. O‘tgan mavzulardan bilamizki, vertikal to‘g‘ri chiziqning og‘ish koeffitsiyentini aniqlashning imkoni yo‘q, shuning uchun bu nuqtada hosila ham mavjud emas.



Bilamizki, $f(x) = |x|$ funksiya I oraliqning har bir nuqtasida uzluksiz funksiya, biroq bu funksiya I oraliqning har bir nuqtasida differensiallanuvchi bo‘lmasligi mumkin ekan. Shundan kelib chiqib aytish mumkinki, funksiyaning uzluksizligi uni differensiallanuvchi bo‘lishini anglatmas ekan. Biroq teskari tasdiq o‘rinli ekan, ya’ni funkdsiya differensiallanuvchi bo‘lsa, u albatta uzluksiz bo‘lar ekan. Demak, agar $f'(a)$ hosila mavjud bo‘lsa, $f(x)$ funksiya $x = a$ nuqtada uzluksizdir. Misol uchun $f(x) = x^2$ funksiya $(-\infty, \infty)$ oraliqda differensiallanuvchi, shuning uchun butun sonlar o‘qida uzluksiz.Shuningdek, agar funksiya oraliqda differensiallanuvchi bo‘lsa, u nafaqat uzluksiz, silliq ham bo‘lar ekan, ya’ni funksiya grafigida “uchlari” bo‘lmas ekan.

3-vazifa. $f(x)=|x+7|$ **funksiya qaysi nuqtada hosilaga ega emas? Nima uchun?**

7-misol. Quyida grafigi berilgan $y=t(x)$ funksiya qaysi nuqtalarda hosilaga ega emas?



Yechilishi: ►Funksiya hosilaga ega bo‘lmaydi, agarda funksiya bu nuqtada

1) uzilishga ega bo‘lsa,

2) uchli (silliq emas) bo‘lsa,

3) vertikal urinma o‘tkazish mumkin bo‘lsa.

Shunga ko‘ra, $y=t(x)$ funksiya $x=b$, $x=c$, $x=f$ nuqtalarda uzilishga ega bo‘lganligi sababli hosilaga ega bo‘lmaydi. $x=d$ va $x=i$ nuqtalarda uchli bo‘lganligi uchun hosilaga ega emas va $x=j$ nuqtada vertikal urinma o‘tkazib bo‘lganligi sababli hosilaga ega emas.

Funksiya $x=a$, $x=e$, $x=g$ va $x=h$ nuqtalarda hosilaga ega bo‘ladi.◄

2.5 bo‘limda nimalarni o‘rgandik?

● **Urinma** – bu shunday to‘g‘ri chiziqki, u funksiya grafigiga berilgan nuqtada urinadi. Berilgan nuqtaga **urinish nuqtasi** deyiladi.

● $f(x)$ **funksiyaning hosilasi** $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ formuladan topiladi.

● $x=a$ nuqtada $y=f(x)$ funksiya grafigiga o‘tkazilgan urinmaning **og‘ish koeffitsiyenti** bu funksiyaning $x=a$ nuqtadagi **hosilasiga teng**.

● Urinmalarning og‘ish koeffitsiyenti **oniy hosila** sifatida tavsiflanadi.

● $y=f(x)$ funksiyaga $x=a$ nuqtada o‘tkazilgan urinma tenglamasi $y-f(a)=f'(a)(x-a)$ ga teng.

● Agar funksiya $x=a$ nuqtada differensiallanuvchi bo‘lsa, u holda funksiya $x=a$ nuqtada uzluksiz bo‘ladi, ya’ni differensiallanuvchanlikdan uzluksizlik kelib chiqadi.

● Funksiyaning $x=a$ nuqtada uzluksiz bo‘lishi uning bu nuqtada differensiallanuvchi bo‘lishini bildirmaydi. Buni $f(x)=|x|$ funksiya misolida ko‘rdik. Funksiyaning $x=a$ nuqtada uzluksiz ekanligi uning shu nuqtada differensiallanuvchi bo‘lishi uchun yetarli emas.

● Funksiya $x=a$ nuqtada hosilaga ega bo‘lmaydi, agarda funksiya

1) $x=a$ nuqtada uzilishga ega bo‘lsa,

2) $x=a$ nuqtada uchli (silliq emas) bo‘lsa,

3) $x=a$ nuqtada vertikal urinma o‘tkazish mumkin bo‘lsa.

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-15 misollarda

a) funksiya grafigini yasang;

b) funksiya grafigiga $x=-2$, $x=0$, $x=1$ **nuqtalarda urinmalar o‘tkazing;**

c) $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ **formuladan funksiya hosilasini toping;**

d) $f'(-2)$, $f'(0)$, $f'(1)$ **nuqtalardagi hosilalarni hisoblang. Bu hosilalar (b) shartdagi urinmalarning og‘ish koeffitsiyentiga teng.**

1. $f(x)=\dfrac{3x^2}{2}$; **2.** $f(x)=\dfrac{x^2}{2}$; **3.** $f(x)=-3x^2$;

4. $f(x)=-5x^2$; **5.** $f(x)=x^3$; **6.** $f(x)=-x^3$;

7. $g(x)=2x-8$; **8.** $g(x)=-2x-7$; **9.** $g(x)=\dfrac{x}{2}-9$;

10. $g(x)=\dfrac{3x}{4}-2$; **11.** $y=x^2+3x$; **12.** $y=x^2-3x$;

13. $y=2x^2+3x-3$; **14.** $y=5x^2-2x+7$; **15.** $y=-\dfrac{1}{x}$.

16-21 misollar funksiya grafigiga o‘tkazilgan urinma tenglamasini tuzish haqida:

16. $f(x) = 3x^2$ funksiya grafigiga quyidagi nuqtalarda o‘tkazilgan urinmalar tenglamalarini tuzing:

a) (3, 9) nuqtada;

b) (-1, 1) nuqtada; c) (10, 100) nuqtada.

17. $f(x) = x^3$ funksiya grafigiga quyidagi nuqtalarda o‘tkazilgan urinmalar tenglamalarini tuzing:

a) (-2, -8) nuqtada;

b) (0, 0) nuqtada;

c) (4, 64) nuqtada.

18. $f(x) = \frac{2}{x}$ funksiya grafigiga quyidagi nuqtalarda o‘tkazilgan urinmalar tenglamalarini tuzing:

a) (1, 2) nuqtada;

b) (-1, -2) nuqtada;

c) (100, 0.2) nuqtada.

19. $f(x) = -\frac{1}{x}$ funksiya grafigiga quyidagi nuqtalarda o‘tkazilgan urinma tenglamasini tuzing:

a) (2, -1/2) nuqtada;

b) (-1, 1) nuqtada;

c) (-5, 1/5) nuqtada.

20. $f(x)=4-x^2$ funksiya grafigiga quyidagi nuqtalarda o‘tkazilgan urinmalar tenglamalarini tuzing:

a) (0, 4) nuqtada; b) (-1, 3 nuqtada; c) (5, -21) nuqtada.

21. $f(x)=x^2-2x$ funksiya grafigiga quyidagi nuqtalarda o‘tkazilgan urinmalar tenglamalarini tuzing:

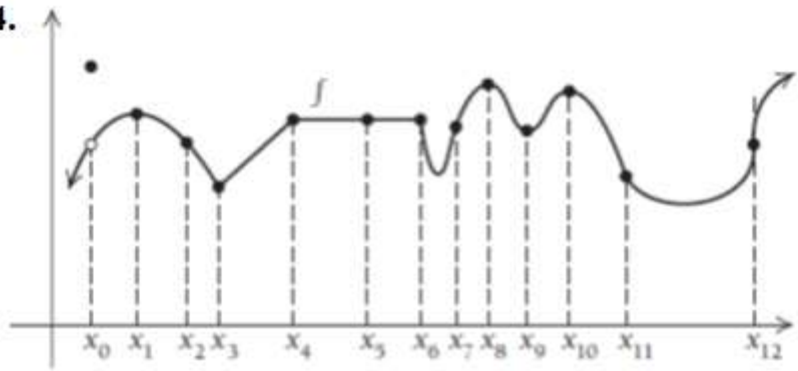
a) (-2, 8) nuqtada; b) (1, -1) nuqtada; c) (4, 8) nuqtada.

22. $f(x)=mx+b$ funksiya uchun $f'(x)$ ni hisoblang.

23. $f(x)=ax^2+bx$ funksiya uchun $f'(x)$ ni hisoblang.

24-27- misollarda funksiya hosilaga ega bo‘lmaydigan nuqtalarni ko‘rsating:

24.



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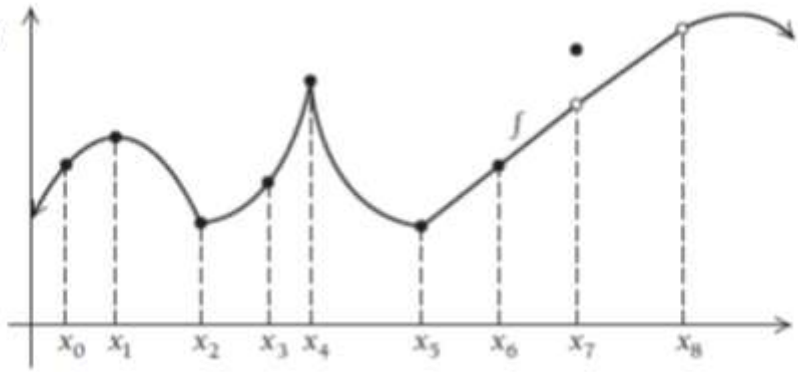

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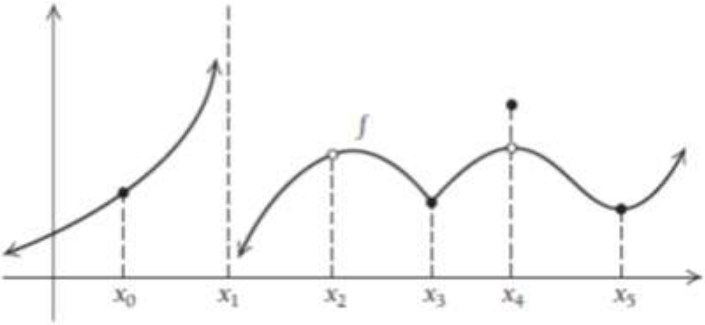

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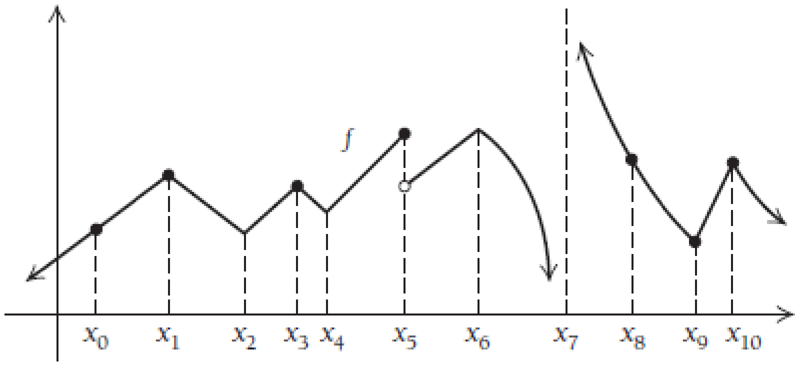


28. $x=3$ nuqtada uzluksiz, lekin differensiallanuvchi bo‘lmagan $f(x)$ funksiyani yozing.

29. Grafigiga $x=5$ nuqtada gorizontal urinma o‘tkazilgan funksiya grafigini yasang.

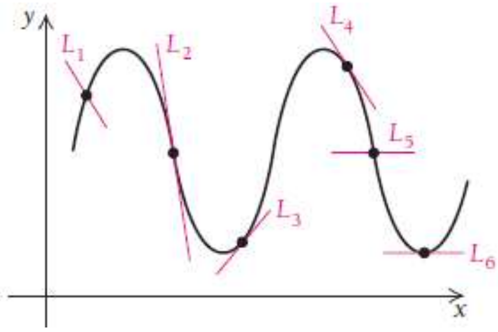
30. Grafigiga $x=0$, $x=2$, $x=4$ nuqtalarda gorizontal urinmalar o‘tkazilgan va differensiallanuvchi funksiya grafigini yasang.

31. Grafigiga $x=2$, $x=5$ nuqtalarda gorizontal urinmalar o‘tkazilgan va $x=3$ nuqtada uzluksiz, lekin differensiallanuvchi bo‘lmagan funksiya grafigini yasang.

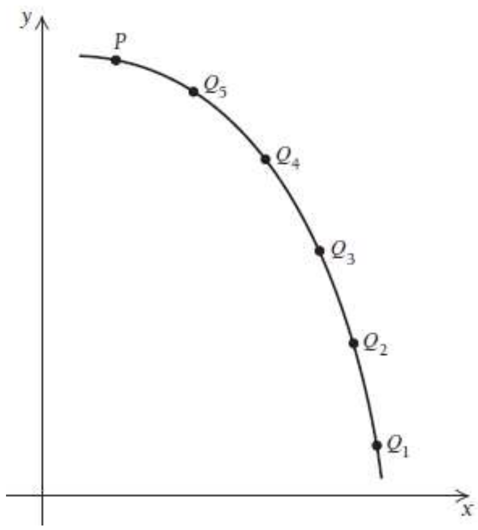
32. Grafigi $x=1$ nuqtada silliq, lekin differensiallanuvchi bo‘lmagan funksiya grafigini yasang.

33. Grafigi x ning barcha qiymatlarida silliq, lekin $x=-1$ va $x=2$ nuqtalarda differensiallanuvchi bo‘lmagan funksiya grafigini yasang.

34. Diagrammadagi qaysi chiziqlar urinma bo‘ladi? Javobingizni tushuntiring.



35. Quyidagi diagrammada P nuqtadan Q nuqtalarga kesuvchilarni o‘tkazing. So‘ngra P nuqtadan grafikka urinma o‘tkazing. Qanday hodisa yuz berishini tushuntiring.



36-44 misollarda limitdan foydalanib, funksiya $f'(x)$ hosilasini toping:

36. $f(x)=x^4$; **37.** $f(x)=\dfrac{1}{1-x}$; **38.** $f(x)=x^5$

39. $f(x)=\dfrac{1}{x^2}$; **40.** $f(x)=\sqrt{x}$; **41.** $f(x)=\sqrt{2x+1}$

42. $f(x)=\dfrac{1}{\sqrt{x}}$; **43.** $f(x)=\sqrt{x}$; **44.** $f(x)=\sqrt{x^3}$.

45. $f(x)=\dfrac{x^2-9}{x+3}$ funksiya

a) x ning qanday qiymatlarida differensiallanuvchi emas?

b) $f'(4)$ hosilasini hisoblang.

46. $f(x)=\dfrac{x^2+x}{2x}$ funksiya

a) x ning qanday qiymatlarida differensiallanuvchi emas?

b) $f'(3)$ hosilani hisoblashning eng oson yo‘lini ko‘rsating.

47. $f(x)=|x-3|+2$ funksiya

a) x ning qanday qiymatlarida differensiallanuvchi emas?

b) $f'(0)$, $f'(1)$, $f'(4)$, $f'(10)$ hosilalarni hisoblashning oson yo‘lini ko‘rsating.

48. $f(x)=2|x+5|$ funksiya

a) x ning qanday qiymatlarida differensiallanuvchi emas?

b) $f'(-10)$, $f'(-7)$, $f'(-2)$, $f'(0)$ hosilalarni hisoblashning oson yo‘lini ko‘rsating.

49. $f(x)=\dfrac{x^2+4x+3}{x+1}$ funksiya berilgan bo‘lsin. Talaba berilgan funksiyani soddalashtirish mumkinligini bildi va soddalashtirdi:

$$f(x)=\frac{x^2+4x+3}{x+1}=\frac{(x+1)(x+3)}{x+1}=x+3.$$

Shunday qilib, $y=x+3$ to‘g‘ri chiziqning og‘ish koeffitsiyenti 1 ga teng deb $f'(-2)=1$, $f'(-1)=1$, $f'(0)=1$ va $f'(1)=1$ deb xulosa qildi. Talaba qayerda xatoga yo‘l qo‘ydi.

50. Talaba $f(x)=\sqrt[3]{x}$ funksiya grafigini chizdi va unga grafik x ning barcha qiymatlarida silliq va uzluksizdek ko‘rindi. Shunda talaba berilgan funksiya barcha x larda differentsiallanuvchan degan noto‘g‘ri xulosaga keldi. U qanday xato qildi? Differentsiallanuvchanlikning to‘g‘ri tasdig‘ini ayting.

51. $f(x)=\begin{cases} x^2+1, & \text{agar } x\le 2 \\ 2x+1, & \text{agar } x>2 \end{cases}$ bo‘lakli aniqlangan funksiyaning

a) $x=2$ nuqtada funksiya uzluksiz ekanligini tekshiring;

b) $x=2$ nuqtada funksiya differentsiallanuvchimi?

52. $f(x)=\begin{cases} x^3, & \text{agar } x\le 1 \\ 3x-2, & \text{agar } x>1 \end{cases}$ bo‘lakli aniqlangan funksiya uchun

a) $x=1$ nuqtada funksiya uzluksiz ekanligini tekshiring;

b) $x=1$ nuqtada funksiya differentsiallanuvchimi?

53. $f(x)=\begin{cases} 2x^2-x, & \text{agar } x\le 3 \\ mx+b, & \text{agar } x>3 \end{cases}$ bo‘lakli aniqlangan funksiya uchun

$x=3$ nuqtada funksiya differentsiallanuvchi bo‘lsa, m va b ning qiymatini toping.

2.6. Differensiallash qoidalari: yig‘indi va ayirmaning hosilasi

2.6.1. Leybnits belgilashi

Faraz qilaylik, y funksiya x ning funksiyasi bo‘lsin. “y ning x bo‘yicha hosilasi”ning keng tarqalgan belgilanishi: $\frac{dy}{dx}$

Hosilaning bunday belgilash nemis matematigi Leybnits tomonidan kiritilgan. Bu belgilashga asosan quyidagini yozishimiz mumkin:

Agar $y = f(x)$ funksiya berilgan bo‘lsa, u holda y ning x bo‘yicha hosilasi $\frac{dy}{dx} = f'(x)$ ga teng.

Amaliyotda biz asosan y' yoki $f'(x)$ belgilashlardan foydalanamiz.

Agar hosilani biror nuqtada aniqlamoqchi bo‘lsak, $\left.\frac{dy}{dx}\right|_{x=2} = f'(2)$ yozuvni ishlatamiz. $x = 2$ vertikal chiziq hosilaning qaysi nuqtada aniqlanganini bildiradi, shunday qilib, yuqoridagi ifoda “$x = 2$ da aniqlangan y ning x bo‘yicha hosilasi $f'(2)$ qiymatga teng” deb o‘qiladi.

Hosilani boshqacha $\frac{d}{dx} f(x)$ deb ham yozish mumkin. Bu yozuv $\frac{dy}{dx}$ belgilash bilan bir xil ma’noni beradi, faqat boshqacha usulda yozilgan. Agar hosila olish belgisi $\frac{d}{dx} f(x)$ funksiyaning oldiga yozib qo‘yilsa, bu hosila olishga berilgan “**buyruq**” sifatida qaraladi.

Misol. $\frac{d}{dx}x^2 = 2x$, $\frac{d}{dx}x^3 = 3x^2$, $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$, …

2.6.2. Hosilani hisoblash qoidalari:

2.4-bo‘limda ba’zi bir funksiyalar hosilalarini ta’rifga ko‘ra hisoblagan edik. Endi shularni umumlashtirib, hosilalar jadvalini tuzib boramiz.

Darajali funksiyaning hosilasini topish:

$$\frac{d}{dx}x^k = k \cdot x^{k-1}$$

1. Daraja ko’rsatkichini funksiya oldiga tushiramiz.

2. Daraja ko’rsatkichidan 1 ni ayiramiz.

Funksiya	Hosila
x^2	$2x^1$
x^3	$3x^2$
x^4	$4x^3$
$\frac{1}{x} = x^{-1}$	$-1 \cdot x^{-2} = \frac{-1}{x^2}$
$\frac{1}{x^2} = x^{-2}$	$-2 \cdot x^{-3} = \frac{-2}{x^3}$
$\sqrt{x} = x^{1/2}$	$\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

1-teorema: Agar har qanday haqiqiy k son uchun $y = x^k$ funksiya berilgan bo‘lsa, u holda bu funksiyaning hosilasi $\frac{d}{dx}x^k = k \cdot x^{k-1}$ ga teng bo‘ladi.

Isboti: ►Aytaylik, $f(x) = x^k$ berilgan bo‘lsin. Ta’rifga ko‘ra, orttirmalar nisbatini topib, argument orttirmasi nolga intilganda limitga o‘tamiz. $f(x+h)=(x+h)^k$ ifoda berilgan funksiyaning kengaytirilgan ko‘rinishi bo‘lib, uning yoyilmasi quyidagi jadvalda keltirilgan:

$$(x+h)^1 = x + h,$$
$$(x+h)^2 = x^2 + 2xh + h^2,$$
$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3,$$
$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4.$$

Jadbalda bo‘yalgan qismdagi hadlarda h ning kvadrati va undan yuqori darajalari qatnashgan:

$$(x+h)^k = x^k + kx^{k-1}h + (\text{bo'yalgan qism}).$$

Orttirmalar nisbatini aniqlaymiz:

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^k - x^k}{h} = \frac{x^k + kx^{k-1}h + (\text{bo'yalgan qism}) - x^k}{h} =$$

$$= \frac{kx^{k-1}h + (\text{bo'yalgan qism})}{h} = \frac{h\left[kx^{k-1} + (h \text{ ni chqargandan keyingi bo'yalgan qism})\right]}{h} =$$

$$= kx^{k-1} + (\text{keltirilgan bo'yalgan qism}).$$

$h \to 0$ da limitga o‘tamiz:

$$f'(x) = \lim_{h\to 0} kx^{k-1} + (\text{keltirilgan bo'yalgan qism}) = kx^{k-1}.$$

Bundan ko‘rinadiki, $\left[(x+h)^k\right]' = kx^{k-1}$ o‘rinli. ◄

Biz darajali funksiya hosilasini k musbat butun son bo‘lgan hol uchun isbotladik, bu qoida k ning ixtiyoriy haqiqiy qiymati uchun ham o‘rinli bo‘ladi.

1-misol. Quyidagi funksiyalarning hosilalarini toping:

1) $y = x^7$; 2) $y = x$; 3) $y = x^{-8}$.

Yechilishi: ► 1) $\frac{d}{dx}x^7 = 7x^6$ yoki $(x^7)' = 7x^6$;

2) $\frac{d}{dx}x = x^0 = 1$ yoki $(x)' = x^0 = 1$;

3) $\frac{d}{dx}x^{-8} = -8x^{-8-1} = -8x^{-9}$ yoki $(x^{-8})' = -8x^{-9}$. ◄

2-misol. Quyidagi funksiyalarning hosilalarini toping:

1) $y = \sqrt[7]{x}$; 2) $y = x^{0.6}$; 3) $y = x^{-0.4}$.

Yechilishi: ►

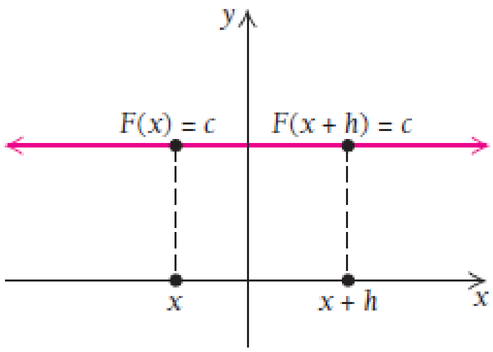
1) $\frac{d}{dx}\sqrt[7]{x} = \frac{d}{dx}x^{\frac{1}{7}} = \frac{1}{7}x^{\frac{1}{7}-1} = \frac{1}{7}x^{-\frac{6}{7}} = \frac{1}{7\sqrt[7]{x^6}}$ yoki $(x^{\frac{1}{7}})' = \frac{1}{7\sqrt[7]{x^6}}$;

2) $\frac{d}{dx}x^{0.6} = 0.6x^{0.6-1} = 0.6x^{-0.4}$ yoki $(x^{0.6})' = 0.6x^{-0.4}$;

3) $\frac{d}{dx}x^{-0.4} = -0.4x^{-0.4-1} = -0.4x^{-1.4}$ yoki $(x^{-0.4})' = -0.4x^{-1.4}$. ◄

O‘zgarmas funksiyaning hosilasini topish:

O‘zgarmas funksiya $F(x) = c$ ni grafigida og‘ish burchagi 0 ga teng bo‘lgan shakldan iborat.



2-teorema: O‘zgarmas funksiyaning hosilasi nolga teng: $\frac{d}{dx}c=0$ yoki

$$c'=0.$$

Isboti: ► $\lim\limits_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim\limits_{h\to 0}\frac{c-c}{h}=\frac{0}{h}=0$, demak, $F'(x)=0$.◄

Funksiya bilan o‘zgarmasning ko‘paytmasi hosilasini topish:

3-misol. Quyidagi funksiyalarning hosilalarini toping:

1) $y=5x^7$; 2) $y=-4x^3$ 3) $y=6x^{-5}$.

Yechilishi: ► 1) $y'=\left(5x^7\right)'=35x^6$;

2) $y'=\left(-4x^3\right)'=-12x^2$;

3) $y'=\left(6x^{-5}\right)'=-30x^{-6}$. ◄

Ayrim funksiyalarning o‘zgarmasga ko‘paytmasidan olingan hosilalar quyidagi jadvalda keltirilgan:

Funksiya	Hosila
$5x^2$	$10x$
$3x^{-1}$	$-3x^{-2}$
$\frac{3}{2}x^2$	$3x$
$1\cdot x^3$	$3x^2$

3-teorema: Funksiya bilan o‘zgarmasning ko‘paytmasidan hosila olishda o‘zgarmas sonni hosila belgisidan tashqariga chiqarish mumkin:

$$\frac{d}{dx}[c\cdot f(x)]=c\cdot\frac{d}{dx}f(x) \quad \text{yoki} \quad [c\cdot f(x)]'=c\cdot f'(x).$$

Isboti: ▶

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{c\cdot f(x+h)-c\cdot f(x)}{h}=\lim_{h\to 0}\frac{c[f(x+h)-f(x)]}{h}=$$

$$=c\cdot\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]=c\cdot f'(x)$$ ◀

4-misol. Quyidagi funksiyalarning hosilalarini toping:

1) $y=\frac{3}{4}x^5$; 2) $y=-4x$; 3) $y=\frac{1}{6x^3}$.

Yechilishi: ▶ 1) $y'=\left(\frac{3}{4}x^5\right)'=\frac{3}{4}\cdot 5x^4=\frac{15}{4}x^4$;

2) $y'=(-4x)'=-4$;

3) $y'=\left(\frac{1}{6x^3}\right)'=\frac{1}{6}\cdot\left(x^{-3}\right)'=\frac{1}{6}\cdot(-3)x^{-4}=-\frac{1}{2x^4}$. ◀

5-misol. Tibbiyot masalasi (O‘sma hajmi). Shar shaklidagi o‘sma (shish) hajmi quyidagicha approksimatsiyalanishi mumkin, bunda r – o‘sma radiusi, (sm): $V(r)=\frac{4}{3}\pi r^3$.

a) Radiusga nisbatan hajm hosilasini toping.

b) $r=1.2$ santimetr bo‘lganda hajm hosilasini toping.

Yechilishi: ▶ a) $V'(r)=\left(\frac{4}{3}\pi r^3\right)'=4\pi r^2$;

b) $V'(1.2)=4\pi(1.2)^2=5.76\pi\approx 18\frac{\text{sm}^3}{\text{sm}}=18\ \text{sm}^2$

Ya’ni $r=1.2$ santimetr bo‘lgan o‘smaning radiusi har 1 sm ga kattalashganda uning hajmi 18 sm^3 ga oshib boradi. ◀

1-vazifa. Quyidagi funksiyalar hosilasini toping:

a) $y = 12x^9$; b) $y = \pi \cdot x^3$; v) $y = \dfrac{2}{7x^5}$.

2.6.3. Yig‘indi va ayirmaning hosilasini topish:

4-teorema: 1) Yig‘indining hosilasi hosilalar yig‘indisiga teng:

$$[f(x)+g(x)]' = f'(x)+g'(x).$$

2) Ayirmaning hosilasi hosilalar ayirmasiga teng:

$$[f(x)-g(x)]' = f'(x)-g'(x).$$

Isboti: ►

1) $$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} =$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right] = f'(x)+g'(x).$$

Ya’ni $[f(x)+g(x)]' = f'(x)+g'(x).$

2) $[f(x)-g(x)]' = [f(x)+(-1)g(x)]' = f'(x)+(-1)g'(x) = f'(x)-g'(x).$ ◄

● Xulosa qilib aytish mumkinki, har qanday funksiyalarning yig‘indi (ayirma) laridan tashkil topgan ifodalarning hosilasi har bir had hosilalari yig‘indisi (ayirmasi)ga teng.

6-misol. Hosilalarini hisoblang:

a) $y = \left(3x - 4\sqrt[3]{x} + 2\right)$; b) $y = \left(x^2 - \dfrac{1}{x^3} + 5\sqrt{x}\right)$.

Yechilishi: ▶ a) $y' = \left(3x - 4\sqrt[3]{x} + 2\right)' = (3x)' - \left(4\sqrt[3]{x}\right)' + 2' = 3 - \dfrac{4}{3\sqrt[3]{x^2}}$;

$$\text{b) } y' = \left(x^2 - \frac{1}{x^3} + 5\sqrt{x}\right)' = 2x + \frac{3}{x^4} + \frac{5}{2\sqrt{x}}. \blacktriangleleft$$

2-vazifa. Quyidagi funksiyalar hosilasini toping:

a) $y = \left(4x^2 - \dfrac{3}{\sqrt{x}} + 4\right)$;

b) $y = \left(x^3 - 4\sqrt[4]{x^3} + 2\right)$;

v) $y = \left(3x^5 - \dfrac{5}{x^3} - 2\right)$.

2.6.4. Urinmaning og‘ish burchagi

Egri chiziqqa o‘tkazilgan urinmaning og‘ish burchagi ma’lum bo‘lsa, shu urinma o‘tkazilgan nuqtani aniqlay olamizmi?

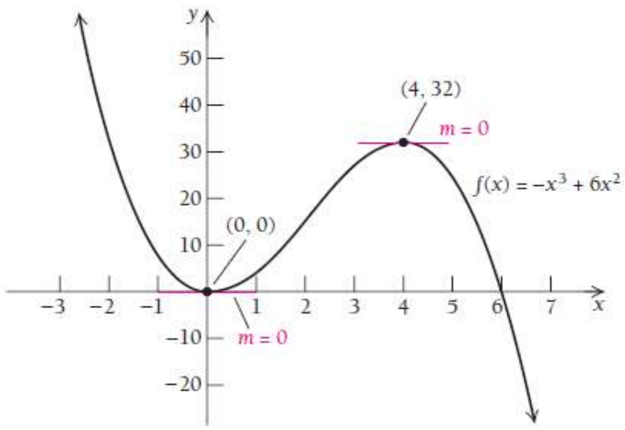
7-misol. $y = -x^3 + 6x^2$ funksiya grafigiga o‘tkazilgan urinmasi gorizontal bo‘ladigan nuqtani toping.

Yechilishi: ▶ Bilamizki, funksiyaning hosilasi urinmaning og‘ish burchagiga teng, ya’ni $y' = tgx_0$. Agar $y' = tgx_0 = 0$ bo‘lsa, shu x_0 nuqtada o‘tkazilgan urinma gorizontal bo‘ladi. Shuning uchun

$$y' = \left(-x^3 + 6x^2\right)' = -3x^2 + 12x = 0$$

$$-3x(x-4) = 0$$

$$x = 0 \quad \text{va} \quad x = 4$$



$$y(0) = -0^3 + 6\cdot 0^2 = 0 \quad \text{va} \quad y(4) = -4^3 + 6\cdot 4^2 = -64 + 96 = 32\,.$$

Demak, 2 ta nuqtada o‘tkazilgan urinma gorizontal bo‘lar ekan, bular $(0;0)$ va $(4;32)$ nuqtalardir. ◄

8-misol. $y = -x^3 + 6x^2$ funksiya grafigiga o‘tkazilgan urinmasi 9 ga teng bo‘ladigan nuqtani toping.

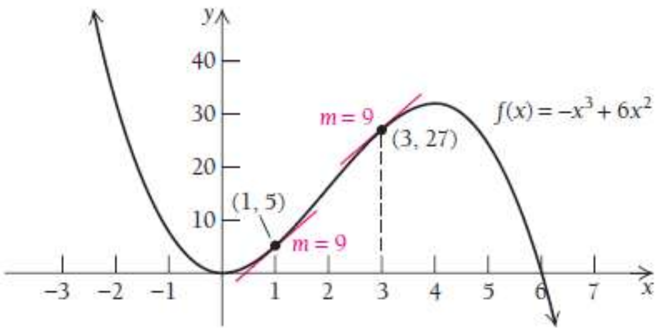
Yechilishi: ► $y' = 9$ bo‘lishi kerak. Shuning uchun

$$y' = \left(-x^3 + 6x^2\right)' = -3x^2 + 12x = 9$$

$$3x^2 - 12x + 9 = 0 \quad \longrightarrow \quad x^2 - 4x + 3 = 0$$

$$(x-1)(x-3) = 0 \quad \longrightarrow \quad x = 1 \;\; \text{va} \;\; x = 3.$$

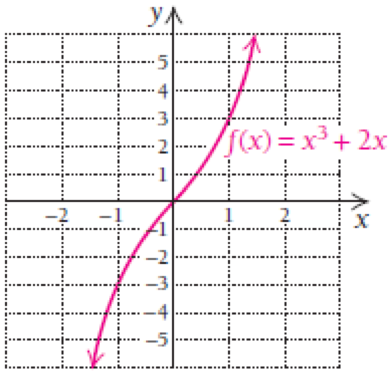
$$y(1) = -1^3 + 6\cdot 1^2 = 5 \quad \text{va} \quad y(3) = -3^3 + 6\cdot 3^2 = -27 + 54 = 27\,.$$



Demak, 2 ta nuqtada o‘tkazilgan urinmaning burchak koeffitsiyenti 9 ga teng bo‘lar ekan, bular $(1;5)$ va $(3;27)$ nuqtalardir.◄

2.6.5. Hosila yordamida funksiyaning o‘sish, kamayishini aniqlash

Ba’zi funksiyalar yoki faqat o‘suvchi yoki faqat kamayuvchi bo‘ladi. Masalan, $f(x)=x^3+2x$ funksiya grafigi butun son o‘qida qat’uy o‘suvchi bo‘lib, uning barcha urinmalari musbat burchak koeffitsiyentiga ega. **Ushbu tasdiqni isbotlash uchun hosiladan qanday foydalanamiz?**

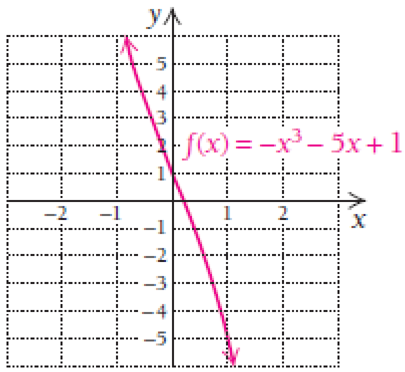


$f(x)=x^3+2x$ funksiyaning hosilasi x ning barcha qiymatlarida musbat bo‘ladi, chunki $f'(x)=3x^2+2$. Noma’lum x ni aniqlash uchun bu hosilani manfiy qiymatga tenglab, yechib bo‘lmaydi (urinib ko‘ring ☺☺☺). Hosila x ning kvadrati qatnashgani va qolgan sonlar ham musbat bo‘lgani uchun berilgan funksiya faqat o‘sadi deb xulosa qilamiz.

9-misol. $f(x)=-x^3-5x+1$ funksiya o‘suvchimi yoki kamayuvchimi?

Yechilishi: ►

Diagrammaga qarab xulosa qiladigan bo‘lsak, funksiya faqat kamayuvchiga o‘xshaydi.



Biz grafikning kichik bir bo‘lagini ko‘rib turibmiz xolos. Biz ko‘ra olmaydigan boshqa bir bo‘lagida balki funksiya o‘suvchidir? Diagrammaning o‘zigina yetarli emas ekan. Hosiladan foydalanib ko‘raylik-chi:

$$f'(x) = -3x^2 - 5 \ .$$

x^2 had faqat 0 yoki musbat bo‘ladi, $-3x^2$ esa faqat manfiy yoki 0 bo‘ladi. $-3x^2$ dan 5 ni ayirsak, yana manfiy qiymat hosil bo‘ladi. Xulosa qilish mumkinki, x ning barcha qiymatlarida hosila manfiy ekan. Bundan ko‘rinadiki, funksiyaga ixtiyoriy nuqtada o‘tkazilgan urinmaning burchak koeffitsiyenti manfiy va faqat kamayuvchi bo‘ladi. Shunga ko‘ra, funksiya grafigi faqat kamayuvchi ekani kelib chiqadi.◄

Funksiya grafigini chizish bilan uning o‘suvchi yoki kamayuvchi ekanini aniqlash noto‘g‘ri natijaga olib kelishi mumkin ekan. Misol uchun, $f(x) = x^3 - x^2$ funksiya faqat o‘suvchi bo‘lib ko‘rinadi. Biroq bu funksiyaning shunday bir kichik oralig‘i borki, funksiya bu oraliqda kamayadi.

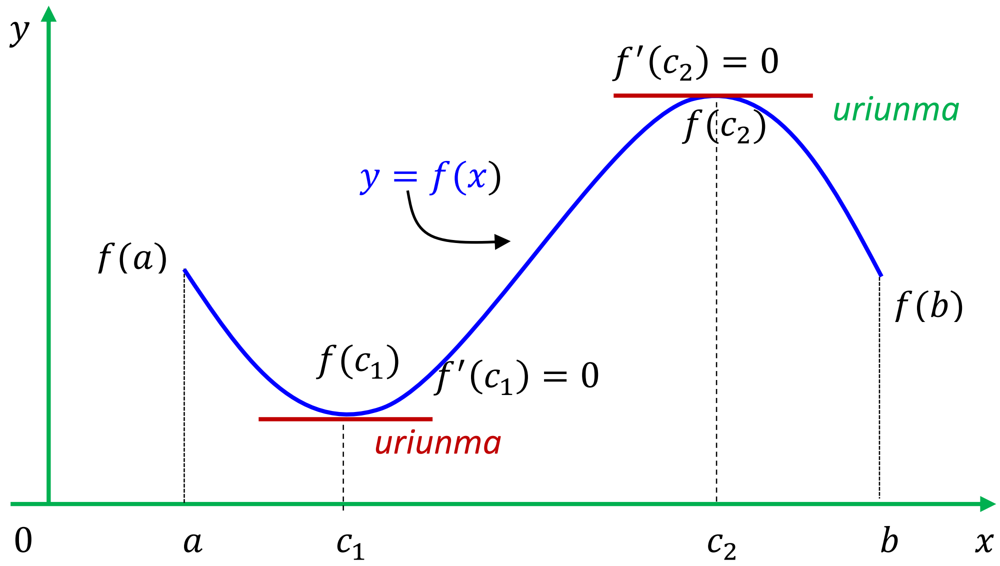
2.6.6. O‘rta qiymat haqidagi teoremalar

(Roll, Lagranj va Koshi teoremalari)

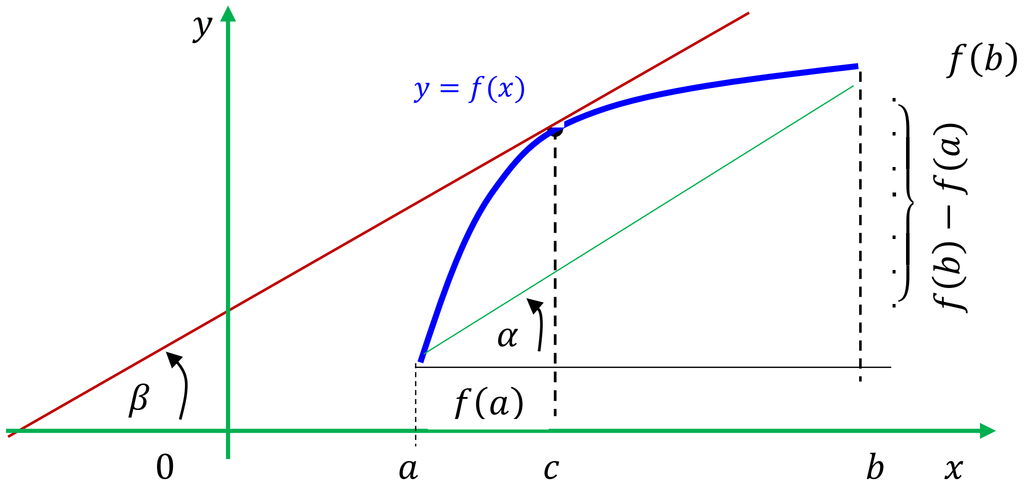
Roll teoremasi (hosilaning nollari haqidagi teorema). Agar $y = f(x)$ funksiya $[a, b]$ kesmada aniqlangan, uzluksiz va differensiallanuvchi bo‘lib, kesmaning oxirlarida teng $f(a) = f(b)$ qiymatlarni qabul qilsa, u holda kesmaning ichida kamida bitta $c \in (a, b)$ nuqta mavjudki, unda hosila nolga teng, ya’ni $f'(c) = 0$.

Teoremaning shartlaridan aqalli bittasining buzilishi teorema tasdig‘ining buzilishiga olib keladi.

Lagranj teoremasi (chekli orttirmalar haqidagi teorema). Agar $y = f(x)$ funksiya $[a, b]$ kesmada aniqlangan, uzluksiz va differensiallanuvchi bo‘lsa , u holda $[a, b]$ kesma ichida kamida bitta $c \in (a, b)$ nuqta topiladiki, bu nuqtada $f(b) - f(a) = f'(c) \cdot (b - a)$ tenglik bajariladi.



Bu teoremaning geometrik ma'nosini aniqlash uchun Lagranj formulasini $\frac{f(b)-f(a)}{b-a} = f'(c)$ ko'rinishda yozamiz.



Koshi teoremasi (ikki funksiya orttirmasining nisbati haqidagi teorema). Agar ikkita $f(x)$ va $\varphi(x)$ funksiya $[a, b]$ kesmada uzluksiz, (a, b) oraliqda differensiallanuvchi, shu bilan birga barcha $x \in (a, b)$ lar uchun $\varphi'(x) \neq 0$ bo'lsa, u holda $[a, b]$ kesma ichida hech bo'lmaganda bitta $c \in (a, b)$ nuqta mavjudki, u nuqtada

$$\frac{f(b) - f(a)}{\varphi(b) - \varphi(a)} = \frac{f'(c)}{\varphi'(c)}$$

tenglik bajariladi, bunda $\varphi(b) \neq \varphi(a)$.

2.6.7. Koshining “o‘rta qiymat haqida”gi teoremasining tatbig‘i

Koshining o‘rta qiymat haqidagi teoremasi mazmunidan sof nazariy tasdiqqa o‘xshaydi, lekin uni ham amaliy tatbiq qilish mumkin. Teoremaning bitta tatbig‘i aniqmasliklarni ochishda qo‘llaniladigan Lopital qoidasini isbotlashdan iborat.

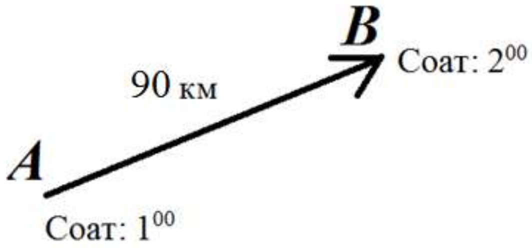
Teoremadan xulosa qilish mumkinki, funksiyaning hosilasi berilgan oraliqning aqalli bitta nuqtasida funksiyaning o‘rtacha qiymatiga teng bo‘ladi.

Bu teoremani qayerda va qanday ishlatish mumkin?

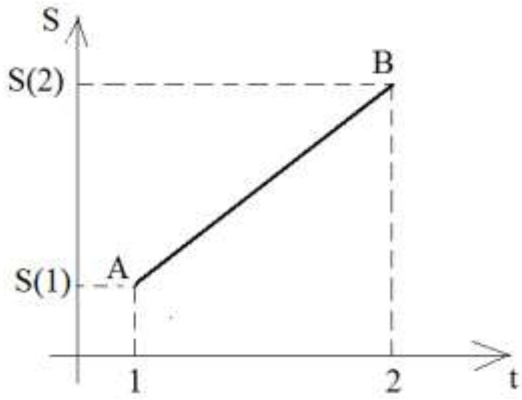
Xorijiy davlatlardagi kabi Respublikamizda ham tirbandliklardan xoli bo‘lgan pullik yo‘l qurilib, ishga tushirildi deylik. Bu yo‘lga kirishdan oldin haydovchi bo‘sh chiptani oladi, yo‘ldan chiqayotganida esa chiptaga yo‘lga kirish badali va agar tezlikni me’yordan oshirgan bo‘lsa, jarima haqidagi ma’lumot kiritilgan bo‘ladi. Haydovchi chiptada ko‘rsatilgan pulni to‘laydi. Albatta haydovchi tezlikni me’yordan oshirmaganligini aytadi. Agar yo‘l patrul xizmati xodimi Koshining o‘rta qiymat haqidagi teoremasini o‘rgangan bo‘lsa, haydovchining tezlikni me’yordan oshirganligini isbotlab bera oladi.

Tezlikning me’yordan oshirilganligini qanday isbotlash mumkin?

Yo‘lga kirish nuqtasini A, chiqish nuqtasini esa B bilan belgilaymiz. Yo‘l uzunligi 90 km bo‘lsin. Yo‘ldagi maksimal tezlik 90 km/soat deb belgilangan. Aytaylik, haydovchi yo‘lga soat 13^{00} da kirib, soat 14^{00} da chiqqan bo‘lsin.



Endi bu chizmani dekart koordinata sistemasida yo‘lni vaqtga bog‘liqlik grafigi sifatida tasvirlaymiz:



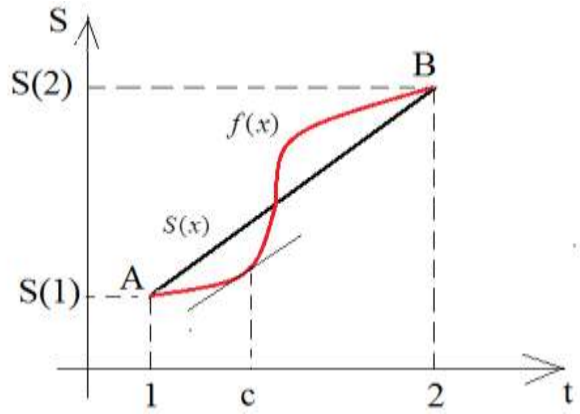
O‘tilgan yo‘lni ΔS bilan belgilaymiz: $\Delta S = S(2) - S(1) = 90$ km bo‘ladi. Vaqtning o‘zgarishini Δt deb belgilasak, $\Delta t = 2 - 1 = 1$ soat ga teng bo‘ladi. $\frac{\Delta S}{\Delta t}$ ni hisoblaymiz:

$$\frac{\Delta S}{\Delta t} = \frac{S(2) - S(1)}{2 - 1} = 90 \text{ km/soat.}$$

Bu natija haydovchining o‘rtacha tezligi 90 km/soat bo‘lganini bildiradi.

$[a,b]$ oraliqda uzluksiz bo‘lgan va hech qachon hosilasi (tezligi) o‘rtacha qiymatga (o‘rtacha tezlikka) teng bo‘lmaydigan $f(x)$ funksiya tuzish mumkinmi, boshqacha qilib aytganda **belgilangan masofani belgilangan vaqtda o‘rtacha tezlikdan past tezlik bilan bosib o‘tish mumkinmi?**

Bu savolga Koshi teoremasi javob beradi: $f(x)$ funksiya ham $S(x)$ funksiya ham $[1,2]$ oraliqda uzluksiz, chunki haydovchining transport vositasini teleport qilish qobiliyati yo‘q. Shuningdek, ikkala funksiya ham differensiallanuvchi, chunki biz har doim yo‘l funksiyasidan hosila olib tezlikni topa olamiz. Koshi teoremasiga ko‘ra esa, agar yuqoridagi shartlar bajarilsa, kamida bitta shunday $c \in [a,b]$ nuqta mavjudki, bu nuqtada hosila $S'(c) = 90$ km/soatga teng bo‘ladi. Bu holat quyidagi chizmada keltirilgan:



Bundan kelib chiqadiki, haydovchining tezligi butun yo‘l davomida hech bo‘lmaganda bir marta o‘rtacha tezlikka teng bo‘ladi. Haydovchi ba’zi joylarda o‘rtacha tezlikdan past tezlikda, ba’zi joylarda esa o‘rtacha tezlikdan yuqori tezlikda yurgan bo‘ladi. Masalan, kirayotganda tezlanish olayotgan paytda va chiqishda to‘xtashdan oldin tezligi pasayadi.

Endi haydovchi tezlikni oshirmadim deya olmaydi, chunki Koshi teoremasi buni isbotlab beradi.

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-24 misollarda $\frac{dy}{dx}$ ni toping:

1. $f(x) = x^3$ **2.** $f(x) = -x^5$ **3.** $f(x) = 2x$

4) $y = -3x$ **5.** $y = 12$ **6.** $y = 9$

7. $f(x) = x^{-6}$ **8.** $f(x) = -x^{-5}$ **9.** $f(x) = 2x^{-2}$

10. $f(x) = 15x^4$ **11.** $f(x) = 16x^4$ **12.** $f(x) = 3x^{10}$

13. $f(x) = x^3 + x^5$ **14.** $f(x) = x^4 - 7x^3$ **15.** $f(x) = 2x + 5x^2$

16. $f(x) = \frac{1}{7}x^{0.7} - \sqrt{2}x^{0.5}$ **17.** $f(x) = x^{-0.5}$ **18.** $f(x) = x^{0.9}$

19. $f(t) = \sqrt[5]{t^2}$ **20.** $f(t) = -12\sqrt[3]{x^{-5}}$ **21.** $f(x) = \frac{2x}{\sqrt[3]{x^2}}$

22. $f(x) = \frac{3x+1}{7}$ **23.** $f(x) = \frac{2x^2 - 16x}{4}$ **24.** $f(x) = \frac{3x}{5}$

25 – 32 misollarda differensialni hisoblang:

25. $\frac{d}{dx}\left(\sqrt[3]{x} + \frac{3}{x}\right)$ **26.** $\frac{d}{dx}\left(\sqrt[4]{x} - \frac{2}{x}\right)$ **27.** $\frac{d}{dx}\left(\sqrt{x} + \frac{3}{\sqrt{x}}\right)$

28. $\frac{d}{dt}\left(\sqrt[3]{t}-\frac{2}{\sqrt{t}}\right)$ **29.** $\frac{d}{dt}\left(-2\sqrt[3]{t^5}\right)$ **30.** $\frac{d}{dt}\left(17-\frac{2}{\sqrt[5]{t^2}}\right)$

31. $\frac{d}{dz}(2z^3+z^2-6z+9)$ **32.** $\frac{d}{dz}\left(\frac{z^3}{2}+\frac{6z}{13}-1\right).$

33– 46 misollarda $f'(x)$ va y' hosilalarni hisoblang:

33. $f(x)=0.5x^3$ **34.** $f(x)=-3.5x^5$

35. $f(x)=\frac{2x}{a}$ **36.** $y=-\frac{6}{7x^4}$

37. $y=x^{12}$ **38.** $y=\frac{9}{5x^3}$

39. $f(x)=\frac{4}{x}-x^{2/3}-0.2$ **40.** $f(x)=\frac{1}{3x}+x^{1/3}+x+1.7$

41. $f(x)=-0.02x^2-0.5x+135.5$ **42.** $f(x)=-0.05x^2-0.1x+570$

43. $y=3x^{-2/3}+x^{3/4}+x^{6/5}+\frac{6}{x^5}$ **44.** $y=-x^{-3/4}+4x^{3/4}-x^{5/4}-\frac{2}{x^4}$

45. $y=\frac{t}{2}+\frac{2}{t}+1$ **46.** $y=\frac{k}{7}-\frac{7}{k}$

47. Agar $f(x)=x^2+4x-5$ bo‘lsa, $f'(10)$ ni toping.

48. Agar $f(x)=\sqrt{x}+2$ bo‘lsa, $f'(4)$ ni toping.

49. Agar $y=\frac{4}{x^2}$ bo‘lsa, $\left.\frac{dy}{dx}\right|_{x=-2}$ ni toping.

50. Agar $y=x+\frac{2}{x^3}-\frac{1}{3}$ bo‘lsa, $\left.\frac{dy}{dx}\right|_{x=1}$ ni toping.

51. Agar $y=x^3+2x-3$ bo‘lsa, $\left.\frac{dy}{dx}\right|_{x=-2}$ ni toping.

52. Agar $y=\sqrt[3]{x}+\sqrt{x}-0.01$ bo‘lsa, $\left.\frac{dy}{dx}\right|_{x=64}$ ni toping.

53. Agar $y=\frac{1}{3x^4}$ bo‘lsa, $\left.\frac{dy}{dx}\right|_{x=-1}$ ni toping.

54. Agar $y=\frac{2}{5x^3}$ bo‘lsa, $\left.\frac{dy}{dx}\right|_{x=4}$ ni toping.

55. Agar $y=\frac{1}{3x^4}+\frac{x^4}{4}-\frac{2}{12}$ bo‘lsa, $\left.\frac{dy}{dx}\right|_{x=0}$ ni toping.

56-59 urinma tenglamasini tuzishga doir:

56. $f(x)=x^3-2x-5$ funksiya grafigiga

a) (2; 5),

b) (-1; 2),

v) (0; 1) nuqtalarda o‘tkazilgan urinma tenglamasini ($y=mx+b$) tuzing.

57. $f(x)=x^2-\sqrt{x}+0.5$ funksiya grafigiga

a) (4; 14),

b) (9; 78),

v) (1; 0) nuqtalarda o‘tkazilgan urinma tenglamasini tuzing.

58. $f(x)=1+\frac{1}{x^2}$ funksiya grafigiga

a) (1; 1),

b) (3; 1/9),

v) (-2; 1/4) nuqtalarda o‘tkazilgan urinma tenglamasini tuzing.

59. $f(x)=\sqrt[3]{x^2}-32$ funksiya grafigiga

a) (-1; 1),

b) (1; 1),

v) (8; 4) nuqtalarda o‘tkazilgan urinma tenglamasini tuzing.

60-69 misollarda funksiya grafigiga o‘tkazilgan urinmasi gorizontal bo‘ladigan nuqtani toping.

60. $f(x)=1-x^3$

61. $f(x)=-x^2+4$

62. $f(x)=x^2-1$

63. $f(x)=3x^2-5x+4$

64. $f(x)=5x^2-3x+4$

65. $f(x)=x^3-2$

66. $f(x)=-0.01x^2-0.5x+7$

67. $f(x)=-0.01x^2+0.4x+50$

68. $y=3$

69. $y=-4$

Funksiyani hosila yordamida tekshirish.

70-72 misollarda $f'(x)$ hosila musbat bo‘ladigan oraliqlarni toping:

70. $f(x)=x^2-4x+1$

71. $f(x)=x^2+7x+2$

72. $f(x)=\frac{1}{3}x^3-x^2-3x+5$

73-77 misollarda funksiyaning o‘sish, kamayish oraliqlarini toping:

73. $f(x)=x^5+x^3+1$

74. $f(x)=x^3+2x-5$

75. $f(x)=\frac{1}{x},\ x\neq 0$

76. $f(x)=\sqrt{x},\ x\geq 0$

77. $f(x)=x^3+ax$ funksiya $x>0$ da faqat o‘suvchi, $x<0$ da faqat kamayuvchi bo‘lishini isbotlang.

Hosilaning tatbiqlariga doir masalalar

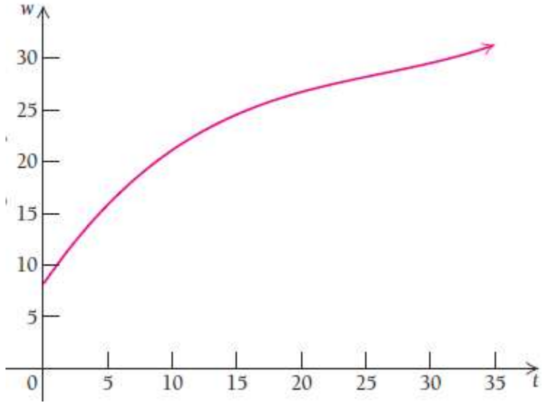
78. Tibbiyot. Yaraning tuzalishi. Tuzalayotgan yara $A(r) = \pi r^2$ tenglik bilan A doira shaklidagi soha (sm^2) ga approksimatsiyalangan, bunda $r -$soha radiusi. Radiusga nisbatan soha hosilasini toping. Topilgan qiymatni tushuntiring.

79. Tibbiyot. Yaraning tuzalishi. Tuzalayotgan yara $C(r) = 2\pi r$ tenglik bilan C aylana (sm)ga approksimatsiyalangan, bunda $r -$aylana radiusi. Radiusga nisbatan aylana hosilasini toping. Topilgan qiymatni tushuntiring.

80. Anatomiya. Chaqaloq vaznining ortishi. Tug'ilganidan, ya'ni 0 dan 36 oylikkacha bo'lgan o'g'il chaqaloqning o'rtacha vazni

$$w(t) = 0.000758t^3 - 0.0596t^2 + 1.82t + 8.15$$

tenglik bilan approksimatsiyalanadi, bunda $t -$bir oyda bir marta o'lchab ko'rilgan w- chaqaloq massasi (funt birlikda).



Tenglikdan foydalanib o'rtacha vazndagi o'g'il chaqaloq uchun quyidagilarni aniqlang:

a) Vaqtga nisbatan vazn hosilasini toping;

b) 10 oylik chaqaloqning vaznini aniqlang;

v) 10 –oyda bola vaznining hosilasi nimaga teng.

81. Tibbiyot. Bemor haroratining ko‘tarilishi. Bemor harorati t kunga nisbatan $T(t) = -0.1t^2 + 1.2t + 98.6$ tenglik bilan approksimatsiyalanadi (harorat Farengeyt shkalasida keltirilgan).

a) Vaqtga nisbatan o‘rtacha haroratni toping.

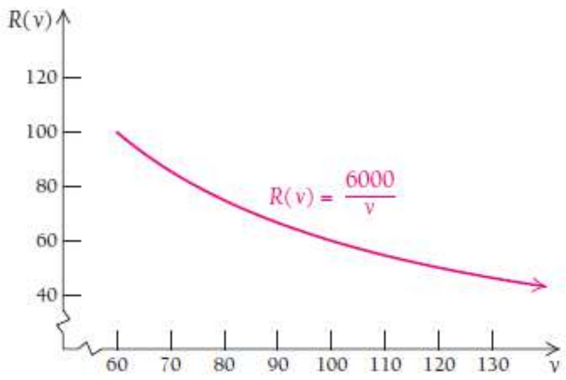
b) 1,5 kundagi harorat qanday bo‘lgan?

v) 1,5 kundagi o‘rtacha harorat qanday bo‘lgan?

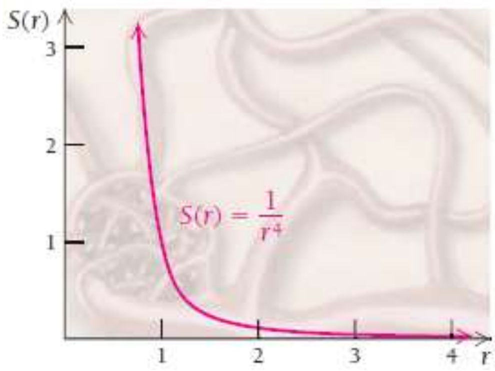
82. Tibbiyot. Odam yuragi ritmi. Daqiqasiga 6000 ml (bir urishda v ml) qonni haydaydigan yurak ritmi $R(v) = \frac{6000}{v}$ tenglik bilan approksimatsiyalash mumkin.

a) Yurak ritmi tenglamasidan hosila oling.

b) Bir urishda $v = 80$ ml qonni haydash ritmini toping.



83. Tibbiyot. Tomirlarda tromblar hosil bo‘lishi. $S(r)=\frac{1}{r^4}$ tenglik



bilan radiusi r mm bo‘lgan tomirlarda qonning harakatlanishidagi qarshiligini ifodalash mumkin.

a) Radiusga bo‘yicha tenglikdan hosila oling;

b) Radiusi $r=1.2$ mm bo‘lsa, qarshilikni aniqlang.

84. Aholi sonining o‘sish sur’ati. Shahardagi aholi soni 100 000 kishidan P tenglik bilan ko‘paymoqda, bunda t yil hisobida:

$$P(t)=100000+2000t^2.$$

a) Aholining o‘sish sur’atini aniqlang;

b) 10 yildan keyin shahardagi aholi soni nechta nafarga yetadi?

v) $t=10$ yilning o‘zida aholi soni qanchaga o‘sadi?

85. Ayollarning oila qurishdagi o‘rtacha yoshi. Ayollarning oila qurish yoshini quyidagi chiziqli funksiya bilan approksimatsiyalash mumkin:

$$A(t)=0.08t+19.7,$$

bunda $A(t)$ qiymat 1950 yildan buyon ayollaning turmush qurishdagi o‘rtacha yoshi. O‘rtacha yosh funksiyasidan hosila oling va hosil bo‘lgan qiymat nimani anglatishini tushuntiring.

86. Gorizont haqida tasavvur. S ko‘zimiz bilan ko‘ra oladigan masofa (mill hisobida), agar tikka turib kuzatsak, h (bo‘yimiz uzunligi)

balandlikdan ko‘ra oladigan masofani quyidagi tenglik bilan approksimatsiyalash mumkin: $S(h) = 1.22\sqrt{h}$.

a) S dan h bo‘yicha hosila oling. Olingan hosila nimani anglatadi?

b) 40 000 fut balandlikda samolyot oynasidan gorizontning qancha uzoqligini ko‘rishi mumkin?

c) h=40 000 dagi o‘rtacha qiymatni toping. Bu qiymat nimani bildiradi?

2.7. Differensiallash qoidalari: ko‘paytma va bo‘linmaning hosilasi

2.7.1. Ko‘paytmaning hosilasi

Ikkita funksiyaning ko‘paytmasi yana funksiya bo‘ladi. Masalan, $F(x)=x^2\cdot x^5$, bu funksiya $f(x)=x^2$ va $g(x)=x^5$ funksiyalarning ko‘paytmasidan tashkil topgan, ya’ni $F(x)=f(x)\cdot g(x)$.

$F(x)$ **ikkita funksiyaning ko‘paytmasi bo‘lganda, uning hosilasini topish uchun qanday amal bajarish kerak?**

Shu savolga javob izlaymiz.

5-teorema: Ko‘paytma qoidasi. $F(x)=f(x)\cdot g(x)$ funksiya berilgan bo‘lsin. U holda

$$F'(x)=\frac{d}{dx}\left[f(x)\cdot g(x)\right]=\left[\frac{d}{dx}f(x)\right]\cdot g(x)+f(x)\cdot\left[\frac{d}{dx}g(x)\right]$$

yoki

$$F'(x)=f'(x)\cdot g(x)+f(x)\cdot g'(x)$$

o‘rinli.

Ikkita funksiya ko‘paytmasining hosilasi 1-funksiya hosilasining 2-funksiyaga ko‘paytmasi bilan 2-funksiya hosilasining 1-funksiyaga ko‘paytmasi yig‘indisiga teng.

Isboti: ►

$$\frac{d}{dx}[f(x)\cdot g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[f(x+h)\cdot\frac{g(x+h)-g(x)}{h}\right] + \lim_{h\to 0}\left[g(x)\cdot\frac{f(x+h)-f(x)}{h}\right]$$

$$= f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

$$= f(x)\cdot\left[\frac{d}{dx}g(x)\right] + g(x)\cdot\left[\frac{d}{dx}f(x)\right]$$

Demak, $f(x)=x^2$ va $g(x)=x^5$ funksiyalar ko‘paytmasidan hosila olish algoritmi 5 qadamdan iborat bo‘lar ekan, ya’ni:

$$\frac{d}{dx}(x^2\cdot x^5)$$

$$\overset{(1)}{=}\ \overset{(2)}{2x\cdot x^5} + \overset{(3)}{x^2}\cdot\overset{(4)}{5x^4}$$

$$= 2x^6 + 5x^6 \quad (5)$$

$$= 7x^6$$

1-qadam: 1-funksiyadan hosila olamiz;

2-qadam: olingan hosilani 2-funksiyaga ko‘paytiramiz;

3-qadam: 2-funksiyadan hosila olamiz;

4-qadam: olingan hosilani 1-funksiyaga ko‘paytiramiz;

5-qadam: Ikkila ko‘paytmani qo‘shamiz.

1-misol. $\left(\frac{1}{3}x^3-x^2-3x+5\right)\cdot\left(4x^2+x+1\right)$ funksiyaning hosilasini toping.

Yechilishi: ▶ $\left[\left(\frac{1}{3}x^3-x^2-3x+5\right)\cdot\left(4x^2+x+1\right)\right]'=$

$$=\left(\frac{1}{3}x^3-x^2-3x+5\right)'\cdot\left(4x^2+x+1\right)+\left(\frac{1}{3}x^3-x^2-3x+5\right)\cdot\left(4x^2+x+1\right)'=$$

$$=\left(x^2-2x-3\right)\cdot\left(4x^2+x+1\right)+\left(\frac{1}{3}x^3-x^2-3x+5\right)\cdot\left(8x+1\right)=$$

$$=4x^4+x^3+x^2-8x^3-2x^2-2x-12x^2-3x-3+\frac{8}{3}x^4+\frac{1}{3}x^3-8x^3-x^2-24x^2-3x+40x+5=$$

$$=\frac{20}{3}x^4-\frac{44}{3}x^3-38x^2+32x+2$$ ◀

1-vazifa. Quyidagi funksiyalar hosilasini toping:

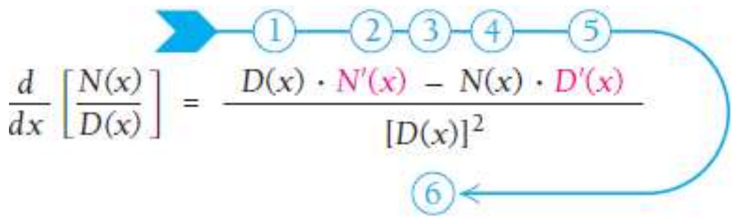
a) $y=\left(2x^5-x^2+x\right)\cdot\left(3x^2-x-2\right)$; b) $y=\left(\sqrt{x}+1\right)\cdot\left(\sqrt[5]{x^2}-x\right)$

2.7.2. Bo‘linmaning hosilasi

6-teorema: Bo‘linma qoidasi. $Q(x)=\frac{f(x)}{g(x)}$ funksiya berilgan bo‘lsin. U holda $Q'(x)=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{\left[g(x)\right]^2}$ o‘rinli.

Ikkita funksiya bo‘linmasining hosilasi 1-funksiya hosilasining 2-funksiyaga ko‘paytmasidan 2-funksiya hosilasining 1-funksiyaga ko‘paytmasini ayirmasini 2-funksiya kvadratiga bo‘linganiga teng.

Demak, bo‘linma hosilasini topish algoritmi 6 qadamdan iborat:



2-misol. $Q(x)=\frac{x^3}{x^5}$ funksiyaning hosilasini toping.

Yechilishi: ► $Q'(x)=\frac{\left(x^3\right)'\cdot x^5 - x^3\cdot\left(x^5\right)'}{\left(x^5\right)^2}=\frac{3x^2\cdot x^5 - x^3\cdot 5x^4}{x^{10}}=\frac{-2x^7}{x^{10}}=-\frac{2}{x^3}$

yoki berilgan ifodani oldin soddalashtirib, keyin $Q(x)=\frac{x^3}{x^5}=x^{-2}$ dan

hosila olamiz: $Q'(x)=\left(x^{-2}\right)'=-2x^{-2-1}=-2x^{-3}=-\frac{2}{x^3}$. ◄

3-misol. $Q(x)=\frac{1-x^2}{x^3+1}$ funksiyaning hosilasini toping.

Yechilishi: ►

$$Q'(x)=\frac{-2x\cdot(x^3+1)-(1-x^2)\cdot 3x^2}{\left(x^3+1\right)^2}=\frac{-2x^4-2x-3x^2+3x^4}{(x^3+1)^2}=\frac{x^4-3x^2-2x}{(x^3+1)^2}$$ ◄

4-misol. $f(x)=\frac{1-\sin x}{\cos x}$ funksiyaning hosilasini toping.

Yechilishi: ► $f'(x)=\left(\frac{1-\sin x}{\cos x}\right)'=\frac{(1-\sin x)'\cos x-(1-\sin x)(\cos x)'}{\cos^2 x}=$

$$=\frac{-\cos x\cos x-(1-\sin x)(-\sin x)}{\cos^2 x}=\frac{-\cos^2 x+\sin x-\sin^2 x}{\cos^2 x}=\frac{\sin x-1}{\cos^2 x}$$ ◄

2-vazifa. Quyidagi funksiyalar hosilasini toping:

a) $y=\dfrac{1-3x}{x^2+2}$; b) $y=\dfrac{ax+1}{bx+1}$

2.7.3. Hosilaning iqtisodiy masalalarga tatbig‘i

Ta’rif: Iqtisod. Agar x mahsulotning ishlab chiqarishdagi tannarxi $T(x)$ bo‘lsa, u holda shu mahsulotning o‘rtacha tannarxi $\dfrac{T(x)}{x}$ ga teng bo‘ladi.

Agar x mahsulotni sotishdan olinadigan daromad $D(x)$ bo‘lsa, u holda shu mahsulotni sotishdagi o‘rtacha daromad $\dfrac{D(x)}{x}$ ga teng bo‘ladi.

Agar x mahsulotni sotishdan olinadigan sof foyda $F(x)$ bo‘lsa, u holda shu mahsulotdan olinadigan o‘rtacha sof foyda $\dfrac{F(x)}{x}$ ga teng bo‘ladi.

5-misol. Tadbirkorlik. “Greenhous” firmasi geran $x=100$ tup gul ko‘chatlari o‘stirib sotadi. Agar 100 ta geran tannarxini $T(x)=200+100\cdot\sqrt[4]{x}$ (dollar) tenglik bilan, ularni sotishdan tushadigan daromad $D(x)=120+90\sqrt{x}$ deb modellashtirish mumkin bo‘lsa, u holda

a) 100 ta ko‘chatning o‘rtacha narxini, o‘rtacha daromad va o‘rtacha foydani toping.

b) 300 ta ko‘chat sotilgan bo‘lsa, o‘rtacha narxini, o‘rtacha daromadni va o‘rtacha foydani toping.

Yechilishi: ► $A_T(x)$, $A_D(x)$, $A_F(x)$ bilan mos ravishda gul ko‘chatlarining o‘rtacha tannarxini, o‘rtacha daromad va o‘rtacha sof foydani belgilaymiz. U holda

a) $x=100$ tup gul ko‘chati uchun quyidagilarni topamiz:

$$A_T(x)=\frac{T(x)}{x}=\frac{200+100\cdot\sqrt[4]{x}}{x}; \qquad A_D(x)=\frac{D(x)}{x}=\frac{120+90\sqrt{x}}{x}$$

$$A_F(x)=\frac{F(x)}{x}=\frac{D(x)-T(x)}{x}=\frac{120+90\sqrt{x}-(200+100\cdot\sqrt[4]{x})}{x}=\frac{90\sqrt{x}-100\cdot\sqrt[4]{x}-80}{x}$$

b) 300 tup gul ko‘chati uchun o‘rtacha tannarx, o‘rtacha daromad va o‘rtacha sof foydaning qiymatlari o‘zgaradi. Buning uchun sof foyda funksiyasidan hosila olamiz:

$$A_F'(x)=\left(\frac{90\sqrt{x}-100\cdot\sqrt[4]{x}-80}{x}\right)'=\frac{\left(90x^{1/2}-100x^{1/4}-80\right)'\cdot x-\left(90x^{1/2}-100x^{1/4}-80\right)}{x^2}=$$

$$=\frac{\left(45x^{-1/2}-25x^{-3/4}\right)\cdot x-90\sqrt{x}+100\cdot\sqrt[4]{x}+80}{x^2}=\frac{-45\sqrt{x}+75\cdot\sqrt[4]{x}+80}{x^2}$$

$$A_F'(x)=\frac{-45\sqrt{x}+75\cdot\sqrt[4]{x}+80}{x^2}$$

Endi $x=100$ uchun yozilgan formulani $x=300$ uchun hisoblaymiz, soddalik uchun 100 ni bir birlik deb olamiz, shunda 300 uch birlik bo‘ladi. Natijada $A_F'(3)=\frac{-45\sqrt{3}+75\cdot\sqrt[4]{3}+80}{3^2}\approx 11.20$

dollar **o‘rtacha sof foyda** olinar ekan. ◄

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-10 misollarda ko‘paytmaning hosilasini oson usulda hisoblang:

1. $F(x) = x^7 \cdot x^5$

2. $F(x) = x^{-2} \cdot x^5$

3. $y = (x^2 + 5) \cdot (x^3 - 1)$

4. $y = (2x + 4) \cdot (3x - 1)$

5. $y = x^2 (x^3 + x - 1)$

6. $y = 3x^4 (x^3 - 7x)$

7. $y = (x^2 + 1) \cdot (\sqrt{x} - 1)$

8. $y = (x\sqrt{x} - \sqrt[3]{x}) \cdot (x + 2)$

9. $y(t) = (\sqrt[3]{t^2} + \sqrt{t} + 1) \cdot (\sqrt{x^3} - \sqrt{t})$

10. $y(z) = (\sqrt[3]{z} + \sqrt{z}) \cdot (\sqrt{z^3} - 2\sqrt{z})$

11-22 misollarda bo‘linmaning hosilasini oson usulda hisoblang:

11. $F(x) = \dfrac{x^7}{x^9}$

12. $F(x) = \dfrac{x^{-2}}{x^3}$

13. $y = \dfrac{x^2 + 5}{x - 1}$

14. $y = \dfrac{2x + 4}{x^2 + 3x - 5}$

15. $y = \dfrac{x^3 + x - 1}{2x + 3}$

16. $y = \dfrac{x^3 - 7x}{5x}$

17. $y = \dfrac{\sqrt{x} - 1}{\sqrt{x}}$

18. $y = \dfrac{x\sqrt{x} - \sqrt[3]{x}}{\sqrt[3]{x} + 2}$

19. $y(t) = \dfrac{\sqrt[3]{t^2} + \sqrt{t} + 1}{\sqrt[3]{t}}$

20. $y(z) = \dfrac{\sqrt[3]{z} + \sqrt{z}}{1 - \sqrt{z}}$

21. $F(x) = \dfrac{7 - \dfrac{3}{2x}}{\dfrac{4}{x^2} + 5}$

22. $F(x) = \dfrac{(x - 1)(x^2 + x + 1)}{x^4 - 3x^3 - 5}$

23-28 misollarda darajali funksiyani funksiyalar ko‘paytmasiga aylantirib, hosilasini hisoblang:

23. $y=(x^2+5)^2$ **24.** $y=(x^2-2x+4)^2$

25. $y=(x^3+x-1)^2$ **26.** $y=3x^4(x^3-7x)$

27. $y=(\sqrt{x}-1)^2$ **28.** $y=(x\sqrt{x}-\sqrt[3]{x})^2$

29-32 misollarda funksiyaga nuqtada o‘tkazilgan urinma tenglamasini tuzing:

29. $f(x)=\dfrac{8}{x^2+4}$ funksiya grafigiga (0; 2) va (-2; 1) nuqtalarda o‘tkazilgan urinma tenglamasini tuzing.

30. $f(x)=\dfrac{\sqrt{x}}{x+1}$ funksiya grafigiga $x=1$ va $x=1/4$ nuqtalarda o‘tkazilgan urinma tenglamasini tuzing.

31. $f(x)=x^2+\dfrac{3}{x-1}$ funksiya grafigiga $x=2$ va $x=3$ nuqtalarda o‘tkazilgan urinma tenglamasini tuzing.

32. $f(x)=\dfrac{4x}{x^2+1}$ funksiya grafigiga (0; 0) va (-1; -2) nuqtalarda o‘tkazilgan urinma tenglamasini tuzing.

33. $f(x)\cdot g(x)\cdot h(x)$ ko‘paytma uchun formula keltirib chiqaring.

HOSILANING TATBIQLARIGA DOIR MISOLLAR

34-42 misollar iqtisodiyot (tadbirkorlik) haqida

34. Mahsulotning o‘rtacha tannarxi. Yoz mavsumida fabrikalar nimchalarni $T(x)=950+15\sqrt{x}$ (dollar) narxda sotishadi. 400 ta nimchaning o‘rtacha tannarxi qancha?

35. Mahsulotning o‘rtacha tannarxi. “O‘rikzor” MCHJ 1 litrlik bankada tomat pastalarini $T(x)=375+0.75\cdot\sqrt[4]{x^3}$ (dollar) narxda sotadi. 81 banka tomat pastasi uchun bu tannarx o‘zgaradi. Mahsulotning o‘rtacha tannarxini toping.

36. O‘rtacha daromad. Yoz mavsumida fabrikalar nimchalarni sotishidan tushadigan daromad $D(x)=85\sqrt{x}$ (dollar) ni tashkil qiladi. 400 ta nimchada foyda qiymati o‘zgaradi, o‘rtacha foydani toping.

37. O‘rtacha daromad. “O‘rikzor” MCHJ 1 litrlik bankada tomat pastalarini sotishdagi daromad $D(x)=7.5x^{0.7}$ (dollar) ni tashkil qiladi. 81 ta banka tomat pastasini sotishda foyda qiymati o‘zgaradi, o‘rtacha foydani toping.

38. O‘rtacha sof foyda. Yoz mavsumida fabrikalar nimchalarni $T(x)=950+15\sqrt{x}$ (dollar) narxda sotishadi va $D(x)=85\sqrt{x}$ daromad olishadi. 400 ta nimchadan olinadigan o‘rtacha sof foyda qancha?

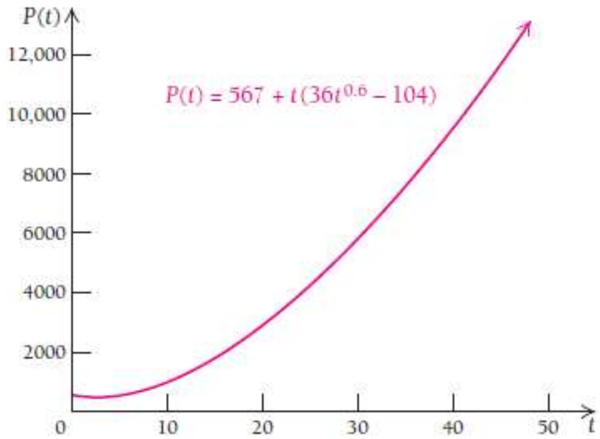
39. O‘rtacha sof foyda. “O‘rikzor” MCHJ 1 litrlik bankada tomat pastalarini $T(x)=375+0.75\cdot\sqrt[4]{x^3}$ (dollar) narxda sotadi va $D(x)=7.5x^{0.7}$

daromad qiladi. 81 banka tomat pastasi sotilganda o‘rtacha sof foydani toping.

40. O‘rtacha sof foyda. “Xorazm usta ” MCHJ ishlab chiqargan o‘ymakorlik buyumlarini $T(x)=4300+2.1x^{0.6}$ (dollar) narxda sotadi va $D(x)=65x^{0.9}$ daromad qiladi. 50 ta mahsulot ssotilgandagi o‘rtacha sof foydani toping.

41. O‘rtacha sof foyda. “Xiva gilami” OAJ 4x5 o‘lchamli gilamlarni $T(x)=900+18x^{0.7}$ (dollar) narxda sotadi va $D(x)=75x^{0.8}$ daromad qiladi. 20 ta gilam sotishdan olinadigan o‘rtacha sof foydani toping.

42. Mehnatga haq to‘lash. Universitet o‘qituvchisining oylik maoshini (dollarda) $P(t)=567+t(36t^{0.6}-104)$ funksiyaga approksimatsiyalash mumkin bo‘lsin. U 1985 yildan buyon ishlayotgan va $t-$uning ishlagan yillari bo‘lsin.

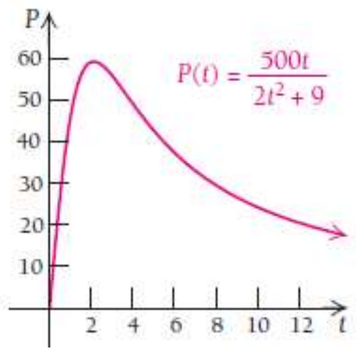


a) $P'(t)$ ni toping.

b) $P'(30)$ ni toping va u nimani anglatishini tushuntiring.

43- 45 misollar ijtimoiy va tabiiy fanlar doirasida:

43. Kichikroq shaharchada aholining ko‘payishini $P(t)=\frac{500t}{2t^2+9}$ formula bilan approksimatsiyalash mumkin, bunda $t-$ yillar hisobi.

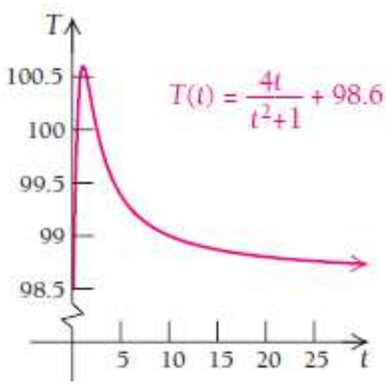


a) Aholining o‘sish sur’atini aniqlang.

b) 12 yildan so‘ng aholi soni qanchaga ko‘payadi?

v) $t=12$ yil uchun aholining o‘sish sur’atini toping.

44. Kasallik paytida haroratning ko‘tarilishi. Kishilar kasallanganda harorati (Farangeyt birligida) $H(t)=\frac{4t}{t^2+1}+98.6$ formula bilan ifodalanadi, bunda $t-$vaqt (soat).



a) Vaqt bo‘yicha harorat funksiyasidan hosila oling.

b) $t=2$ soatdagi haroratni toping

v) $t=2$ soatda harorat funksiyasi hosilasini aniqlang.

45. Sezgirlik. Davolashda dorining Q dozasiga inson tanasining reaksiyasi $R(Q)=Q^2\left(\frac{k}{2}-\frac{Q}{3}\right)$, bunda $k-$koeffitsiyent. Agar reaksiya qon bosimining oshishi bilan ro‘y bersa, $R-$ millimetr simob ustuni (mmHg) bilan, agar haroratning o‘zgarishi bilan ketsa, Farengeyt (^{0}F) bilan o‘lchanadi. Davolashga tananing sezgirligini aniqlash uchun funksiyadan $\frac{dR}{dQ}$ hosila olinadi. Sezgirlik formulasini toping va u nimani anglatishini tushuntiring.

46. $f(x)=\frac{x}{x+1}$ va $g(x)=\frac{-1}{x+1}$ funksiyalar berilgan.

a) $f'(x)$ hosilani toping.

b) $g'(x)$ hosilani toping.

v) Yuqoridagilar asosida grafik chizing, bu grafik nimani bildiradi?

47. $f(x)=\frac{x^2}{x^2-1}$ va $g(x)=\frac{1}{x^2-1}$ funksiyalar berilgan.

a) $f'(x)$ hosilani toping.

b) $g'(x)$ hosilani toping.

v) Yuqoridagilar asosida grafik chizing, bu grafik nimani bildiradi?

2.8. Murakkab funksiyaning hosilasi

Ba’zi funksiyalarni sodda funksiya deb hisoblaymiz. Sodda funksiyalarga misollar keltirish bilan ularning qanday ko‘rinishda bo‘lishini tushuntirish mumkin. 2.5 va 2.6 bo‘limlarda biz sodda funksiyalarni qarab chiqqan edik:

$$f(x)=2x, \qquad g(x)=5x^3-2, \qquad h(x)=4\sqrt{x}, \qquad y=\frac{2x+1}{x^2-3}.$$

Quyidagi funksiyalarni sodda funksiyalar deb bo‘lmaydi:

$$f(x)=\left(x^3+2x\right)^3, \qquad g(x)=\sqrt{3x+7}, \qquad h(x)=\left(\frac{3x-5}{4x^2+1}\right)^3.$$

Chunki bu misollarda x o‘zgaruvchi bitta yoki bir nechta funksiyalarning algebraik ifodalarini biror darajaga ko‘tarishdan hosil bo‘lgan funksiyaning argumenti sifatida kelmoqda. Bunday funksiyalarga 2.5 va 2.6 mavzularda o‘rganilgan differensiallash qoidalarini qanday qo‘llaymiz? Shu va boshqa savollarga javob topish uchun oldin murakkab funksiya tushunchasini kiritamiz. Avvalo xususiy holdan boshlaymiz.

2.8.1. Darajali funksiya hosilasining umumlashgan formulasi

$y=x^3+1$ funksiya sodda funksiya bo‘lib, uning hosilasini bevosita darajali funksiyaning hosilasidan foydalanib, hisoblash mumkin:

$$y' = 3x^2 .$$

Agar bu funksiyani qandaydir darajaga ko‘tarsak, u murakkab funksiyaga aylanadi, masalan: $y = (x^3+1)^3$.

● **Ushbu funksiyaning hosilasi qanday topiladi?**

Hosilani $y' = 3(x^3+1)^2$ shunday deb faraz qilamiz. ☺☺☺

- Bu yechim to‘g‘rimi ?

Bu farazni to‘g‘riligini tekshirish uchun dastlab funksiyani yoyib yozamiz:

$$y = (x^3+1)^3 = (x^3+1)(x^3+1)(x^3+1) = (x^6+2x^3+1)(x^3+1) = x^9+3x^6+3x^3+1$$

Endi undan hosila olamiz: $y' = 9x^8 + 18x^5 + 9x^2$.

$9x^2$ ni qavsdan tashqariga chiqaramiz:

$$y' = 9x^2(x^6+2x^3+1) = 3(x^3+1)^2 \cdot 3x^2$$

Bundan ko‘rinadiki, funksiya hosilasi $y' = 3(x^3+1)^2 \cdot 3x^2$ ga teng. Yuqorida faraz qilganimiz to‘g‘ri yechimga yaqinroq, lekin unda qavs ichidan olingan hosila qiymati $3x^2$ ko‘pytuvchi yetishmayotgan ekan.

Shunday qilib, $y = (x^3+1)^3$ funksiya hosilasi $y' = 3(x^3+1)^2 \cdot 3x^2$ ga teng.

7-teorema: Darajali funksiya hosilasining umumlashgan formulasi. $u(x)$ differensiallanuvchi funksiya x o‘zgaruvchining funksiyasi bo‘lsin. U holda quyidagi tenglik o‘rinli:

$$\left([u(x)]^k\right)' = k \cdot [u(x)]^{k-1} \cdot (u(x))'$$

Darajali funksiya hosilasining umumlashgan formulasi bizga daraja ko‘rsatkichi juda katta bo‘lgan funksiyalardan oson hosila olish imkonini beradi. Masalan, $y=\left(x^5+1\right)^{97}$ ning hosilasini topishda biz uni 97-darajagacha yoyib o‘tirmaymiz. Shuningdek, $y=\sqrt[6]{1+x^5}$ dan hosila olishda funksiyani $y=\left(x^5+1\right)^{1/6}$ darajali murakkab funksiya sifatida qarab, oson hisoblash mumkin.

Keling, darajali murakkab funksiyadan hosila olishning 3 qadamdan iborat algoritmini o‘rganamiz.

Bizga $y=\left(x^5+1\right)^{97}$ funksiya berilgan bo‘lsin. Funksiya hosilasini topamiz:

1-qadam: Dilimizda “ichki” funksiyani ajratib olamiz: $x^5+1=t$

2-qadam: “Tashqi” funksiyadan hosila olamiz:

$$\left(x^5+1\right)^{97}=t^{97} \quad \Rightarrow \quad 97t^{96}=97\left(x^5+1\right)^{96}$$

3-qadam: “Ichki” funksiya hosilasini ko‘paytiramiz:

$$97\left(x^5+1\right)^{96}\cdot 5x^4=485x^4(x^5+1)$$

3 ta qadamdan foydalanib, murakkab funksiyalarning hosilalarini topishga doir misollar qaraymiz.

1-misol. $y=\left(3+x^2\right)^5$ funksiya hosilasini toping.

Yechilishi: ► $y'=\left(\left(3+x^2\right)^5\right)'=5\left(3+x^2\right)^4\cdot 2x=10x\left(3+x^2\right)^4$ ◄

2-misol. $y=\sqrt{1+x^3}$ funksiya hosilasini toping.

Yechilishi: ► $y=\sqrt{1+x^3}=\left(1+x^3\right)^{1/2}$ deb yozib olamiz, so‘ngra hosila olamiz: $y' = \frac{1}{2}\left(1+x^3\right)^{\frac{1}{2}-1} \cdot 3x^2 = \frac{3x^2}{2}\left(1+x^3\right)^{-\frac{1}{2}} = \frac{3x^2}{2\sqrt{1+x^3}}$ ◀

3-misol. $y=(1-x^2)^3+(5+4x)^2$ funksiya hosilasini toping.

Yechilishi: ▶

$$y' = 3(1-x^2)^2 \cdot (-2x) + 2(5+4x) \cdot 4 = -6x(1-x^2)^2 + 8(5+4x) =$$

$$= -6x + 12x^3 - 6x^5 + 40 + 32x = -6x^5 + 12x^3 + 26x + 40$$ ◀

1-vazifa. Quyidagi funksiyalar hosilasini toping:

a) $y=\left(2x^5-x^2+x\right)^4$**;**

b) $y' = \left(\left(x^4-4x^2+1\right)^3\right)' = 3(x^4-4x^2+1)\cdot 4x^3 - 8x$ **tenglik to'g'rimi?**

4-misol. $y=(3x-5)^4(7-x)^{10}$ funksiya hosilasini toping.

Yechilishi: ▶ $y' = 4(3x-5)^3 \cdot 3 \cdot (7-x)^{10} + (3x-5)^4 \cdot 10(7-x)^9 \cdot (-1) =$

$$= 12(3x-5)^3(7-x)^{10} - 10(3x-5)^4(7-x)^9 =$$

$$= 2(3x-5)^3(7-x)^9\left[6(7-x) - 5(3x-5)\right] =$$

$$= 2(3x-5)^3(7-x)^9(42-6x-15x+25) =$$

$$= 2(3x-5)^3(7-x)^9(67-21x)$$ ◀

5-misol. $y=\sqrt[4]{\frac{x+1}{x-3}}$ funksiya hosilasini toping.

Yechilishi: ▶ $y=\sqrt[4]{\frac{x+1}{x-3}} = \left(\frac{x+1}{x-3}\right)^{\frac{1}{4}}$ deb yozib olamiz, so'ngra hosila

olamiz: $y' = \frac{1}{4}\left(\frac{x+1}{x-3}\right)^{\frac{1}{4}-1} \cdot \left(\frac{x+1}{x-3}\right)' = \frac{1}{4}\left(\frac{x+1}{x-3}\right)^{-\frac{3}{4}} \cdot \left(\frac{1\cdot(x-3)-(x+1)\cdot 1}{(x-3)^2}\right) =$

$$= \frac{1}{4}\left(\frac{x+1}{x-3}\right)^{-\frac{3}{4}} \cdot \frac{x-3-x-1}{(x-3)^2} = \frac{1}{4}\frac{(x+1)^{-\frac{3}{4}}}{(x-3)^{-\frac{3}{4}}} \cdot \frac{-4}{(x-3)^2} = -\frac{1}{(x+1)^{3/4}(x-3)^{5/4}}$$

yoki $y' = -\dfrac{1}{\sqrt[4]{(x+1)^3(x-3)^5}}$ ◀

2-vazifa. Funksiya hosilasini toping: $f(x) = \dfrac{3x^2-4}{(2x^4+1)^2}$

2.8.2. Funksiyalar kompozitsiyasi (Murakkab funksiya)

Dastlab funksiyalar kompozitsiyasi (murakkab funksiya) ni tushunib olamiz.

Sobir aka haftada uch marta fitness klubiga boradi. U yaqinda sotib olgan krossovkasi tagidagi sonlar 5 ta davlatning ekvivalent o‘lchamlarini bildirishini payqab qoldi.

Bu belgilarga ko‘ra shunday funksiyani topish mumkinki, u bir davlat poyafzal o‘lchamini boshqa davlatning o‘lchamiga o‘tkazib bersin. Haqiqatan, AQSh poyafzal o‘lchamini Fransiya o‘lchamiga o‘tkazuvchi funksiya mavjud: $g(x)=\frac{4x+92}{3}$,

bunda x AQSh o‘lchamini, $g(x)$ esa Fransiya o‘lchamini bildiradi.

$$g(11\frac{1}{2})=\frac{4x+92}{3}=\frac{4\cdot 11\frac{1}{2}+92}{3}=\frac{46+92}{3}=46.$$

Demak, oyoq kiyimida AQSh ning 11.5 o‘lchami Fransiyaning 46 o‘lchamiga to‘g‘ri kelar ekan. Xuddi shuningdek, Fransiya poyafzal o‘lchamini Yaponiya o‘lchamiga o‘tkazuvchi funksiya mavjud:

$$f(x)=\frac{15x-100}{2},$$

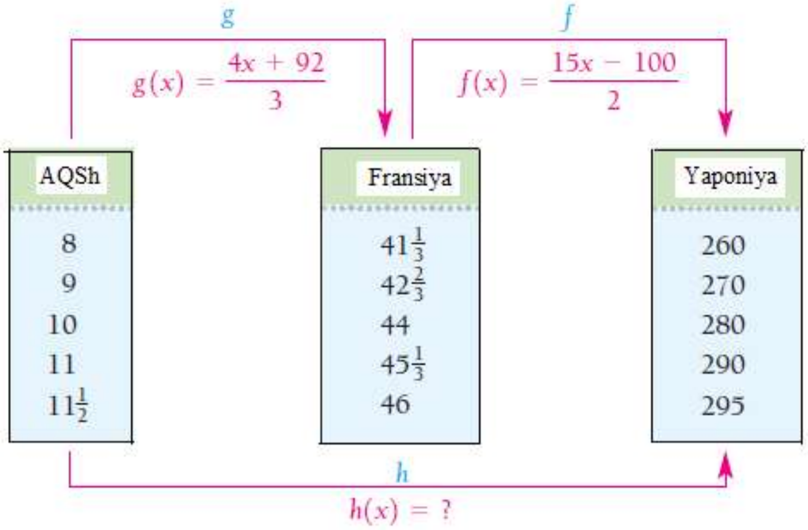
bunda x Fransiya o‘lchamini, $f(x)$ esa Yaponiya o‘lchamini bildiradi.

$$f(46)=\frac{15x-100}{2}=\frac{15\cdot 46-100}{2}=295.$$

Demak, oyoq kiyimida Fransiyaning 46 o‘lchami Yaponiyaning 295 o‘lchamiga to‘g‘ri kelar ekan.

Shu ma’lumotlar asosida AQSh 11.5 poyafzal o‘lchamidan Yaponiya 295 o‘lchamiga o‘tkazadigan funksiyani topish mumkinmi? Albatta, bu funksiyani $h(x)$ deb belgilaymiz.

Bilamizki, AQSh ning x o‘lchami Fransiyaning $g(x)$ o‘lchamiga mos keladi, $g(x)=\frac{4x+92}{3}$.



Shunga ko‘ra, agar x ning o‘rniga $g(x)$ ni keltirib qo‘ysak, Amerika o‘lchamini Yaponiya o‘lchamiga o‘tkazish mumkin:

$$f(g(x)) = f\left(\frac{4x+92}{3}\right) = \frac{15\cdot\frac{4x+92}{3}-100}{2} = \frac{5(4x+92)-100}{2} = 10x+180$$

Bu tenglikni h deb bilgilasak, u holda h funksiya AQSh poyafzal o‘lchamini Yaponiya o‘lchamiga o‘tkazib beradigan funksiya bo‘ladi:

$h:\quad h(x) = 10x+180$. $\qquad h(11\frac{1}{2} = 10\cdot 11\frac{1}{2} + 180 = 295$.

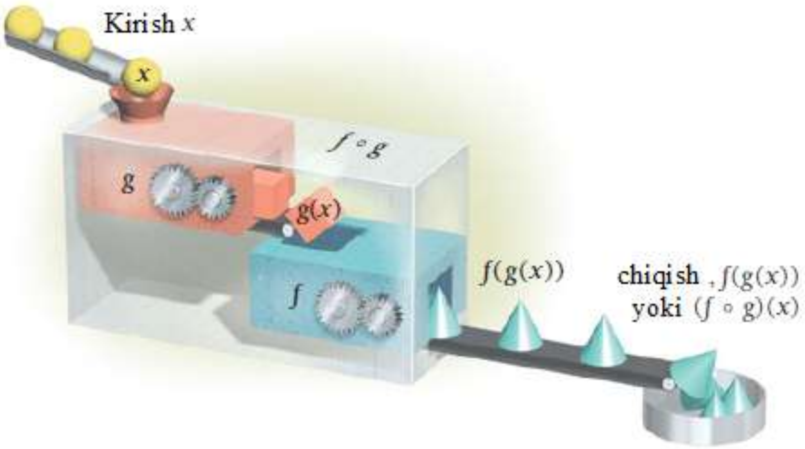
h funksiya f va g ning kompozitsiyasidan iborat bo‘ladi va **“ f funksiya g ning funksiyasi” yoki “ f kompozitsiya g ”** deb o‘qiladi.

Ta’rif. f **va** g **funksiyalarning kompozitsiyasi** deb,

$$(f\circ g)(x) = f(g(x))$$

tenglik bilan ifodalanadigan funksiyalarga aytiladi.

Kompozitsiyani hosil qiladigan mashinani quyidagicha tasvirlashimiz mumkin:



$f(g(x))$ ni topish uchun $f(x)$ da x ning o'rniga $g(x)$ funksiyani qo'yamiz. $g(x)$ funksiya $f(x)$ ning aniqlanish sohasida bo'lishi kerak.

6-misol. $f(x) = x^3$ va $g(x) = 1 - x^2$ funksiyalarning $f(g(x))$ va $g(f(x))$ kompozitsiyalarini toping.

Yechilishi: ► 1) $f(g(x)) = f(1-x^2) = (1-x^2)^3 = 1 - 3x^2 + 3x^4 - x^6$;

2) $g(f(x)) = g(x^3) = 1 - \left(x^3\right)^2 = 1 - x^6$. ◄

7-misol. $f(x) = \sqrt{x}$ va $g(x) = x - 1$ funksiyalarning $f(g(x))$ va $g(f(x))$ kompozitsiyalarini toping.

Yechilishi: ► 1) $f(g(x)) = f(x-1) = \sqrt{x-1}$;

2) $g(f(x)) = g(\sqrt{x}) = \sqrt{x} - 1$. ◄

8-misol. $f(x) = \sqrt{x^3}$ va $g(x) = 1 + 2x$ funksiyalarning $f(g(x))$ va $g(f(x))$ kompozitsiyalarini toping.

Yechilishi: ► 1) $f(g(x)) = f(1+2x) = \sqrt{(1+2x)^3}$;

2) $g(f(x)) = g(\sqrt{x^3}) = 1 + 2\sqrt{x^3}$. ◀

Yuqorida keltirilgan misollardan ko‘rinadiki, $f(g(x)) \neq g(f(x))$ har doim o‘rinli, ya’ni g ning f ga kompozitsiyasi bilan f ning g ga kompozitsiyasi teng kuchli emas.

3-vazifa. 7- va 8- misollarda berilgan funksiyalarning $(f \circ f)(x)$ va $(g \circ g)(x)$ kompozitsiyalarini toping.

2.8.3. Murakkab funksiyaning hosilasi

Funksiyalar kompozitsiyasini qanday differensiallaymiz? Quyidagi teorema shu haqida:

8-teorema: Zanjir qoidasi: Murakkab funksiyaning hosilasi

$$[(f \circ g)(x)]' = [f(g(x))]' = f'(g(x)) \cdot g'(x)$$ ga teng.

Darajali funksiya hosilasining umumlashgan formulasi

$$\left([u(x)]^k\right)' = k \cdot [u(x)]^{k-1} \cdot (u(x))'$$

murakkab funksiya hosilasining xususiy holi hisoblanadi.

Zanjir qoidasi ko‘pincha boshqacha yoziladi. Faraz qilaylik, $y = f(u)$ va $u = g(x)$ bo‘lsin. U holda $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ tenglik o‘rinli bo‘ladi.

Zanjir qoidasini yaxshi tushunib olish uchun faraz qiling, dasturchi Ali video o'yinlar yaratishda 1 minut davom qiladigan o'yindan tushadigan foydani (dollar hisobida) hisoblamoqchi. Foydani hisoblashning bir usuli quyidagicha:

Foydani vaqtning funksiyasidan olinadigan hosila deb qaraydi, ya'ni foyda o'yinlar soniga bo'gliq, o'yinlar soni esa vaqtga bo'g'liq.

$$\left[\begin{array}{l}\text{vaqtga nisbatan}\\ \text{foydaning o'zgarishi}\\ \dfrac{dy}{dx}\end{array}\right] = \left[\begin{array}{l}\text{ishlab chiqilgan o'yinlar}\\ \text{soniga nisbatan foydaning}\\ \text{o'zgarishi}\quad \dfrac{dy}{du}\end{array}\right] \cdot \left[\begin{array}{l}\text{vaqtga nisbatan o'yinlar}\\ \text{sonini o'zgarishi}\\ \dfrac{du}{dx}\end{array}\right]$$

9-misol. $y = 2 + \sqrt{u}$ va $u = x^3 + 1$ funksiyalar berilgan $\dfrac{dy}{du}$, $\dfrac{du}{dx}$ va $\dfrac{dy}{dx}$ hisoblang.

Yechilishi: ► 1) $\dfrac{dy}{du} = \left[2 + \sqrt{u}\right]' = \dfrac{1}{2\sqrt{u}}$;

2) $\dfrac{du}{dx} = (x^3 + 1)' = 3x^2$;

3) $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{1}{2\sqrt{u}} \cdot 3x^2 = \dfrac{3x^2}{2\sqrt{x^3 + 1}}$. ◄

4-vazifa. Agar $y = u^2 + u + 1$ va $u = x^2 + x + 1$ bo'lsa, $\dfrac{dy}{dx}$ ni toping.

10-misol. **Tadbirkorlik.** Yangi qurilma yaratildi va bozorga chiqarildi. U hammaga manzur bo‘la boshladi. t vaqtda (hafta hisobida) qurilmaning N donasi sotilgan bo‘lsa va quyidagi funksiyani qanoatlantirsa,

$$N(t)=\frac{250000t^2}{(2t+1)^2},\ \text{bunda}\ \ t>0$$

funksiya hosilasini toping. Funksiyaning $N'(52)$ va $N'(208)$ qiymatlardagi hosilalari nimani anglatadi?

Yechilishi: ▶ $N'(t)$ hosilani topamiz.

$$N'(t)=\left(\frac{250000t^2}{(2t+1)^2}\right)'=\frac{500000t\cdot(2t+1)^2-250000t^2\cdot 2(2t+1)\cdot 2}{(2t+1)^4}=$$

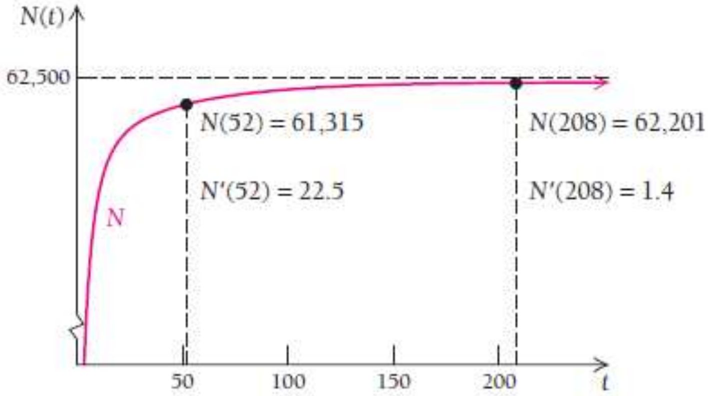
$$=\frac{500000t\cdot(4t^2+4t+1-4t^2-2t)}{(2t+1)^4}=\frac{500000t(2t+1)}{(2t+1)^4}=\frac{500000t}{(2t+1)^3};$$

$t=52$ da $N'(t)$ ni hisoblaymiz: $N'(52)=\frac{500000t}{(2t+1)^3}=\frac{500000\cdot 52}{(2\cdot 52+1)^3}\approx 22.5$

Bu qiymat qurilmaning 52 haftadan so‘ng (1-yilda) haftasiga 22.5 dona ko‘p sotilib boshlashini bildiradi.

$t=208$ da $N'(t)$ ni hisoblaymiz: $N'(52)=\frac{500000t}{(2t+1)^3}=\frac{500000\cdot 208}{(2\cdot 208+1)^3}\approx 1.4$

Topilgan qiymat qurilmaning 4 yildan so‘ng haftasiga 1.4 ta ko‘p sotilib boshlashini bildiradi. Nimaga bunday? Nima ro‘y beradi?



Grafikdan ko‘rish mumkinki, egri chiziqqa o‘tkazilgan urinmalar sotilgan qurilmaning sonidagi o‘zgarishni ko‘rsatadi va haftalar o‘tgani sayin urinmalar ham tenglasha boshladi. Ehtimol, bozor bu qurilmadan to‘lib qolgandir. ◄

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-48 misollarda funksiyalar hosilasini hisoblang:

1. $y=(5x+1)^2$

2. $y=(3-5x)^2$

3. $y=(7-x)^{56}$

4. $y=(8-x)^{100}$

5. $y=\sqrt{2+8x}$

6. $y=\sqrt{1-x}$

7. $y=\sqrt{3x^2-4}$

8. $y=\sqrt{6x^2-1}$

9. $y=(8x^2-6)^{-40}$

10. $y=(4x^2+6)^{-50}$

11. $y=(x-4)^8(2x+3)^6$

12. $y=(x+7)^5(3x-2)^8$

13. $y=\dfrac{2}{(3x+13)^2}$

14. $y=\dfrac{3}{(5x-4)^2}$

15. $y=\dfrac{4x^2}{(7-5x)^3}$

16. $y=\dfrac{2x^3}{(4-9x)^5}$

17. $y=(x^3-4)^3-(x^8+3)^4$

18. $y=(x^3+3)^5-(x^7+1)^4$

19. $y=x^3-(300-x)^2$

20. $y=x^2+(200-x)^2$

21. $f(x)=\sqrt{x}+(3-x)^2$

22. $f(x)=\sqrt[3]{3x-1}+(4-x)^2$

23. $f(x)=-5x(3-2x)^4$

24. $f(x)=-3x(4+5x)^6$

25. $f(x)=(3x-4)^7(2x+1)^5$

26. $f(x)=(5x+2)^4(2x-3)^8$

27. $g(t)=t^7\sqrt{4t-1}$

28. $g(t)=t^3\sqrt{5t+2}$

29. $g(t)=\sqrt[3]{t^5+6t}+1$

30. $g(t)=\sqrt[4]{t^2-5t+2}$

31. $F(x)=\left(\dfrac{3x-1}{5x+2}\right)^2$

32. $F(x)=\left(\dfrac{3x}{x^2+2}\right)^4$

33. $G(x)=\sqrt{\dfrac{2x-1}{3x+1}}$

34. $G(x)=\sqrt{\dfrac{3x+2}{5-x}}$

35. $y=(2x^3-3x^2+4x+1)^{100}$

36. $y=(7x^4-6x^3+x)^{254}$

37. $F(x)=\left(\dfrac{4x-1}{5x+3}\right)^{-4}$

38. $F(x)=\left(\dfrac{2-7x}{3+5x}\right)^{-6}$

39. $G(x)=\sqrt{\dfrac{x^2+x}{x^2-x}}$

40. $G(x)=\sqrt[3]{\dfrac{4-x^3}{x-x^2}}$

41. $y=\dfrac{(3+2x)^4}{(2-5x)^5}$

42. $y=\dfrac{(5x-4)^7}{(4x+3)^3}$

43. $f(x)=14(3x-4)^{3/4}(2x+1)^{2/3}$

44. $f(x)=6\cdot\sqrt[3]{3x-5}(x^4-6x)^3$

45. $y=(2x^3+(4x-5)^2)^6$

46. $y=(-x^5+4x+\sqrt{2x+1})^3$

47. $g(t)=\sqrt{t^2+\sqrt{1-3t}}$

48. $g(t)=\sqrt[3]{2t+(t^2+t)^4}$

49-54 misollarda funksiyalar uchun $\frac{dy}{du}$, $\frac{du}{dx}$ va $\frac{dy}{dx}$ differensiallarni toping:

49. $y=\sqrt{u}$ va $u=x^2-1$

50. $y=\frac{15}{u^3}$ va $u=2x+1$

51. $y=u^{50}$ va $u=4x^3-2x^2+1$

52. $y=\frac{u+1}{u-1}$ va $u=\sqrt{x}+1$

53. $y=u(2+u)$ va $u=x^3-2x$

54. $y=(u-1)(u+1)$ va $u=x^3+1$

55-58 misollarda funksiyalar uchun $\frac{dy}{dx}$ differensialni toping:

55. $y=3u+5u^2$ va $u=x^3+2$

56. $y=u^3-7u^2$ va $u=x^2+3$

57. $y=\sqrt[3]{2u+5}$ va $u=x^2-x$

58. $y=\sqrt{u(2+u)}$ va $u=x^2-7$

59-60 misollarda funksiyalar uchun $\frac{dy}{dt}$ differensialni toping:

59. $y=\frac{3}{u^2+u}$ va $u=3t+5$

60. $y=\frac{1}{3u^5-7}$ va $u=7t^2-1$

61-64 misollarda funksiya grafigiga berilgan nuqtada o‘tkazilgan urinma tenglamasini tuzing:

61. $y=\sqrt{x^2+3x}$ funksiyaga (1; 2) nuqtada;

62. $y=(x^3-4x)^{10}$ funksiyaga (2; 0) nuqtada;

63. $y=x\sqrt{2x+3}$ funksiyaga (3; 9) nuqtada;

64. $y=\left(\frac{2x+3}{x-1}\right)^3$ funksiyaga (2; 343) nuqtada.

65-68 misollarda $h(x)=(f\circ g)(x)$ **bo‘lsa,** $f(x)$ **va** $g(x)$ **larni ajrating:**

65. $h(x)=(3x^2-7)^5$

66. $h(x)=\frac{1}{\sqrt{9x+2}}$

67. $h(x)=\frac{x^3+1}{x^3-1}$

68. $h(x)=(\sqrt{x}+5)^4$

69-72 misollarda $(f\circ g)(x)$ **ni berilgan qiymatlarda hisoblang:**

69. $f(u)=u^3,\quad g(x)=u=2x^4+1$ **bo‘lsa,** $(f\circ g)'(-1)$ **;**

70. $f(u)=\frac{u+1}{u-1},\quad g(x)=u=\sqrt{x}$ **bo‘lsa,** $(f\circ g)'(4)$ **;**

71. $f(u)=\sqrt[3]{u},\quad g(x)=u=3x^2+1$ **bo‘lsa,** $(f\circ g)'(2)$ **;**

72. $f(u)=2u^5,\quad g(x)=u=\frac{3-x}{4+x}$ **bo‘lsa,** $(f\circ g)'(-10)$**.**

73-76 misollarda $[(f\circ f)(x)]'$ **va** $[(f\circ f\circ f)(x)]'$ **ni hisoblang:**

73. $f(x)=x^2+1$ uchun $[(f\circ f)(x)]'$ ni hisoblang;

74. $f(x)=x+\sqrt{x}$ uchun $[(f\circ f)(x)]'$ ni hisoblang;

75. $f(x)=x^2+1$ uchun $[(f\circ f\circ f)(x)]'$ ni hisoblang;

76. $f(x)=\sqrt[3]{x}$ uchun $[(f\circ f\circ f)(x)]'$ ni hisoblang;

77- 84 misollarda iqtisodiyot va tadbirkorlikdagi tadbiqlari:

77. Umumiy daromad. Umumiy daromad $D(x)=1000\sqrt{x^2-0.1x}$ funksiyasi savdo nuqtalaridan tushgan daromadni ifodalaydi (1000 dollar hisobida). Agar 20 savdo nuqtasi sotilsa, umumiy daromad o‘zgaradi. O‘rtacha daromadni toping.

78. Umumiy tannarx. Umumiy tannarx $T(x)=2000(x^2+2)^{1/3}+700$ funksiyasi savdo nuqtalaridagi mahsulotlar tannarxini ifodalaydi (1000 dollar hisobida). Agar 20 ta savdo nuqtasi qo‘shilsa, umumiy tannarx o‘zgaradi. O‘rtacha tannarxni toping.

79. Umumiy foyda. Savdo nuqtalaridagi umumiy tannarx $T(x)=2000(x^2+2)^{1/3}+700$ va umumiy daromad $D(x)=1000\sqrt{x^2-0.1x}$ funksiyalari (1000 dollar hisobida) bo‘lsa, 20 ta savdo nuqtasi qo‘shilganda yoki kamayganda, o‘rtacha foydani toping.

80. Umumiy tannarx. Agar kompaniya x savdo nuqtasi ochsa, umumiy tannarx $T(x)=\sqrt{5x^2+60}$ funksiya bilan approksimatsiyalanadi (1000 dollar hisobida). Shunga ko‘ra, hozirdan boshlab t oyda ishlab chiqarishni ko‘paytirishni rajalashtirdi: $x(t)=20t+40$. 4 oy ichida sarf-harajat qanchaga oshadi?

81. Murakkab foiz. Agar kvartaliga i foiz bilan 1000$ investitsiya kiritilgan bo‘lsa, 5 yilda u $A=1000\left(1+\frac{i}{4}\right)^{20}$ ga oshadi. $\frac{dA}{di}$ ni hisoblang va u nimani anglatishini tushuntiring.

82. Murakkab foiz. Agar yiliga i foiz bilan 1000$ investitsiya kiritilgan bo‘lsa, 3 yilda u $A = 1000(1+i)^3$ ga oshadi. $\frac{dA}{di}$ ni hisoblang va u nimani anglatishini tushuntiring.

83. Talab. Faraz qiling, biror mahsulotga talab funksiyasi $P(x) = \frac{80000}{x}$ bo‘lsin. Mahsulot narxi vaqt funksiyasidan $x(t) = 1.6t + 9$ iborat bo‘lsin.

a) Talab funksiyasini vaqt funksiyasi sifatida aniqlang;

b) $t = 100$ kundan keyingi mahsulotga bo‘lgan talabni aniqlang.

84. Foyda. Kompaniya noutbuk savdosi bilan shug‘ullanadi va umumiy foyda $F(x) = 0.08x^2 + 80x$ funksiya bilan hisoblanadi. Bunda $x -$ ishlab chiqarilgan va sotilgan noutbuklar soni $x(t) = 5t + 1$ bo‘lsa,

a) Umumiy foydani vaqt funksiyasi sifatida aniqlang;

b) $t = 48$ oydan keyingi o‘rtacha foydani aniqlang.

2.9. Yuqori tartibli differensiallar

2.9.1. Yuqori tartibli hosila va differensial

Bizga $y = f(x) = x^5 - 3x^4 + x + 252$ funksiya berilgan bo‘lsin. Undan $f'(x)$ hosila olamiz: $y' = f'(x) = 5x^4 - 12x^3 + 1$.

Funksiyaning hosilasi yana differensiallanuvchi ekan, yana hosila olish mumkin. Keyingi hosilalar uchun belgilashlar kiritamiz. U holda 2-marta olinadigan hosilani $f''(x) = [f'(x)]'$ deb belgilaymiz:

$$y'' = f''(x) = (5x^4 - 12x^3 + 1)' = 20x^3 - 36x^2.$$

Xuddi shuningdek, 3-, 4- va h.k.hosilalarni olish mumkin.

$y''' = f'''(x) = (20x^3 - 36x^2)' = 60x^2 - 72x$;

$y^{(IV)} = f^{(IV)}(x) = (60x^2 - 72x)' = 120x - 72$;

$y^{(V)} = f^{(V)}(x) = (120x - 72)' = 120$;

$y^{(VI)} = f^{(VI)}(x) = 120' = 0$;

$\ldots$

$y^{(n)} = 0$, $n \geq 6$ butun sonlar uchun.

Endi yuqori tartibli differensial belgisini kiritamiz.

Birinchi tartibli differensial $\frac{dy}{dx}$ ko‘rinishda: $\frac{dy}{dx} = 5x^4 - 12x^3 + 1$;

Ikkinchi tartibli differensial $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 20x^3 - 36x^2$;

Uchinchi tartibli differensial $\frac{d^3y}{dx^3}=60x^2-72x$;

To‘rtinchi tartibli differensial $\frac{d^4y}{dx^4}=120x-72$;

Beshinchi tartibli differensial $\frac{d^5y}{dx^5}=120$ ko‘rinishda yoziladi.

1-misol. $y=\frac{1}{x}$ funksiya uchun $\frac{d^2y}{dx^2}$ ni toping.

Yechilishi: ► Dastlab, 1-tartibli differensialni $\frac{dy}{dx}=\left(x^{-1}\right)'=-1\cdot x^{-2}=-\frac{1}{x^2}$ topamiz. Endi 2-tartibli differensialni aniqlaymiz:

$$\frac{d^2y}{dx^2}=\left(-x^{-2}\right)'=-(-2)x^{-3}=\frac{2}{x^3}. \qquad ◄$$

2-misol. $y=(x^2+10x)^{20}$ funksiyaning y' va y'' hosilalarini toping.

Yechilishi: ► 1-tartibli hosila: $y'=20(x^2+10x)^{19}\cdot(2x+10)=$

$$=20(x^2+10x)^{19}\cdot 2(x+5)=40(x^2+10x)^{19}(x+5);$$

2-tartibli hosila:

$$y''=\left[40(x^2+10x)^{19}(x+5)\right]'=40(19(x^2+10x)^{18}(2x+10)(x+5)+(x^2+10x)^{19}\cdot 1=$$

$$=40(x^2+10x)^{18}(38(x+5)^2+x^2+10x)=.$$

$$=40(x^2+10x)^{18}(38x^2+380x+950+x^2+10x)=40(x^2+10x)^{18}(39x^2+390x+950) \;◄$$

3-misol. $y=\sin x$ funksiyaning $n-$ tartibli hosilasini toping.

Yechilishi: ► Berilgan funksiyani $n-$ tartibli hosilasini topamiz. Umumiy formula chiqarishga harakat qilamiz:

$$y' = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$y'' = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + 2\cdot\frac{\pi}{2}\right),$$

$$y''' = \cos\left(x + 2\cdot\frac{\pi}{2}\right) = \sin\left(x + 3\cdot\frac{\pi}{2}\right),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$y^{(n)} = \cos\left(x + (n-1)\frac{\pi}{2}\right) = \sin\left(x + n\cdot\frac{\pi}{2}\right). \quad ◀$$

2.9.2. Tezlik va tezlanish

Biz ko‘rdikki, funksiya hosilasi oniy o‘zgarishni ifodalaydi.

Agar funksiya vaqt o‘zgarishi bilan masofa o‘zgarishini bildirsa, u holda oniy hosilaga **tezlik** deyiladi. Tezlikni v harfi bilan belgilaymiz.

Ta’rif. Tezlik – yo‘lning vaqtga nisbatan o‘zgarishidir.
Ya’ni moddiy nuqtaning tezligi, boshlang‘ich nuqtadan t vaqt o‘tguncha bosib o‘tilgan yo‘lni ifodalaydi:

$$v(t) = s'(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t}$$

4-misol. Fizika. Tezlik. Faraz qiling, ob’yekt shunday harakatlanmoqdaki, uning jo‘nagan nuqtasidan boshlab bosib o‘tgan masofasi (mil hisobida) vaqtning funksiyasini ifodalaydi: $s(t) = 10t^2$.

a) $t=2$ soat va $t=5$ soat orasidagi o‘rtacha tezlikni toping;

b) $t=4$ soatdagi oniy tezlikni toping.

Yechilishi: ▶ a) $t=2$ soat va $t=5$ soat orasidagi o‘rtacha tezlikni topish uchun

$$\frac{\Delta s}{\Delta t}=\frac{s(5)-s(2)}{5-2}=\frac{10\cdot 5^2-10\cdot 2^2}{3}=\frac{210}{3}=70\frac{mil}{soat};$$

b) $\lim\limits_{\Delta t\to 0}\frac{s(t+\Delta t)-s(t)}{\Delta t}=s'(t)$ biz bu limitni hisoblashning oson usulini 2.5 mavzuda o‘rgangan edik. Shunga ko‘ra, $s'(t)=\left(10t^2\right)'=20t$.

$$s'(4)=20\cdot 4=80\frac{mil}{soat}$$ ◀

Tezlik – vaqt funksiyasi hisoblanadi. Samolyot ko‘tarilayotganda yoki transport vositasi birdaniga to‘xtaganida yo‘lovchilar tezlik o‘zgarishini sezishadi. Mana shu tezlik o‘zgarishiga **tezlanish** deyiladi. Aytaylik, avtomobil A 8.4 sekundda 65 mil/soat tezlikka erishdi, avtomobil B esa 8 sekundda 65 mil/soat tezlikka erishdi, bundan ko‘rinadiki, B avtomobil A ga qaraganda kattaroq tezlanish bilan harakatlanayotgan ekan. Tezlanishni a harfi bilan belgilaymiz. Tezlanishni tezlikning o‘rtacha o‘zgarishi deb qarash mumkin.

Ta’rif. Tezlanish – tezlikning vaqtga nisbatan o‘zgarishidir:

$$a(t)=v'(t)=s''(t)=\lim_{\Delta t\to 0}\frac{v(t+\Delta t)-v(t)}{\Delta t}$$

5-misol. Fizika. Tezlanish. $s(t)=10t^2$ uchun $v(t)$ va $a(t)$ ni toping. Bunda s ob'yektning jo'nagan nuqtasidan boshlab bosib o'tgan yo'li (mil hisobida). $t=4$ soatdagi bosib o'tgan yo'lini, tezligini va tezlanishini aniqlang.

Yechilishi: ▶ $s(t)=10t^2$ ma'lum. Bundan

$v(t)=s'(t)=20t$ va $a(t)=v'(t)=s''(t)=(20t)'=20$ ni topamiz.

Endi $t=4$ bo'lganda ob'yekt

$s(4)=10t^2=10\cdot 4^2=160$ mil masofani o'tgan;

$v(t)=s'(t)=20t=20\cdot 4=80$ mil/soat tezlikka ega;

$a(t)=v'(t)=20$ mil/soat^2 tezlanish bilan harakatlangan. ◀

4-misoldan **tezlanish birligi** quyidagiga teng:

$$\text{Tezlanish}=\frac{\text{tezlik o'zgarishi}}{\text{vaqt o'zgarishi}}=\frac{\text{mil}/\text{soat}}{\text{soat}}=\frac{\text{mil}}{\text{soat}^2}$$

6-misol. Fizika. Erkin tushish. Agar havoning qarshiligini hisobga olmasak, biror balandlikdan tashlab yuborilgan jism $s(t)=4.905t^2$ (metr) masofani o'tib, t sekundda yerga tushadi. Aytaylik jism biror balandlikdan tushib ketdi.

a) tushayotib, 5 sekundda qancha masofani o'tadi?

b) 5 sekundda qanday tezlik bilan tushayotgan bo'ladi?

v) 5 sekundda jism qanday tezlanish olgan?

Yechilishi: ▶ a) $s(t)=4.905t^2=4.905\cdot 5^2=122.625$ m;

b) $v(t)=s'(t)=2\cdot 4.905t=9.81t$

$$v(t) = 9.81 \cdot 5 = 49.05 \text{ m/sek};$$

v) $a(t) = v'(t) = 9.81 \text{ m/sek}^2$. ◀

2.8 bo‘limdagi 10-misolda t haftada ishlab chiqarilgan va bozorda sotilgan mahsulotlar sonini N bilan belgilagan edik. Uning 1-tartibli hosilasi har doim musbat (faqat o‘suvchi) bo‘lishi, savdoning o‘sib borayotganini bildiradi. Lekin ma’lum muddatdan keyin mahsulot savdosi nolga teng bo‘lib qildi. Quyidagi misolda biz 2-tartibli hosila yordamida bu nimani anglatishini tushunib olamiz.

7-misol. Tadbirkorlik. 2.7 bo‘limdagi 10-misolda t haftada ishlab chiqarilgan va bozorda sotilgan mahsulotlar soni $N(t) = \dfrac{250000t^2}{(2t+1)^2}$ (bunda $t > 0$) tenglik bilan ifodalanadi. $N'(t) = \dfrac{500000t}{(2t+1)^3}$ kattalik haftasiga sotilgan o‘rtacha miqdorni bildiradi. $N''(t)$ ni toping va $N''(52)$ ni hisoblang. Topilgan qiymat nimani bildirishini tushuntiring.

Yechilishi: ▶ $N''(t) = \left(\dfrac{500000t}{(2t+1)^3}\right)' = 500000\dfrac{(2t+1)^3 - t \cdot 3(2t+1)^2 \cdot 2}{(2t+1)^6} =$

$$= 500000\frac{(2t+1)^2[2t+1-6t]}{(2t+1)^6} = \frac{500000(1-4t)}{(2t+1)^4}$$

$t = 52$ haftada $N''(52)$ ni topamiz:

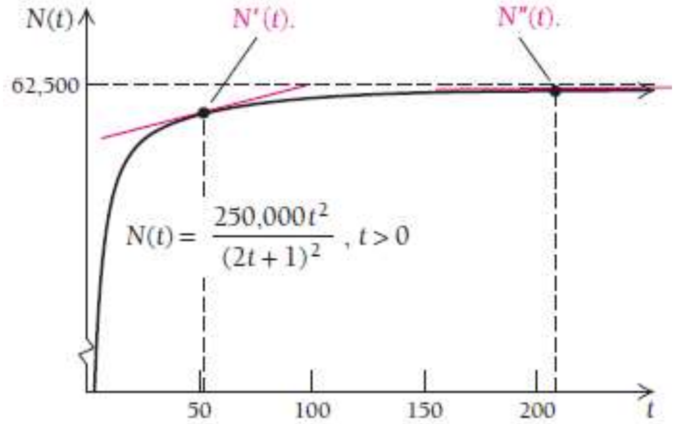
$$N''(52) = \frac{500000(1-4t)}{(2t+1)^4} = \frac{500000(1-4 \cdot 52)}{(2 \cdot 52+1)^4} \approx -0.852$$

-0.852 qiymat 52 haftadan keyin har hafta mahsulot savdosi tushib borishini bildiradi. 52-haftada ham savdo o‘sib borayotgan edi, buni

$N'(52) > 0$ dan ko‘rish mumkin, lekin oldingidek emasdi.

$t = 208$ haftada $N''(208)$ ni topamiz:

$$N''(208) = \frac{500000(1-4t)}{(2t+1)^4} = \frac{500000(1-4\cdot 208)}{(2\cdot 208+1)^4} \approx -0.014$$



208 haftada (4 yildan so‘ng) qurilma savdosi o‘smay qoldi. Buni 2-tartibli hosila yordamida aniqlash mumkin. 2-tartibli hosila nolga yaqinlashib bormoqda. ◄

2.9.3. Hosilalar jadvali

1) $y = C,$ $\quad y' = C' = 0;$ $\quad C$ – const;

2) $y = x\,,$ $\quad y' = x' = 1\,,$ $\quad x - erkli\ o‘zgaruvchi;$

3) $y = u^{\alpha}\,,$ $\quad y' = (u^{\alpha})' = \alpha \cdot u^{\alpha-1} \cdot u';$

4) $y = \sqrt{u}\,,$ $\quad y' = \left(\sqrt{u}\right)' = \frac{1}{2\sqrt{u}} \cdot u'$

5) $y = \frac{1}{u}\,,$ $\quad y' = \left(\frac{1}{u}\right)' = -\frac{1}{u^2} \cdot u'$

6) $y = a^u\,,$ $\quad y' = (a^u)' = a^u \cdot lna \cdot u',$ $\quad a - const, a > 0, a \neq 1;$

7) $y = e^u\,,$ $\quad y' = (e^u)' = e^u \cdot u';$

8) $y = u^v\,,$ $\quad y' = (u^v)' = v \cdot u^{v-1} \cdot u' + u^v \cdot lnu \cdot v';$

$$9)\ y = \log_a u, \quad y' = (\log_a u)' = \frac{1}{u \cdot lna} \cdot u', \quad a - const, a > 0, \quad a \neq 1;$$

$$10)\ y = lnu, \qquad y' = (lnu)' = \frac{1}{u} \cdot u';$$

$$11)\ y = sinu, \qquad y' = (sinu)' = cosu \cdot u';$$

$$12)\ y = cosu, \qquad y' = (cosu)' = -sinu \cdot u';$$

$$13)\ y = tgu, \qquad y' = (tgu)' = \frac{1}{cos^2u} \cdot u';$$

$$14)\ y = ctgu, \qquad y' = (ctgu)' = -\frac{1}{sin^2u} \cdot u';$$

$$15)\ y = arcsinu, \qquad y' = (arcsinu)' = \frac{1}{\sqrt{1-u^2}} \cdot u';$$

$$16)\ y = arccosu, \qquad y' = (arccosu)' = -\frac{1}{\sqrt{1-u^2}} \cdot u';$$

$$17)\ y = arctgu, \qquad y' = (arctgu)' = \frac{1}{1+u^2} \cdot u';$$

$$18)\ y = arcctgu, \qquad y' = (arcctgu)' = -\frac{1}{1+u^2} \cdot u';$$

$$19)\ y = shu, \qquad y' = (shu)' = chu \cdot u';$$

$$20)\ y = chu, \qquad y' = (chu)' = shu \cdot u';$$

$$21)\ y = thu, \qquad y' = (thu)' = \frac{1}{ch^2u} \cdot u';$$

$$22)\ y = cthu, \qquad y' = (cthu)' = -\frac{1}{sh^2u} \cdot u';$$

$$23)\ y = \frac{1}{u^n}, \qquad y' = \left(\frac{1}{u^n}\right)' = -\frac{n}{u^{n+1}} \cdot u'.$$

2.9.4. Parametrik ko‘rinishda berilgan va teskari funksiya hosilalari

x va y o‘zgaruvchilar orasidagi funksional bog‘lanishni har doim ham $y=f(x)$ oshkor ko‘rinishda yoki $F(x,y)=0$ oshkormas ko‘rinishda yozish qulay bo‘lmaydi. Ba’zan yordamchi o‘zgaruvchi t ni kiritib, x va y o‘zgaruvchilarni t ning funksiyasi sifatida ifodalash qulay bo‘ladi:

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}.$$

Bu tenglama funksiyaning parametrik berilishi bo‘lib, t ning ixtiyoriy qiymatiga x ning aniq qiymati va y ning aniq qiymati mos keladi. x va y ning qiymatlari juftiga tekislikda $M(x,y)$ nuqta mos keladi. t parametr aniqlanish sohasidan hamma qiymatlarni qabul qilganda $M(x,y)$ nuqta xOy tekislikda biror chiziqni chizadi. Yuqoridagi tenglamani shu **chiziqning parametrik tenglamasi** deyiladi. y ning x ga oshkor bog‘liqligini topish uchun sistema tenglamalaridan t parametrni topish kerak. Buning uchun bu sistemaning birinchi tenglamasidan t ni x ning funksiyasi sifatida ifodalaydi: $t = u(x)$, buni ikkinchi tenglamaga qo‘yib, $y = \psi(u(x))$ ga yoki $y = f(x)$ ga ega bo‘lamiz.

8-misol. ► To‘g‘ri chiziqning tekislikdagi ushbu $\begin{cases} x = x_0 + mt \\ y = y_0 + nt \end{cases}$ (bunda $m, n -$ yo‘naltiruvchi vektor koordinatalari) parametrik tenglamalarni quyidagicha yozamiz:

$$\frac{x-x_0}{m}=t\,, \qquad \frac{y-y_0}{n}=t\,.$$

Bundan

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}$$

to‘g‘ri chiziqning kanonik tenglamasi kelib chiqadi. ◄

9-misol. ► Aylananing parametrik tenglamasi $\begin{cases} x=Rcost \\ y=Rsint \end{cases}$ berilgan bo‘lsin. Undan t ni topamiz, buning uchun tenglamaning har birini kvadratga ko‘taramiz: $\begin{cases} x^2=R^2cos^2t \\ y^2=R^2sin^2t \end{cases}$ va ularni qo‘shamiz, natijada $x^2+y^2=R^2$ aylana tenglamasi kelib chiqadi. ◄

Parametrik berilgan $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$ funksiya hosilasini topish uchun formula chiqaramiz; bunda $x=\varphi(t)$ funksiya teskari funksiyaga ega. Bu yerda u ni x ning murakkab funksiyasi deb hisoblash mumkin, bunda t oraliq argument. Shu sababli murakkab funksiyani differensiallash qoidasiga ko‘ra: $y'_x=y'_t\cdot t'_x$. Ammo bunda x o‘zgaruvchining t funksiyasi emas, balki t o‘zgaruvchining x funksiyasi berilgan, shu sababli teskari funksiyani differensiallash qoidasiga ko‘ra $t'_x=\frac{1}{x'_t}$ buni yuqoridagi tenglikka qo′yib,

$$y'_x=y'_t\cdot\frac{1}{x'_t}$$

ni hosil qilamiz.

10-misol. Quyidagi funksiya hosilasini toping: $\begin{cases} x = Rcost \\ y = Rsint \end{cases}$.

Yechilishi: ► $y'_x = \frac{Rcost}{-Rsint} = -ctgt$. ◄

11-misol. Quyidagi funksiya hosilasini toping:

$$\begin{cases} x = a(t - sint) \\ y = a(1 - cost) \end{cases}.$$

Yechilishi: ► formulaga ko'ra, $y'_x = \frac{a \cdot sint}{a(1-cost)} = ctg\frac{t}{2}$. ◄

12-misol. $\begin{cases} y = t^3 + t^2 + 1 \\ x = \dfrac{1}{t} \end{cases}$ tenglama bilan berilgan funksiyaning ikkinchi tartibli hosilasini toping.

Yechilishi: ► Parametrik ko\`rinishda berilgan funksiya hosilasini topish uchun

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'_t}{x'_t}$$

formuladan foydalanamiz. U holda ikkinchi tartibli hosila

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}}{dx} = \frac{d(y'_x)}{dx} = \frac{(y'_x)'_t}{x'_t}$$

formula orqali topiladi. Demak, $\frac{dy}{dx} = \frac{(t^3 + t^2 + 1)'_t}{\left(\frac{1}{t}\right)'_t} = \frac{3t^2 + 2t}{-\frac{1}{t^2}} = -3t^4 - 2t^3$

$$\frac{d^2y}{dx^2}=\frac{\left(-3t^4-2t^3\right)'_t}{\left(\frac{1}{t}\right)'_t}=\frac{-12t^3-6t^2}{-\frac{1}{t^2}}=12t^5+6t^4=6t^4\left(2t+1\right).$$ ◄

Teskari funksiyaning hosilasi.

Agar $y=f(x)$ funksiya uchun teskari funksiya mavjud bo`lsa, u holda $x=g(y)$ teskari funksiyaning differensiali quyidagiga teng bo‘ladi (bu yerda $\frac{dg}{dy}=g'(y)\neq 0$):

$$f'(x)=\frac{1}{g'(x)}$$

13-misol. $\left(arctg\,x\right)'=\frac{1}{1+x^2}$ tenglikni isbotlang.

Yechilishi: ► Tenglikni isbot qilish uchun, $y=arctg\,x$ funksiya hosilasini teskari funksiya hosilasi yordamida hisoblaymiz.

$$y=arctg\,x \quad\Rightarrow\quad \text{tg}(y)=\text{tg}(arctg\,x) \quad\Rightarrow\quad x=\text{tg}\,y$$

$x=\text{tg}\,y$ tenglikning ikkala tomonidan hosila olamiz:

$$(x)'=(\text{tg}\,y)' \quad\Rightarrow\quad 1=\frac{1}{\cos^2 y}\cdot y'$$

Bu tenglikdan $y'=\cos^2 y$ tenglik kelib chiqadi. Bundan, $y'=\frac{1}{1+tg^2y}$ va $x=\text{tg}\,y$ ni hisobga olsak, $\left(arctg\,x\right)'=\frac{1}{1+x^2}$ tenglik kelib chiqadi. ◄

MUSTAQIL YECHISH UCHUN MISOLLAR:

1-36 misollarda funksiyalarning 2-tartibli hosilasi va 2-tartibli differensialini toping:

1. $y = x^5 + 9$
2. $y = x^4 - 7$
3. $y = 2x^4 - 5x$
4. $y = 5x^3 + 4x$
5. $y = 4x^2 + 3x - 1$
6. $y = 4x^2 - 5x + 7$
7. $y = 7x + 2$
8. $y = 6x - 3$
9. $y = \frac{1}{x^2}$
10. $y = \frac{1}{x^3}$
11. $y = \sqrt{x}$
12. $y = \sqrt[4]{x}$
13. $f(x) = x^4 + \frac{3}{x}$
14. $f(x) = x^3 - \frac{5}{x}$
15. $f(x) = x^{1/5}$
16. $f(x) = x^{1/3}$
17. $f(x) = 4x^{-3}$
18. $f(x) = 2x^{-2}$
19. $f(x) = (x^2 + 3x)^7$
20. $f(x) = (x^3 + 2x)^6$
21. $f(x) = (2x^2 - 3x + 1)^{10}$
22. $f(x) = (3x^2 + 2x + 1)^5$
23. $f(x) = \sqrt[4]{(x^2 + 1)^3}$
24. $f(x) = \sqrt[3]{(x^2 - 1)^2}$

25. $y = x^{2/3} + 4x$

26. $y = x^{3/2} - 5x$

27. $y = (x^3 - x)^{3/4}$

28. $y = (x^4 + x)^{2/3}$

29. $y = 2x^{5/4} + x^{1/2}$

30. $y = 3x^{4/3} - x^{1/2}$

31. $y = \dfrac{2}{x^3} + \dfrac{1}{x^2}$

32. $y = \dfrac{3}{x^4} - \dfrac{1}{x}$

33. $y = (x^2 + 3)(4x - 1)$

34. $y = (x^3 - 2)(5x + 1)$

35. $y = \dfrac{3x + 1}{2x - 3}$

36. $y = \dfrac{2x + 3}{5x - 1}$

37-44 misollarda funksiyalarning yuqori tartibli hosilalarini toping:

37. $y = x^4, \quad d^4y/dx^4.$

38. $y = x^5, \quad d^4y/dx^4.$

39. $y = x^6 - x^3 + 2x, \quad d^5y/dx^5.$

40. $y = x^7 - 8x^2 + 2, \quad d^6y/dx^6.$

41. $f(x) = x^{-2} - x^{1/2}, \quad f^{(4)}(x).$

42. $f(x) = x^{-3} + 2x^{1/3}, \quad f^{(5)}(x).$

43. $g(x) = x^4 - 3x^3 - 7x^2 - 6x + 9, \quad g^{(6)}(x).$

44. $g(x) = 6x^5 + 2x^4 - 4x^3 + 7x^2 - 8x + 3, \quad g^{(7)}(x).$

HOSILANING TATBIQLARIGA DOIR MISOLLAR

45. Fizika. Agar jism t sekundda $s(t)=t^3+t$ qonuniyat bilan harakat qilayotgan bo‘lsa, uning tezligi va tezlanishi uchun formulalarni toping. 4 sekundda bu jismning tezligi va tezlanishi qanday bo‘ladi?

46. Fizika. Agar jism t sekundda $s(t)=-10t^2+2t+5$ qonuniyat bilan harakat qilayotgan bo‘lsa, uning tezligi va tezlanishi uchun formulalarni toping. 2 sekundda bu jismning tezligi va tezlanishi qanday bo‘ladi?

47. Fizika. Agar jism t sekundda $s(t)=3t+15$ qonuniyat bilan harakat qilayotgan bo‘lsa, uning tezligi va tezlanishi uchun formulalarni toping. 15 sekundda bu jismning tezligi va tezlanishi qanday bo‘ladi?

48. Erkin tushish. Jism biror balandlikdan $s(t)=16t^2$ qonuniyat bilan tushib ketayotgan bo‘lsin.

a) tushayotib, 5 sekundda qancha masofani o‘tadi?

b) 5 sekundda qanday tezlik bilan tushayotgan bo‘ladi?

v) 5 sekundda jism qanday tezlanish olgan?

49- 60 misollarda parametrik ko‘rinishda berilgan funksiyalaring 1- va 2-tartibli hosilalarini toping:

49. $\begin{cases} x = \cos\left(\frac{t}{2}\right) \\ y = t - \sin t \end{cases}$

50. $\begin{cases} x = t^3 + 8 \\ y = t^5 + 2t \end{cases}$

51. $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$

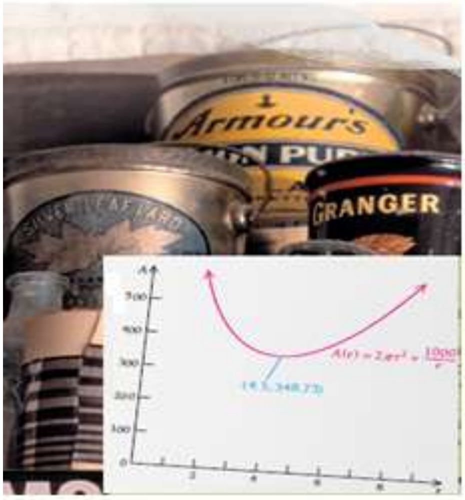
52. $\begin{cases} x = e^{2t} \\ y = \cos t \end{cases}$

53. $\begin{cases} x = 3\cos^2 t \\ y = 2\sin^3 t \end{cases}$

54. $\begin{cases} x = \ln t \\ y = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right) \end{cases}$

55. $\begin{cases} x = \dfrac{1}{t+1} \\ y = \left(\dfrac{t}{t+1}\right)^2 \end{cases}$

56. $\begin{cases} x = \dfrac{2at}{1+t^2} \\ y = \dfrac{a(1-t^2)}{1+t^2} \end{cases}$



III BOB. DIFFERENSIALNING TATBIQLARI

Bu bobni o‘rganishdan maqsad nima?

Ushbu bobda biz differensiallashning tatbiqlarini o‘rganamiz. Funksiyaning maksimal va minimal qiymatlarini topishni o‘rganish bilan biz real hayotdagi ko‘p turdagi masalalarga yechim topamiz. Shuningdek, funksiya grafigini chizish bo‘yicha differensiallash borasida olgan bilimlarimizdan foydalanib, bir-biriga yaqin bo‘lgan funksiyalarni tuza olamiz, ya’ni jadval yoki grafik bilan berilgan funksiyaning taxminiy analitik ifodasini tuzishni o‘rganamiz.

Ishlab chiqarishda asosiy maqsad kam xom ashyo sarflab, ko‘p mahsulot tayyorlashdan iborat, ya’ni sarf-harajatni kamaytirib, samaradorlikni oshirish maqsad qilinadi. Masalan, sanoatda ko‘p partiyalarda ishlab chiqariladigan konserva bankalarini olaylik.

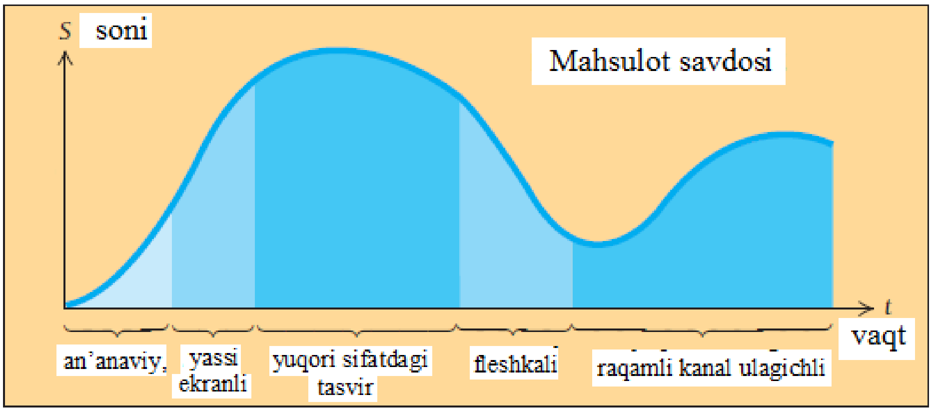
Faraz qiling, quyultirilgan sut uchun mo‘ljallangan hajmi 0.5 lirt bo‘lgan silindrik quti yasash kerak. Uning radiusi va balandligini qanday optimal o‘lchamini olganda, ishlab chiqarishga ketadigan xom ashyo sarfi minimum bo‘ladi? Buni amalga oshira olamizmi? Qancha xom ashyoni iqtisod qila olamiz? Bu bobda ana shu savollarga javob topamiz.

3.1. Birinchi tartibli hosila yordamida funksiyaning ekstremumlarini topish va grafigini yasash

Ushbu bo‘limda biz ko‘p partiyalarda mahsulot ishlab chiqaradigan korxonalarning sanoat ko‘rsatkichlarini diagrammalarda tasvirlashni o‘rganamiz.

Esda saqlang! Biz sotuvga chiqarilgan (yoki sotilgan) mahsulotlar sonini vaqtning funksiyasi deb qaraymiz. Savdoning dastlabki kunlarida mahsulotimiz kam sonlarda sotiladi va asta-sekin savdo maksimal darajagacha oshib boradi, qaysidir vaqtdan keyin esa savdo miqdori tushib boradi.

Nimaga? Chunki bozorda bizning mahsulot bilan raqobatlasha oladigan yangi mahsulot paydo bo‘ladi. Bunday paytda biz mahsulot turini yangilashimiz kerak. Bozorga yangi mahsulot chiqarishni o‘ylashimiz kerak.



Aytaylik, televizor ishlab chiqaramiz. Televizorlar turiga qarab har xil: an’anaviy, yassi ekranli yoki yuqori sifatdagi tasvir tiniqligiga ega, ovoz yozuvchi, kompakt diskli, fleshkali, raqamli kanal ulagichli va h.k. Ularning har birining o‘zining mavjudlik davri bo‘lgan. Har biri bozorda o‘z egri chizig‘ini chizgan.

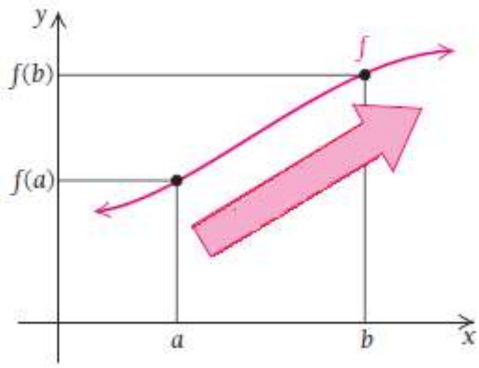
Funksiyaning eng katta va eng kichik, ya’ni maksimum va minimum qiymatlarini topish masalasi amaliyotda keng qo‘llaniladi. Funksiyaning birinchi va ikkinchi tartibli hosilalari – hisob fanining ish qurollari bo‘lib, ular funksiya grafiklarini chizish va minimal, maksimal qiymatlarini aniqlash uchun axborotlar to‘plashda muhim o‘rin tutadi.

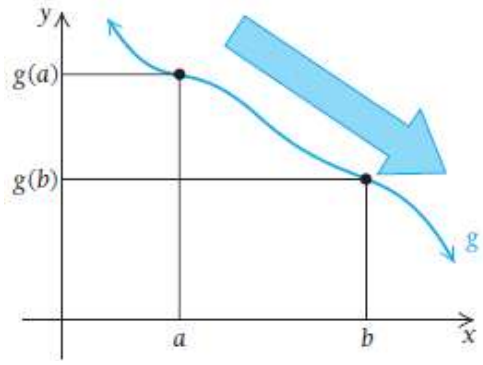
Agar funksiyaning uzilishga ega ekanligi ko‘rsatilmagan bo‘lsa, uni uzluksiz deb hisoblaymiz.

Esda saqlang! Funksiyaning uzluksiz ekanligi uning birinchi va ikkinchi tartibli hosilalari uzluksiz ekanligini bildirmaydi.

3.1.1. O‘suvchi va kamayuvchi funksiyalar

Agar funksiya grafigi I oraliqda chapdan o‘ngga tomon ko‘tarilib borsa, funksiya bu oraliqda o‘suvchi bo‘ladi.





Agar funksiya grafigi I oraliqda chapdan o‘ngga tomon pasayib borsa, funksiya bu oraliqda kamayuvchi bo‘ladi.

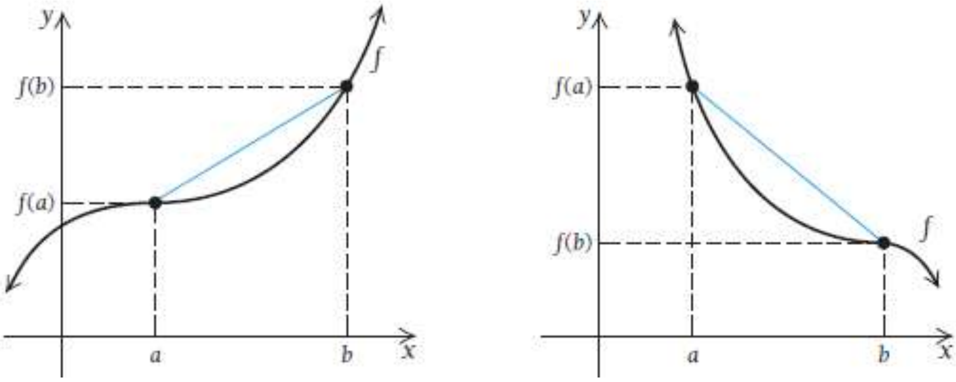
Ta’rif. Agar I oraliqdagi barcha a, b sonlar uchun $a<b$ bo‘lganda $f(a)<f(b)$ tengsizlik bajarilsa, u holda $f(x)$ funksiya I oraliqda **o‘suvchi funksiya** deyiladi.

Agar I oraliqdagi barcha a, b sonlar uchun $a<b$ bo‘lganda $g(a)>g(b)$ tengsizlik bajarilsa, u holda $g(x)$ funksiya I oraliqda **kamayuvchi funksiya** deyiladi.

Oldingi mavzularda keltirilgan ta’riflarga ko‘ra, kesuvchi yordamida tushuntiramiz. Agar funksiya o‘suvchi va $a<b$ bo‘lsa, funksiya grafigiga $x=a$ va $x=b$ nuqtalar orqali o‘tkazilgan kesuvchining burchak koeffitsiyenti musbat bo‘ladi:

$$\frac{f(b)-f(a)}{b-a}>0.$$

Agar funksiya kamayuvchi va $a<b$ bo‘lsa, funksiya grafigiga $x=a$ va $x=b$ nuqtalar orqali o‘tkazilgan kesuvchining burchak koeffitsiyenti manfiy bo‘ladi: $\frac{f(b)-f(a)}{b-a}<0$

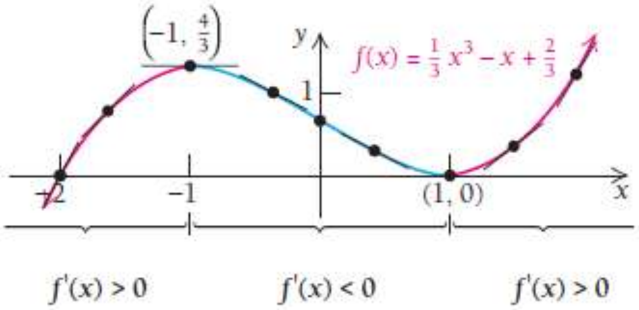


Quyidagi teorema funksiya o‘suvchi yoki kamayuvchi ekanligini hosiladan foydalanib, anqlash imkonini beradi.

1-teorema. Agar (a,b) oraliqning barcha x nuqtalari uchun $f'(x)>0$ bo‘lsa, $f(x)$ funksiya shu oraliqda o‘suvchi bo‘ladi.

Agar (a,b) oraliqning barcha x nuqtalari uchun $f'(x)<0$ bo‘lsa, $f(x)$ funksiya shu oraliqda kamayuvchi bo‘ladi.

Teorema tasdig‘ini chizmadan ko‘rish mumkin:



$(-\infty;-1)$ va $(1;\infty)$ oraliqda funksiya o‘suvchi,

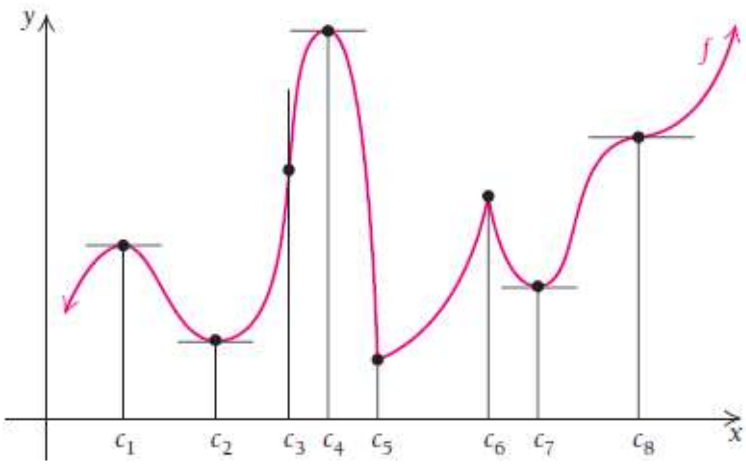
$(-1;1)$ oraliqda funksiya kamayuvchi.

Hosila yordamida funksiyaning o‘suvchi yoki kamayuvchi ekanligi ochiq oraliqda tekshiriladi, ya’ni oraliq chetidagi nuqtalar qaralmaydi.

E’tibor bering, chizmadagi funksiyaning o‘sish, kamayish oraliqlari yozilganda oraliqqa $x=-1$ va $x=1$ nuqtalar kiritilmadi. Bu nuqtalarga **kritik nuqtalar** deyiladi.

3.1.2. Kritik nuqtalar

1-misol. Bizga biror $f(x)$ funksiyaning grafigi berilgan bo‘lsin:



Grafikdan ko‘rinadiki,

1) $x=c_1,\ c_2,\ c_4,\ c_7,\ c_8$ nuqtalarda $f'(c)=0$ ga teng. Shuning uchun bu nuqtalarda grafikka o‘tkazilgan urinmalar gorizontal.

2) $x=c_3,\ c_5,\ c_6$ nuqtalarda $f'(c)$ mavjud emas va bu nuqtalarda o‘tkazilgan urinmalar vertikal (Bunday holatlarni 2.5 bo‘limda ko‘rib chiqqan edik).

Ta’rif. $f(x)$ funksiyaning **kritik nuqtasi** deb, uning aniqlanish sohasiga tegishli qandaydir c nuqtaga aytiladiki, bu nuqtada funksiya hosilasi nolga teng (funksiya grafigiga o‘tkazilgan urinma gorizontal) bo‘ladi yoki bu nuqtada funksiya hosilasi mavjud bo‘lmaydi.

1-misoldagi grafikdan ko'rinadiki,

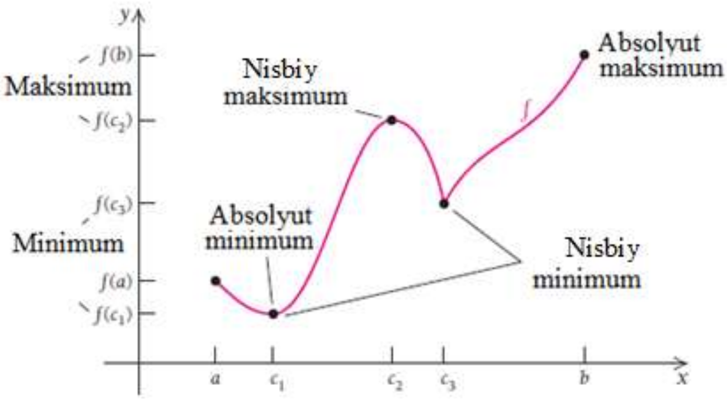
1) $c_1,\ c_2,\ c_4,\ c_7,\ c_8$ nuqtalar kritik nuqtalar, chunki bu nuqtalarda $f'(c)=0$.

2) $c_3,\ c_5,\ c_6$ nuqtalar ham kritik nuqtalar, chunki bu nuqtalarda $f'(c)$ mavjud emas.

Shuni ham bilingki, uzluksiz funksiya faqat ktirik nuqtalarda o'sishdan kamayishga yoki kamayishdan o'sishga o'tishi mumkin. Funksiya o'sishdan kamayishga va kamayishdan o'sishga o'tadigan oraliqlarini $c_1,\ c_2,\ c_4,\ c_5,\ \ c_6,\ c_7$ nuqtalari bilan ajratish mumkin. $c_3,\ c_8$ nuqtalar ham kritik nuqtalar, lekin bu nuqtalarda funksiya o'sishdan kamayishga yoki kamayishdan o'sishga o'tmaydi.

3.1.3. Maksimum va minimum nuqtalarni aniqlash

2-misol. Bizga biror $f(x)$ funksiyaning grafigi berilgan bo'lsin:



Chizmadan ko'rinadiki, funksiyaning bir nechta maksimumi yoki minimumi bo'lishi mumkin ekan. c_2 nuqta funksiyaning $(c_1,\ c_3)$ oraliqdagi eng katta nuqtasi va $f(c_2)$ funksiyaning $(c_1,\ c_3)$ **oraliqdagi eng katta qiymati** (nisbiy maksimum) bo'ladi. c_3 nuqta funksiyaning

$[c_2,\ c_3]$ oraliqdagi eng kichik nuqtasi va $f(c_2)$ funksiyaning $[c_2,\ c_3]$ **oraliqdagi eng kichik qiymati** (nisbiy minimum) bo‘ladi.

Ta’rif. I oraliq $f(x)$ funksiyaning aniqlanish sohasi bo‘lsin.

Agar I oraliqda c nuqtani o‘z ichiga olgan I_1 ochiq oraliqning barcha x nuqtalari uchun $f(c) \leq f(x)$ tengsizlik o‘rinli bo‘lsa, $f(c)$ qiymatga funksiyaning **oraliqdagi eng kichik qiymati** (nisbiy minimum) deyiladi.

Agar I oraliqda c nuqtani o‘z ichiga olgan I_2 ochiq oraliqning barcha x nuqtalari uchun $f(c) \geq f(x)$ tengsizlik o‘rinli bo‘lsa, $f(c)$ qiymatga funksiyaning **oraliqdagi eng katta qiymati** (nisbiy maksimum) deyiladi.

Esda saqlang! Nisbiy minimum funksiyaning aniqlanish sohasidagi eng kichik qiymati (minimum) bo‘lmasligi mumkin. Xuddi shuningdek, nisbiy maksimum ham funksiyaning aniqlanish sohasidagi eng katta qiymati (maksimum) bo‘lmasligi mumkin.

Maksimum va minimum qiymatlarga **ekstremum qiymatlar** deyiladi.

Funksiya hosilasi nolga teng bo‘lgan yoki funksiya hosilasi mavjud bo‘lmaydigan nuqtalarni **kritik nuqtalar** deb nomlagan edik. Xuddi o‘sha ktirik nuqtalar **ekstremum nuqtalar** deyiladi.

2-teorema. Agar $f(x)$ funksiya biror ochiq oraliqda $f(c)$ ekstremum qiymatga ega bo‘lsa, u holda c kritik nuqta bo‘ladi va bu nuqtada $f'(c) = 0$ yoki $f'(c)$ mavjud bo‘lmaydi.

2-teorema shuni bildiradiki, biz ekstremum nuqtalarni topish uchun faqat hosila holga teng bo‘ladigan yoki hosila mavjud bo‘lmaydigan nuqtalarni izlashimiz kerak ekan.

● Kritik nuqta bilan ekstremum nuqtaning farqi nimada?

Ktirik nuqtalar ekstremum nuqtalarga shubhali nuqtalar, ya’ni kritik nuqta ekstremum bo‘lishi ham, bo‘lmasligi ham mumkin. Chunki, 2-misolda chizmadan ko‘rish mumkinki, hamma kritik nuqtalar ham **maksimum (absolyut maksimum)** yoki **minimum (absolyut minimum)** bo‘la olmaydi.

Hamma kritik nuqtalar ham ekstremum bo‘la olmaydi.

3-misol. $f(x)=(x-1)^3+2$ funksiyaning maksimum yoki minimum qiymatlarini toping.

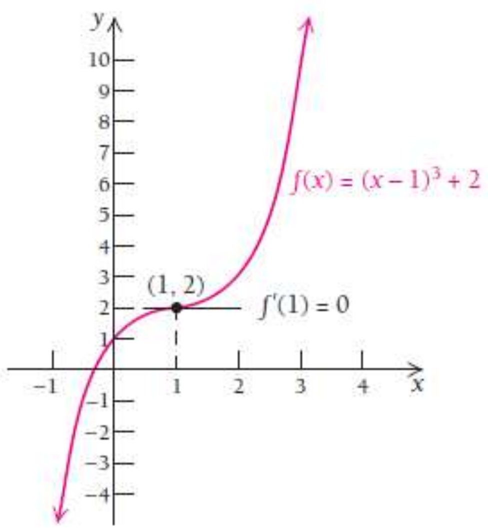
Yechilishi: ►Funksiyadan hosila olamiz: $f'(x)=3(x-1)^2$.

Hosilani nolga tenglaymiz: $f'(x)=3(x-1)^2=0$

va yechimlarni izlaymiz: $x-1=0$

$x=1$ nuqta kritik nuqta bo‘ladi.

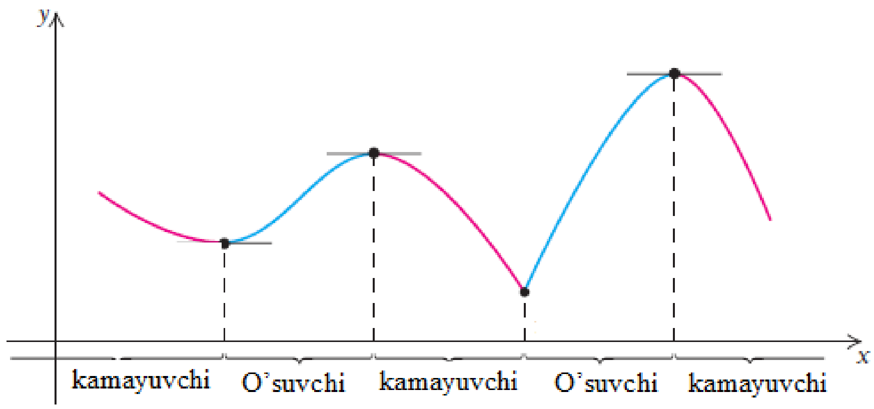
Bu nuqta aniqlanish sohasini 2 ta ochiq oraliqqa ajratadi: $(-\infty;\ 1)\cup(1;\ \infty)$.



Lekin funksiya grafigidan ko‘rish mumkinki, $f(1)=2$ qiymat maksimum ham, minimum ham emas. ◄

2-teorema bizga agar funksiya maksimum yoki minimumga ega bo‘lsa, u holda shu ekstremumning 1-koordinatasi kritik nuqta bo‘ladi, degan edi.

Kritik nuqta qachon ekstremum nuqta bo‘la oladi?

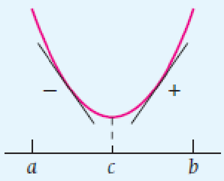
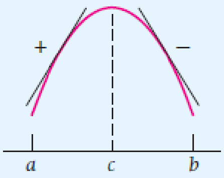
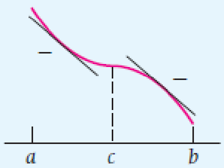
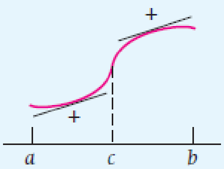


Agar kritik nuqta minimum bo‘lsa, bu nuqtaning chap tomonida funksiya kamayadi, o‘ng tomonida o‘sadi. Agar kritik nuqta maksimum bo‘lsa, bu nuqtaning chap tomonida funksiya o‘sadi, o‘ng tomonida esa kamayadi. Ikkala holda ham ktirik nuqtaning atrofida hosila ishorasini o‘zgartiradi.

Agar kritik nuqtada funksiya kamayishdan o‘sishga o‘tsa, bu nuqta minimum nuqta bo‘ladi.

Agar kritik nuqtada funksiya o‘sishdan kamayishga o‘tsa, bu nuqta maksimum nuqta bo‘ladi.

Funksiyaning o‘sishi yoki kamayishini hosila yordamida aniqlaymiz va quyidagi jadvalga tushiramiz:

interval (a, b)	$f(c)$	$f'(x)$ (a, c)	$f'(x)$ (c, b)	o’suvchi kamayuvchi
	minimum	−	+	kamayuvchi (a, c); o’suvchi (c, b)
	maksimum	+	−	o’suvchi (a, c); kamayuvchi (c, b)
	maksimum ham minimum ham emas	−	−	kamayuvchi (a, b)
	maksimum ham minimum ham emas	+	+	o’suvchi (a, b)

3-teorema. Ekstremumni topish uchun 1-tartibli hosiladan foydalanish. (a,b) oraliqda biror uzluksiz $f(x)$ funksiyaning bitta c kritik nuqtasi bo‘lsa, u holda

1) Hosilaning ishorasi (a,c) da $f'(x)<0$ va (c,b) da $f'(x)>0$ bo‘lsa, funksiya c nuqtada minimumga erishadi;

2) Hosilaning ishorasi (a,c) da $f'(x)>0$ va (c,b) da $f'(x)<0$ bo‘lsa, funksiya c nuqtada maksimumga erishadi;

3) Hosilaning ishorasi (a,c) va (c,b) oraliqlarda bir xil bo‘lsa, funksiya maksimumga ham minimumga ham erishmaydi.

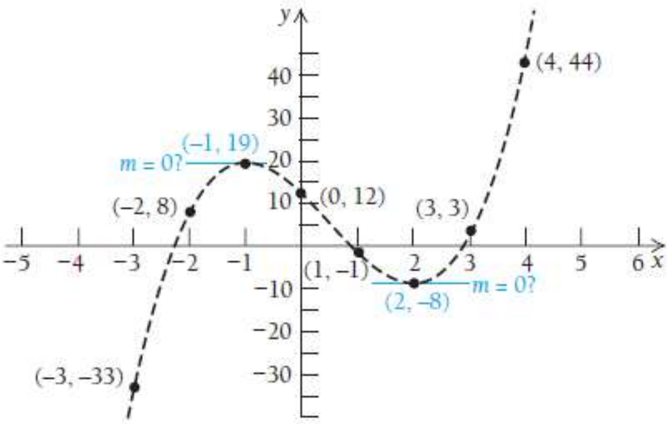
Endi shu teoremani misollarda qo‘llaymiz.

4-misol. $f(x)=2x^3-3x^2-12x+12$ funksiyaning ekstremumlarini toping va grafigini yasang.

Yechilishi: ► Funksiyaning grafigini chizish so‘ralgan bo‘lsin. Grafikni to‘g‘ri chizish uchun juda ko‘p nuqtalarda funksiya qiymatlarini hisoblash kerak bo‘ladi. Hisoblash esa juda murakkab jarayon.

Nima qilsak bo‘ladi? Grafik eskizini chizish uchun uning egilish nuqtalarini topsak yetarli.

x	$f(x)$
−3	−33
−2	8
−1	19
0	12
1	−1
2	−8
3	3
4	44



Dastlab funksiya hosilasini topamiz:

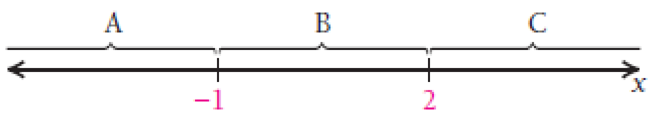
$$f'(x)=\left(2x^3-3x^2-12x+12\right)'=6x^2-6x-12$$

Topilgan ifodani nolga tenglaymiz: $f'(x)=0$.

Tenglamani yechamiz: $6x^2-6x-12=0$

$$x^2-x-2=0 \quad \rightarrow \quad x=2 \text{ va } x=-1.$$

Bu nuqtalarni sonlar o‘qiga joylab, uchta oraliq hosil qilamiz:

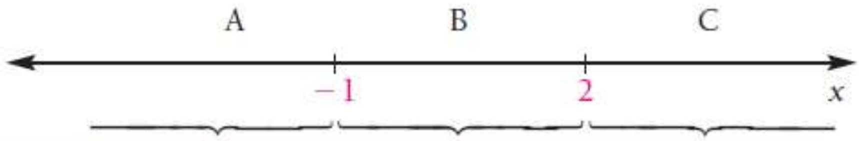


A oraliq $(-\infty;\ -1)$, B oraliq $(-1;\ 2)$ va C oraliq $(2;\ \infty)$ dan iborat. Endi har bir interval ichida funksiya ishorasini tekshiramiz.

A oraliqda: $x=-2$, $f'(-2)=6\cdot(-2)^2-6\cdot(-2)-12=24>0$;

B oraliqda: $x=0$, $f'(0)=6\cdot 0^2-6\cdot 0-12=-12<0$;

C oraliqda: $x=4$, $f'(4)=6\cdot 4^2-6\cdot 4-12=60>0$.

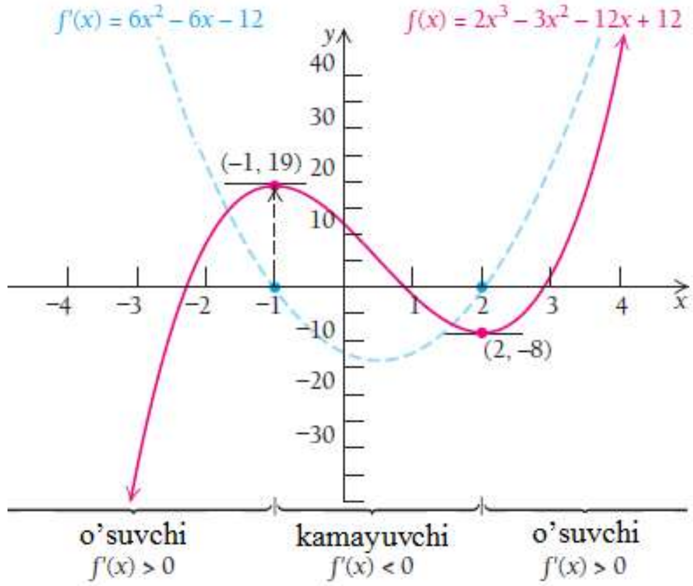


	$x=-2$	$x=0$	$x=4$
$f'(x)$	$f'(-2)>0$	$f'(0)<0$	$f'(4)>0$
	$(-\infty,-1)$	$(-1,2)$	$(2,\infty)$

Jadvalga qarab, funksiya $x=-1$ da maksimumga erishishini bilib olishimiz mumkin: $f(-1)=2\cdot(-1)^3-3\cdot(-1)^2-12\cdot(-1)+12=19$

$x=2$ da funksiya minimumga erishadi: $f(-1)=2\cdot 2^3-3\cdot 2^2-12\cdot 2+12=-8$

Demak, $(-1,\ 19)$ nuqta (nisbiy) maksimum, $(2,\ -8)$ nuqta (nisbiy) minimum ekan.



$(-\infty;\ -1)$ oraliqda funksiya oʻsuvchi;

$(-1;\ 2)$ oraliqda funksiya kamayuvchi;

$(2;\ \infty)$ oraliqda funksiya oʻsuvchi. ◀

1-vazifa. $f(x)=x^3-27x-6$ funksiyaning ekstremumlarini toping va grafigini yasang.

5-misol. $f(x)=2x^3-x^4$ funksiyaning ekstremumlarini toping va grafigini yasang.

Yechilishi: ► Funksiyadan hosila olamiz:

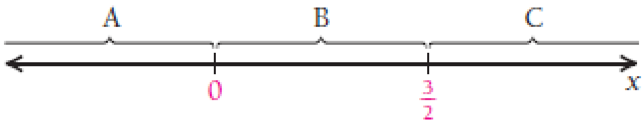
$$f'(x)=\left(2x^3-x^4\right)'=6x^2-4x^3$$

Topilgan ifodani nolga tenglaymiz: $f'(x)=0$.

Tenglamani yechamiz: $6x^2-4x^3=0$

$$2x^2(3-2x)=0 \quad \rightarrow \quad x=0 \text{ va } x=\frac{3}{2}.$$

Bu nuqtalarni sonlar o‘qiga joylab, uchta oraliq hosil qilamiz:



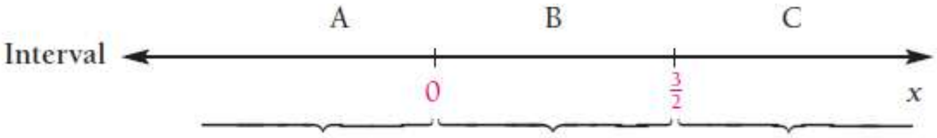
A oraliq $(-\infty;\ 0)$, B oraliq $\left(0;\ \frac{3}{2}\right)$ va C oraliq $\left(\frac{3}{2};\ \infty\right)$ dan iborat.

Endi har bir interval ichida funksiya ishorasini tekshiramiz.

A oraliqda: $x=-1$, $f'(-1)=6\cdot(-1)^2-4\cdot(-1)^3=10>0$;

B oraliqda: $x=1$, $f'(1)=6\cdot 1^2-4\cdot 1^3=2>0$;

C oraliqda: $x=2$, $f'(2)=6\cdot 2^2-4\cdot 2^3=-8<0$



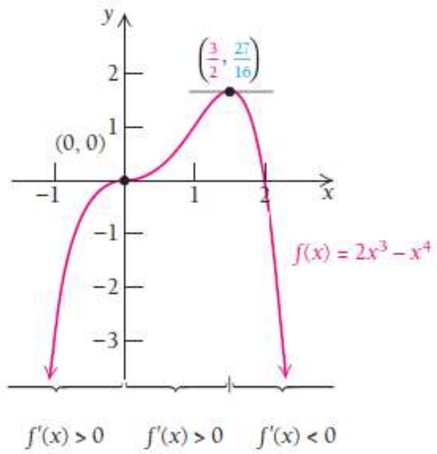
	$x=-1$	$x=1$	$x=2$
$f'(x)$	$f'(-1)>0$	$f'(1)>0$	$f'(2)<0$
	f $(-\infty,0)$	f $\left(0,\frac{3}{2}\right)$	f $\left(\frac{3}{2},\infty\right)$

Jadvalga qarab, funksiya $x = 0$ da maksimumga ham minimumga ham ega emas, chunki bu nuqtaning chap tomonida ham, o‘ng tomonida ham musbat bir xil ishorali. $f(0) = 2 \cdot 0^3 - 0^4 = 0$.

$x = \frac{3}{2}$ da funksiya maksimumga erishadi: $f\left(\frac{3}{2}\right) = 2 \cdot \left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^4 = \frac{27}{16}$.

Demak, $\left(\frac{3}{2}, \frac{27}{16}\right)$ nuqta maksimum nuqta ekan.

x	$f(x)$
−1	−3
−0.5	−0.31
0	0
0.5	0.19
1	1
1.25	1.46
2	0



$(-\infty;\ 0)$ oraliqda funksiya o‘suvchi;

$\left(0;\ \frac{3}{2}\right)$ oraliqda funksiya o‘suvchi;

$\left(\frac{3}{2};\ \infty\right)$ oraliqda funksiya kamayuvchi.

$(-\infty;\ 0)$ va $\left(0;\ \frac{3}{2}\right)$ oraliqlarda funksiya o‘suvchi, shuning uchun funksiyani $\left(-\infty;\ \frac{3}{2}\right)$ oraliqda o‘suvchi deb aytishimiz mumkin. Chunki bu oraliqda funksiya grafigiga uning ixtiyoriy 2 nuqtasi orqali o‘tkazilgan kesuvchining burchak koeffitsiyenti musbat bo‘ladi. ◄

2-vazifa. $f(x)=x^4-\frac{8}{3}x^3$ funksiyaning ekstremumlarini toping va grafigini yasang.

6-misol. $f(x)=\sqrt[3]{(x-2)^2}+1$ funksiyaning ekstremumlarini toping va grafigini yasang.

Yechilishi: ► Funksiyadan hosila olamiz:

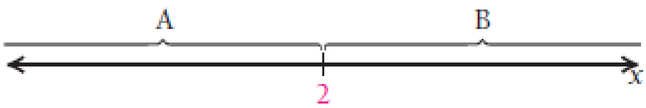
$$f'(x)=\left((x-2)^{2/3}+1\right)'=\frac{2}{3}(x-2)^{\frac{2}{3}-1}=\frac{2}{3\sqrt[3]{x-2}}$$

Topilgan ifodani nolga tenglaymiz: $f'(x)=0$.

Tenglamani yechamiz: $\frac{2}{3\sqrt[3]{x-2}}=0$

$$x-2\neq 0 \quad \rightarrow \quad x\neq 2.$$

Bu nuqtani sonlar o'qiga joylab, ikkita oraliq hosil qilamiz:

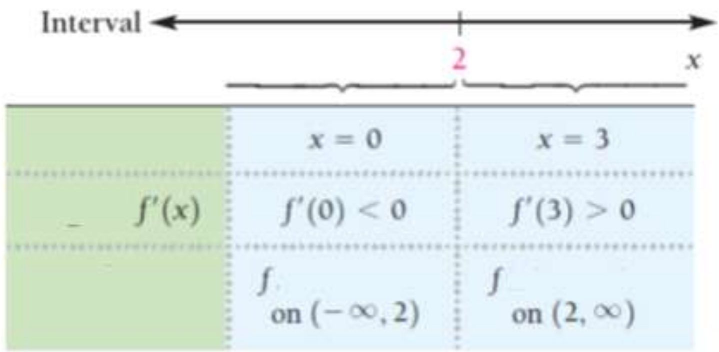


A oraliq $(-\infty;\ 2)$, B oraliq $(2;\ \infty)$ dan iborat.

Endi har bir interval ichida funksiya ishorasini tekshiramiz.

A oraliqda: $x=0$, $\qquad f'(0)=\frac{2}{3\sqrt[3]{0-2}}<0$;

B oraliqda: $x=3$, $\qquad f'(3)=\frac{2}{3\sqrt[3]{3-2}}>0$.

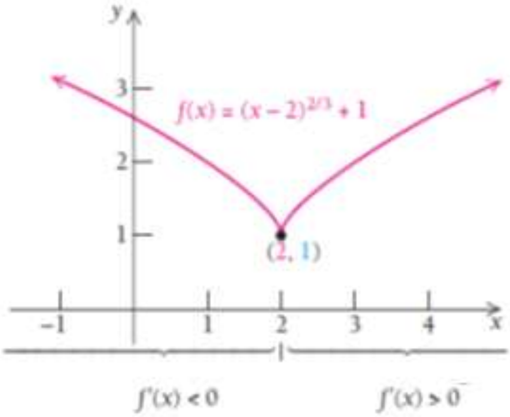


	$x=0$	$x=3$
$f'(x)$	$f'(0)<0$	$f'(3)>0$
	f on $(-\infty,2)$	f on $(2,\infty)$

$x=2$ da funksiya minimumga erishadi: $f(x)=\sqrt[3]{(2-2)^2}+1=1$,

bundan $(2;1)$ nuqta minimum nuqta ekanini aniqlaymiz.

x	$f(x)$
−1	3.08
−0.5	2.84
0	2.59
0.5	2.31
1	2
1.5	1.63



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MUSTAQIL YECHISH UCHUN MASALALAR

1-34 misollarda funksiyaning ekstremumlarini, o‘sish va kamayish oraliqlarini toping, so‘ngra ma’lumotlar asosida grafigini yasang.

1. $f(x) = x^2 + 4x + 5$

2. $f(x) = x^2 + 6x - 3$

3. $f(x) = 5 - x - x^2$

4. $f(x) = 2 - 3x - 2x^2$

5. $g(x) = 1 + 6x + 3x^2$

6. $F(x) = 0.5x^2 + 2x - 11$

7. $G(x) = x^3 - x^2 - x + 2$

8. $g(x) = x^3 + \frac{1}{2}x^2 - 2x + 5$

9. $f(x) = x^3 - 3x + 6$

10. $f(x) = x^3 - 3x^2$

11. $f(x) = 3x^2 + 2x^3$

12. $f(x) = x^3 + 3x$

13. $g(x) = 2x^3 - 16$

14. $F(x) = 1 - x^3$

15. $G(x) = x^3 - 6x^2 + 10$

16. $f(x) = 12 + 9x - 3x^2 - x^3$

17. $g(x) = x^3 - x^4$

18. $f(x) = x^4 - 2x^3$

19. $f(x) = \frac{1}{3}x^3 - 2x^2 + 4x - 1$

20. $F(x) = -\frac{1}{3}x^3 + 3x^2 - 9x + 2$

21. $g(x) = 2x^4 - 20x^2 + 18$

22. $f(x) = 3x^4 - 15x^2 + 12$

23. $F(x) = \sqrt[3]{x - 1}$

24. $G(x) = \sqrt[3]{x + 2}$

25. $f(x) = 1 - x^{2/3}$

26. $f(x) = (x + 3)^{2/3} - 5$

27. $G(x) = \dfrac{-8}{x^2 + 1}$

28. $F(x) = \dfrac{5}{x^2 + 1}$

29. $g(x) = \dfrac{4x}{x^2 + 1}$

30. $g(x) = \dfrac{x^2}{x^2 + 1}$

31. $f(x) = \sqrt[3]{x}$

32. $f(x) = (x + 1)^{1/3}$

33. $g(x) = \sqrt{x^2 + 2x + 5}$

34. $F(x) = \dfrac{1}{\sqrt{x^2 + 1}}$

35-50 misollarda berilgan ma’lumotlarga mos funksiya grafigining eskizini chizing:

35. $f(x)$ funksiya $(-\infty;\ 2)$ da o‘suvchi, $(2;\ \infty)$ da kamayuvchi;

36. $g(x)$ funksiya $(-\infty;\ -3)$ da o‘suvchi, $(-3;\ \infty)$ da kamayuvchi;

37. $h(x)$ funksiya $(-\infty;\ -4)$ va $(10;\ \infty)$ da kamayuvchi, $(-4;\ 10)$ da o‘suvchi;

38. $F(x)$ funksiya $(-\infty;\ -1)$ va $(-1;\ 4)$ da o‘suvchi, $(-1;\ 4)$ da kamayuvchi;

39. $H(x)$: $(-\infty;\ -3)$ da funksiya hosilasi musbat, $(-3;\ \infty)$ da funksiya hosilasi manfiy;

40. $G(x)$: $(-\infty;\ -1)$ da funksiya hosilasi manfiy, $(-1;\ \infty)$ da funksiya hosilasi musbat;

41. $f(x)$: $(-\infty;\ 2)$ va $(5;\ 9)$ oraliqlarda funksiya hosilasi manfiy, $(2;\ 5)$ va $(9;\ \infty)$ oraliqlarda funksiya hosilasi musbat;

42. $F(x)$: $(-\infty;\ -1)$ va $(3;\ 7)$ oraliqlarda funksiya hosilasi manfiy, $(-1;\ 3)$ va $(7;\ \infty)$ oraliqlarda funksiya hosilasi musbat;

43. $f(x)$: $(-\infty;\ 3)$ va $(3;\ 9)$ oraliqlarda funksiya hosilasi musbat, $(9;\ \infty)$ oraliqda funksiya hosilasi manfiy, $x=3$ nuqtada $f'(3)=0$;

44. $h(x)$: $(-\infty;\ 4)$ va $(4;\ 7)$ oraliqlarda funksiya hosilasi musbat, $(7;\ \infty)$ oraliqda funksiya hosilasi manfiy, $x=4$ nuqtada funksiya hosilasi nolga teng;

45. $g(x)$: $(-\infty;\ -1)$ da funksiya hosilasi manfiy, $(-1;\ \infty)$ da funksiya hosilasi musbat va $x=-1$ nuqtada $g'(-1)$ hosila mavjud emas;

46. $f(x)$: $(-\infty;\ 0)$ va $(3;\ \infty)$ oraliqlarda funksiya hosilasi musbat, $(0;\ 3)$ oraliqda funksiya hosilasi manfiy, biroq $x=0$ va $x=3$ nuqtalarning hech birida $f'(0)$, $f'(3)$ hosilalar mavjud emas;

47. $f(x)$: $(-\infty;\ -2)$ va $(1;\ \infty)$ oraliqlarda funksiya hosilasi manfiy, $(-2;\ 1)$ oraliqda funksiya hosilasi musbat, $x=-2$ nuqtada $f'(-2)=0$, biroq $x=0$ nuqtada $f'(0)$ hosila mavjud emas;

48. $f(x)$: $(-\infty;\ -3)$ va $(0;\ 3)$ oraliqlarda funksiya hosilasi musbat, $(-3;\ 0)$ va $(3;\ \infty)$ oraliqlarda funksiya hosilasi manfiy, $x=-3$ va $x=3$ nuqtalarda funksiya hosilasi nolga teng $f'(-2)=0$, biroq $f'(0)$ hosila mavjud emas;

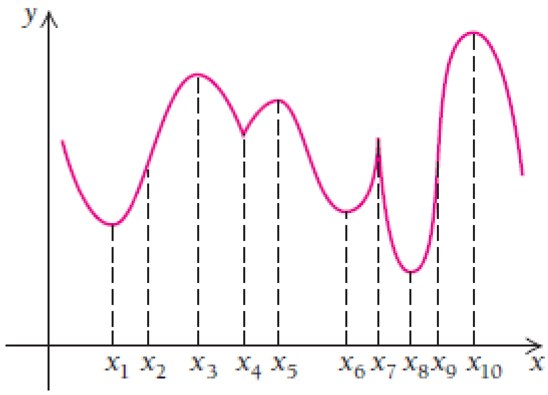
49. $G(x)$ funksiya $(-\infty;\ \infty)$ da o‘suvchi, biroq $x=4$ nuqtada funksiya hosilasi mavjud emas;

50. $G(x)$ funksiya $(-\infty;\ \infty)$ da kamayuvchi, biroq $x=0$ va $x=2$

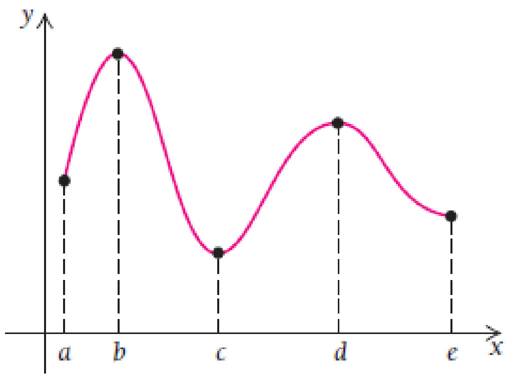
nuqtalarda funksiya hosilasi mavjud emas.

51. $f(x)$: $(-\infty;\ 0)$ va $(0;\ 3)$ oraliqlarda funksiya hosilasi musbat, $(3;5)$ oraliqlarda funksiya hosilasi manfiy, $x=-3$ va $x=3$ nuqtalarda funksiya hosilasi nolga teng $f'(-2)=0$, biroq $f'(0)$ hosila mavjud emas;

52. Kritik nuqta ta’rifini bering. Grafikdagi nuqtalarning qaysilari kritik nuqtalar va nima uchun?



53. Berilgan funksiya grafigi va belgilangan oraliqlardan foydalanib, funksiyaning o‘suvchi va kamayuvchi oraliqlarini yozing. Javoblarni funksiyaning 1-tartibli hosilasining qiymatlari bilan bog‘lang.



AMALIY TATBIQLAR:

54. Bandlikka ko‘maklashish. Toshkent sahri bandlikka ko‘maklashish markazining statistikasiga ko‘ra 2010-2019 yillar davomida kasb egasi bo‘lgan mutaxassislarni ishga joylashtirish amaliyotlari sonini

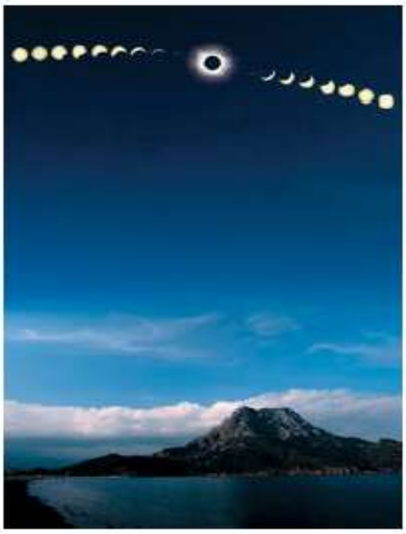
$$B(t) = -28.31t^3 + 381.86t^2 - 1162.07t + 16905.87$$

tenglik bilan ifodalash mumkin. Bunda $t-$yil, $B-$bandlikka ko‘maklashish markaziga murojaat qilgan kishilar soni. Funksiyaning ekstremum qiymatlarini toping va grafigini chizing. Javobingizni tushuntiring.

55. Reklama. “Artel” korxonasi rekalamaga a dollar sarflagandah keyin N dona televizorni $N(a) = -a^2 + 300a + 6$, bunda $0 \le a \le 300$ funksiya bo‘yicha sotishini hisobladi. Ekstremum qiymatlarni toping va funksiya grafigini yasang.

56. Tana haroratining ko‘tarilishi. Kishi ichak kasalligi bilan og‘rigan bo‘lsa, uning harorati $T(t) = -0.1t^2 + 1.2t + 98.6$, bunda $0 \le t \le 12$ bo‘ladi. Haroratning ekstremum qiymatlarni toping va funksiya grafigini yasang.

57. Quyosh tutilishi. 2010 yil 15 yanvarda quyosh tutilishining eng uzun (3040 km) halqasi Afrika va Hind okeani ustida yuz berdi (quyosh oyni orqasida qolib, halqa hosil qildi).



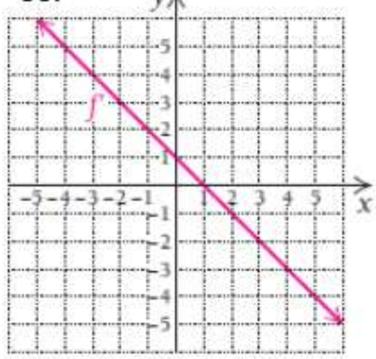
To‘liq tutilish yo‘li yerda quyidagi funksiyaga approksimatsiyalandi:

$$f(x) = 0.0125x^2 - 1.157t + 22.864, \text{ bunda } 15 \le x \le 90.$$

$x-$markaziy meridiandan sharqiy uzunlik o‘lchami, ekvatordan shimoliy kenglikni (musbat) yoki janubiy kenglikni (manfiy) deb oling. To‘liq tutilish ko‘rinadigan eng janubiy nuqtaning kenglik va uzunligini toping.

58-63 misollarda funksiya hosilasining grafigi keltirilgan. Ma’lumotlar asosida funksiyaning o‘sish va kamayish oraliqlarini, ekstremumlarini toping. Funksiya grafigining eskizini chizing.

58.



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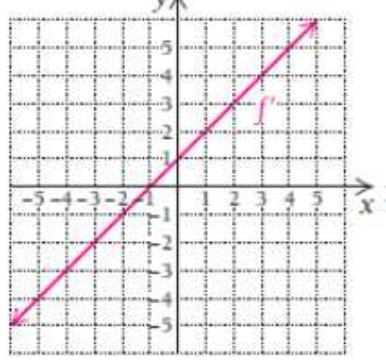

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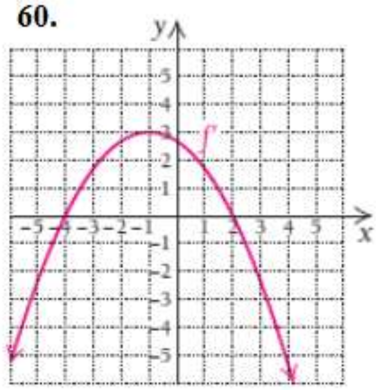

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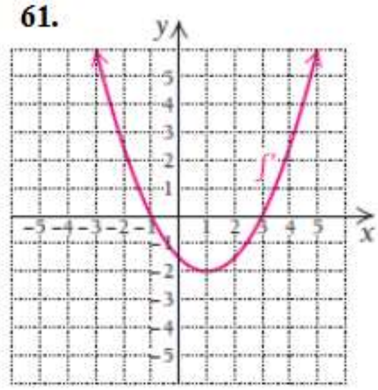
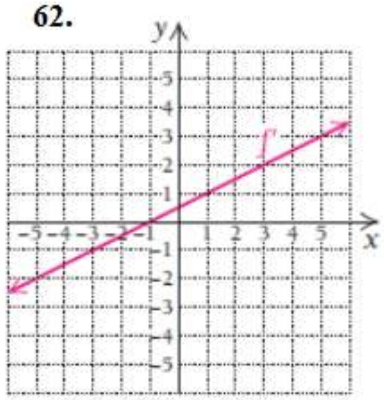


62.



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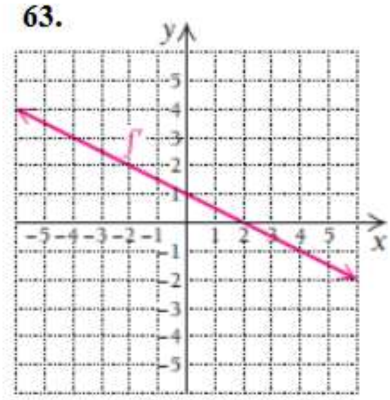
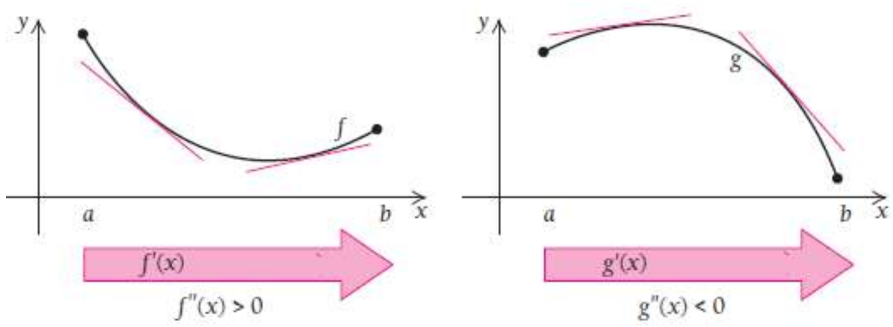


3.2. Ikkinchi tartibli hosila yordamida funksiyaning ekstremumlarini topish va grafigini yasash

3.2.1. Funksiya grafigining botiq, qavariqligi

Ikkinchi tartibli hosila funksiya grafigining egriligini tahlil qilishda katta ahamiyatga ega. Chizmaga qarang: $f(x)$ funksiyaning grafigi botiq, $g(x)$ ning grafigi esa qavariq ko‘rinishda.



Grafiklarning botiq, qavariqligini hosila bilan bog‘lash mumkinmi?

Qo‘limizga chizg‘ichni olamiz. Oldin $f(x)$ funksiya grafigini chap tomonidan boshlab bir nechta urinma chiziqlar chizamiz.

Urinmalar qanday holatlarda joylashdi? Xuddi shu amaliyotni $g(x)$ funksiya grafigi ustida ham bajaramiz. Urinmalarning egilishini ko‘ring.

$f(x)$ ning grafigiga o‘tkazilgan urinmalar kamayuvchidan o‘suvchi chiziqlarga o‘ta boshladi. Bu oraliqda $f'(x)$ ning grafigi o‘suvchi bo‘ladi. $f''(x)$ 2-tartibli hosilani hisoblasak, u musbat bo‘ladi. $f'(x)$ va $f''(x)$ lar orasidagi munosabatni $f(x)$ va $f'(x)$ lar orasidagi munosabatga o‘tkazamiz. Barcha urinmalar $f(x)$ ning grafigidan pastda yotganini hisobga olib qo‘yamiz.

$g(x)$ ning grafigiga o‘tkazilgan urinmalar o‘suvchidan kamayuvchiga o‘ta boshladi. Barcha urinmalar $g(x)$ ning grafigidan tepada yotibdi.

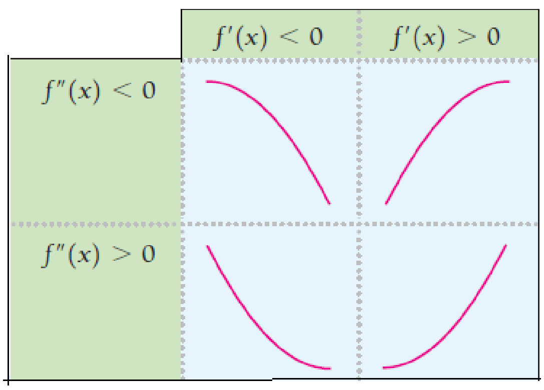
Ta’rif. $f(x)$ funksiya (a,b) oraliqning har bir nuqtasida hosilaga ega bo‘lsin. U holda agar 1-tartibli hosila $f'(x)$ bu oraliqda kamayishdan o‘sishga o‘tsa, $f(x)$ funksiyaning grafigi **botiq** deyiladi.

Agar 1-tartibli hosila $f'(x)$ bu oraliqda o‘sishdan kamayishga o‘tsa, $f(x)$ funksiyaning grafigi **qavariq** deyiladi.

Endi quyidagi teoremada qavariq, botiqlik tushunchasini 2-tartibli hosila bilan bog‘laymiz:

4-teorema (Funksiya grafigining egilishi haqida).

1) Agar (a,b) oraliqda $f''(x)>0$ bo‘lsa, $f(x)$ ning grafigi **botiq** bo‘ladi;
2) Agar (a,b) oraliqda $f''(x)<0$ bo‘lsa, $f(x)$ ning grafigi **qavariq** bo‘ladi.

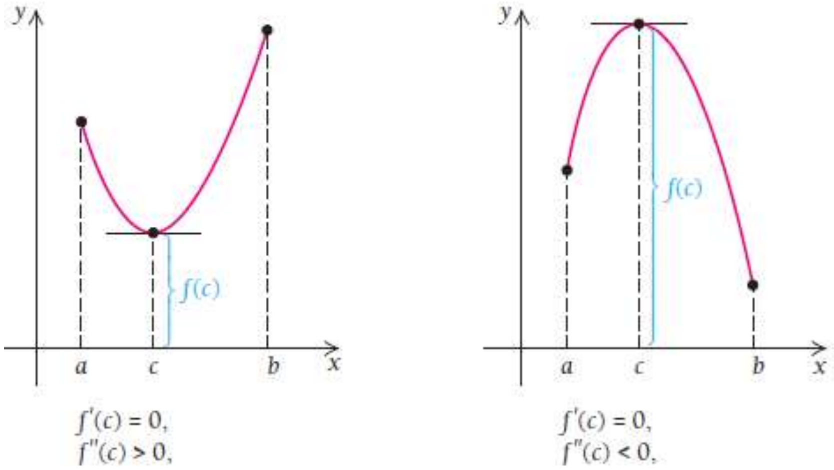


3.2.2. Ikkinchi tartibli hosiladan foydalanib, ekstremumlarni topish

Keling, 2-tartibli hosila yordamida funksiyaning biror (a,b) oraliqda ekstremumi bor yoki yo‘qligini tekshirib ko‘ramiz.

Quyidagi grafiklarda botiq va qavariq funksiyalar berilgan.

2-tartibli hosila musbat bo‘lsa, funksiya grafigi botiq va bu yerda – maksimal nuqta mavjud, 2-tartibli hosila manfiy bo‘lsa, funksiya grafigi qavariq – minimal nuqta mavjud bo‘ladi.



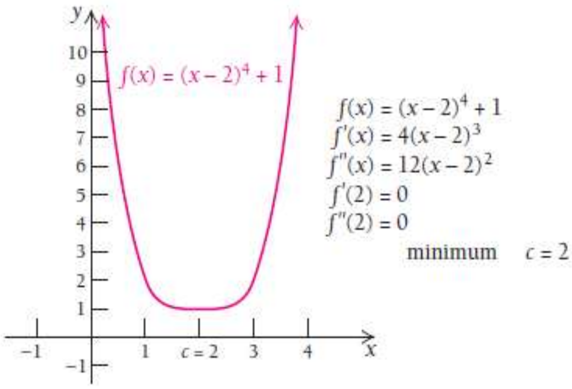
5-teorema (Funksiyaning ekstremumlari haqida).

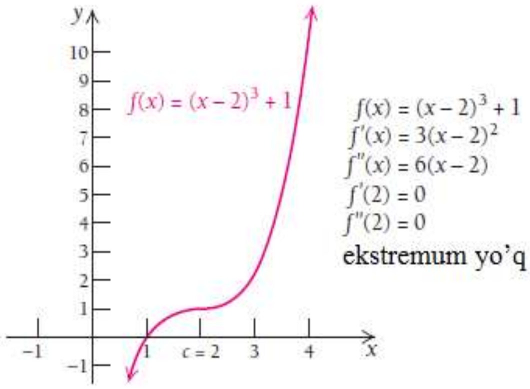
$f(x)$ funksiya (a,b) oraliqning har bir x nuqtasida differensiallanuvchi va oraliqda $f'(x)=0$ shartni qanoatlantiradigan c kritik nuqtaga ega bo‘lsin. U holda

1) Agar $f''(c)>0$ bo‘lsa, $f(c)$ minimum nuqta;
2) Agar $f''(c)<0$ bo‘lsa, $f(c)$ maksimum nuqta bo‘ladi.

Quyidagi chizmani qaraylik. Ikkala grafikda ham $f'(c)=0$, $f''(c)=0$, lekin birinchi funksiyaning ekstremumi mavjud, ikkinchi funksiyada ekstremum yo‘q. Agar c kritik nuqta bo‘lsa va $f''(c)=0$ bo‘lsa, c nuqtada ekstremum bo‘lishi ham bo‘lmasligi ham mumkin. Shuningdek, agar c

kritik nuqtada $f'(c)$ mavjud bo‘lmasa, u holda $f''(c)$ ham mavjud bo‘lmaydi. Boshqa hollarda $f(c)$ ning ekstremum ekanligini albatta 2-tartibli hosila bilan tekshirib ko‘rish zarur.





Demak, 2-tartibli hosiladan ekstremumni aniqlab olish va funksiyaning grafigini to‘liq va to‘g‘ri chizish uchun foydalanilar ekan. Buni quyidagi misolda ham ko‘ramiz.

1-misol. $f(x) = x^3 + 3x^2 - 9x - 13$ funksiyaning ekstremumlarini toping va grafigini yasang.

Yechilishi: ► Funksiyadan hosila olamiz:

$$f'(x) = \left(x^3 + 3x^2 - 9x - 13\right)' = 3x^2 + 6x - 9$$

Topilgan hosilani nolga tenglaymiz: $f'(x) = 0$.

Tenglamani yechamiz va kritik nuqtalarni topamiz:

$$3x^2 + 6x - 9 = 0$$

$$x^2 + 2x - 3 = 0$$

$$(x+3)(x-1) = 0$$

$$x = -3 \text{ va } x = 1.$$

Endi kritik nuqtalarda funksiyaning qiymatlarini topamiz:

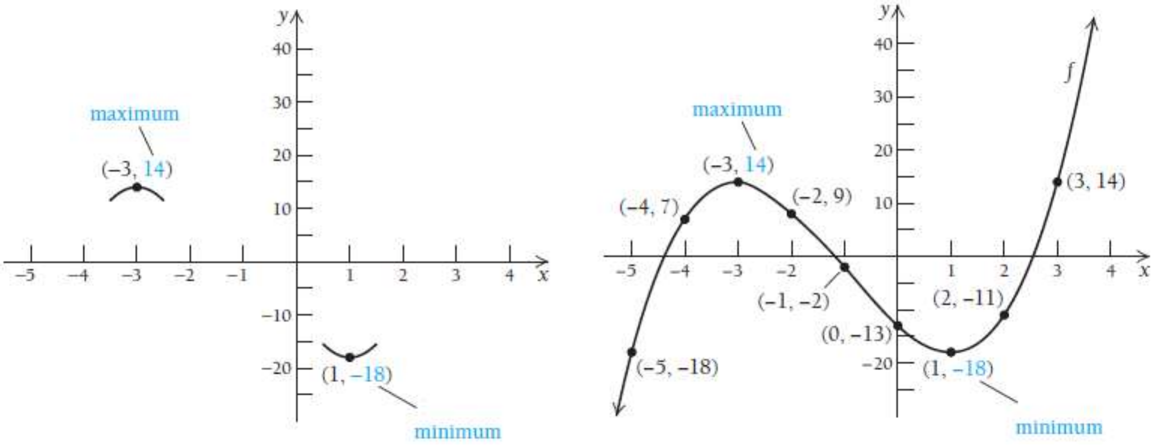
$$f(-3) = (-3)^3 + 3\cdot(-3)^2 - 9\cdot(-3) - 13 = 14;$$

$$f(1) = 1^3 + 3\cdot 1^2 - 9\cdot 1 - 13 = -18.$$

Shunda $(-3;\ 14)$ va $(1;\ -18)$ ekstremum nuqtalar bo‘ladimi?

2-tartibli hosilani qaraymiz. $f''(x) = \left(3x^2 + 6x - 9\right)' = 6x + 6$ hosilaga ktirik qiymatlarni qo‘yib, tekshiramiz: $f''(-3) = 6 \cdot (-3) + 6 = -12 < 0$ (nisbiy) maksimum $f''(1) = 6 \cdot 1 + 6 = 12 > 0$ (nisbiy) minimum nuqtalarni topamiz. Shunday qilib, $f(-3) = 14$ funksiyaning **maksimum nuqtasi**, $f(1) = -18$ funksiyaning **minimum nuqtasi** ekanligini aniqlab oldik.

Endi shu ma’lumotlar asosida funksiya grafigini chizamiz:



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2-misol. $f(x) = 3x^5 - 20x^3$ funksiyaning ekstremumlarini toping va grafigini yasang.

Yechilishi: ► Funksiyaning 1- va 2- tartibli hosilalarini olamiz:

$$f'(x) = \left(3x^5 - 20x^3\right)' = 15x^4 - 60x^2$$

$$f''(x) = \left(15x^4 - 60x^2\right)' = 60x^3 - 120x$$

Birinchi tartibli hosilani nolga tenglaymiz: $f'(x) = 0$.

Tenglamani yechamiz va kritik nuqtalarni topamiz:

$$15x^4 - 60x^2 = 0, \qquad 15x^2(x^2 - 4) = 0,$$

$$x^2(x+2)(x-2) = 0,$$

$$x = -2, \quad x = 0 \text{ va } x = 2.$$

Endi kritik nuqtalarda funksiyaning qiymatlarini topamiz:

$$f(-2) = 3 \cdot (-2)^5 - 20 \cdot (-2)^3 = 64$$

$$f(0) = 3 \cdot 0^5 - 20 \cdot 0^3 = 0$$

$$f(2) = 3 \cdot 2^5 - 20 \cdot 2^3 = -64$$

$-2,\ 0,\ 2$ qiymatlar ekstremumga shubhali nuqtalar hisoblanadi. Ularni 2-tartibli hosilaga qo‘yib, tekshirib ko‘ramiz:

$f''(-2) = 60 \cdot (-2)^3 - 120 \cdot (-2) = -240 < 0$ (nisbiy) maksimum

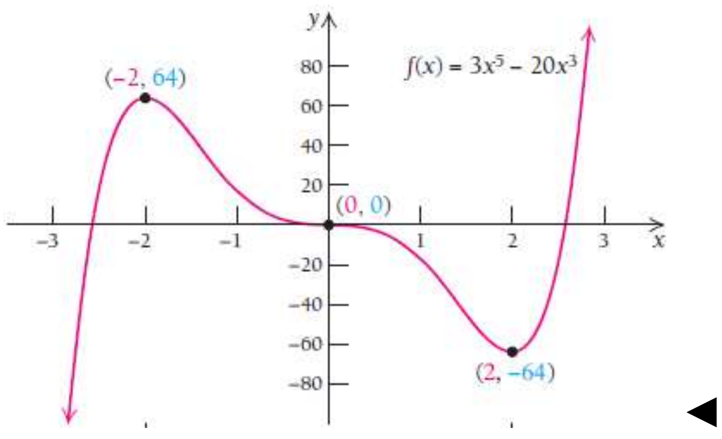
$f''(0) = 60 \cdot 0^3 - 120 \cdot 0 = 0$ bo‘lgani uchun bu nuqtada 1-tartibli hosilani tekshirib ko‘ramiz.

$f''(2) = 60 \cdot 2^3 - 120 \cdot 2 = 240 > 0$ (nisbiy) minimumni topamiz.

Shunday qilib, $f(-2) = 64$ **maksimum**, $f(2) = -64$ **minimum** nuqtalarni topdik. Endi $x = 0$ ning atroflarini o‘rganamiz:

$f'(-1) = 15 \cdot (-1)^4 - 60 \cdot (-1)^2 = -45 < 0$ va $f'(1) = 15 \cdot 1^4 - 60 \cdot 1^2 = -45 < 0$.

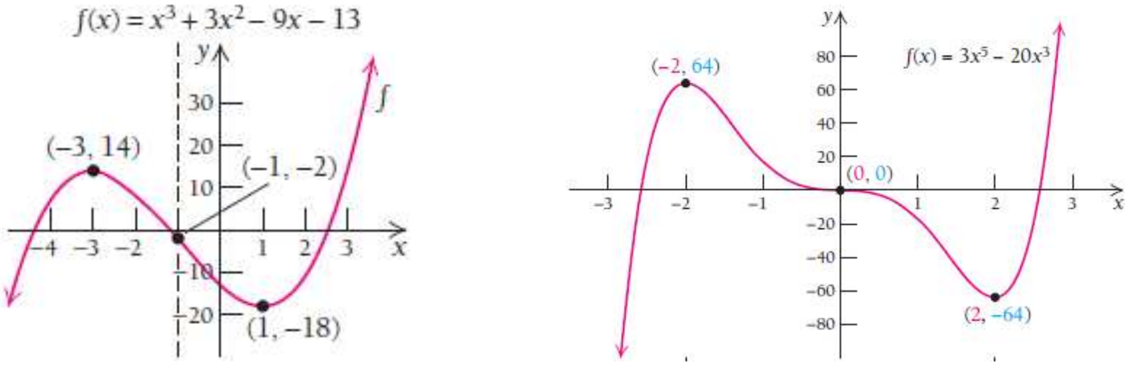
Bundan ko‘rinadiki, funksiya $(-2,\ 0)$ va $(0,\ 2)$ oraliqlarda kamayuvchi ekan. Ma’lumotlar asosida funksiya grafigini chizamiz:



1-vazifa. $f(x) = -6x^5 - 10x^3$ funksiyaning ekstremumlarini toping va grafigini chizing.

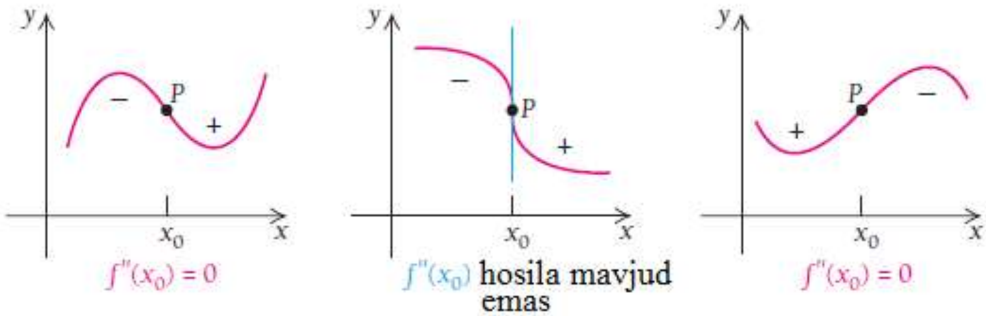
3.2.3. Funksiya grafigining egilish nuqtalari

1-misolda $(-1,\ -2)$ nuqtada, 2-misoldagi esa $(0,\ 0)$ nuqtada funksiyalarning grafiklari qavariqlikdan botiqlikka o‘tmoqda.



Egilish nuqtasi botiqlik va qavariqlik orasida joylashib, ularni bir-biridan ajratib turadigan nuqtadir.

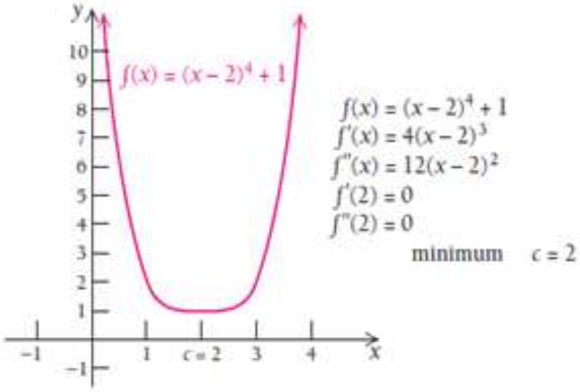
Masalan, quyidagi shakllarda P nuqta egilish nuqtasidir: $f''(x)$



2-tartibli hosilaning ishorasi P nuqtaning ikki tomonidagi botiqlik, qavariqlik turini aniqlab beradi. Chap va o‘ng tomondagi shakllarda $f''(x_0)$ nolga teng, o‘rtadagi shaklda esa 2-tartibli hosila mavjud emas. Har uchchala holda ham P egilish nuqtasi hisoblanadi.

6-teorema. Agar x_0 nuqta $f(x)$ funksiyaning egilish nuqtasi bo‘lsa, u holda bu nuqtada $f''(x_0) = 0$ bo‘ladi yoki $f''(x_0)$ mavjud bo‘lmaydi.

Teskari tasdiq har doim ham o‘rinli emas, ya’ni agar $f''(x_0)=0$ bo‘lsa yoki $f''(x_0)$ mavjud bo‘lmasa, u holda x_0 nuqta $f(x)$ funksiyaning egilish nuqtasi bo‘lavermaydi. Misol uchun, $f(x)=(x-2)^4+1$ funksiya grafigidan ko‘rinadiki, $x=2$ nuqta egilish nuqtasi bo‘la olmaydi, lekin bu nuqtada $f''(2)=0$.



Demak, egilish nuqtasiga shubhali nuqtani topish uchun biz shunday x_0 nuqtani izlashimiz kerakki, bu nuqtada $f''(x_0)=0$ bo‘lsin yoki $f''(x_0)$ 2-tartibli hosila mavjud bo‘lmasin.

Egilish nuqtasi haqidagi 6-teorema kritik nuqta haqidagi 2-teoremaga o‘xshaydi. 2-teoremada agar c kritik nuqta bo‘lsa, bu nuqtada $f'(c)=0$ yoki $f'(c)$ mavjud bo‘lmaydi deyilgan edi. 6-teoremada esa agar x_0 nuqta $f(x)$ funksiyaning egilish nuqtasi bo‘lsa, bu nuqtada $f''(x_0)=0$ yoki $f''(x_0)$ mavjud bo‘lmaydi deyilmoqda.

3-misol. 2-misoldagi $f(x)=3x^5-20x^3$ funksiyaning 2-tartibli hosilasidan foydalanib, egilish nuqtasini toping.

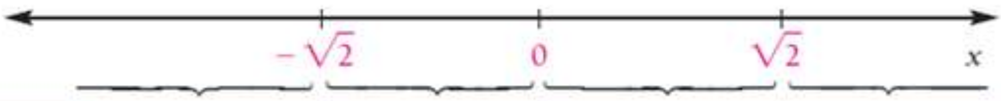
Yechilishi: ► Bizda 2- tartibli hosila $f''(x)=60x^3-120x$ ga teng.

Uni nolga tenglab, yechimlarni topamiz:

$$60x^3-120x=0$$

$$x(x^2-2)=0$$

Bundan, $x=0$ va $x=-\sqrt{2}$ hamda $x=\sqrt{2}$ qiymatlarni hosil qilamiz. So‘ngra shu 3 ta nuqta yordamida ajratilgan oraliqlarda 2-tartibli hosilaning ishoralarini tekshiramiz.



	$x=-2$	$x=-1$	$x=1$	$x=2$
$f''(x)$	$f''(-2)<0$	$f''(-1)>0$	$f''(1)<0$	$f''(2)>0$

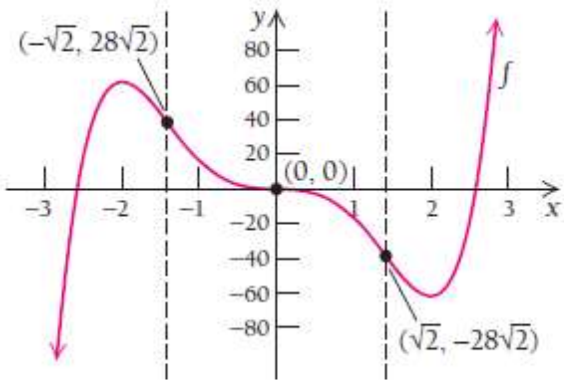
Endi funksiyaning bu nuqtalardagi qiymatlarini hisoblaymiz:

$$f(-\sqrt{2})=3\cdot(-\sqrt{2})^5-20\cdot(-\sqrt{2})^3=28\sqrt{2}$$

$$f(0)=3\cdot 0^5-20\cdot 0^3=0$$

$$f(\sqrt{2})=3\cdot(\sqrt{2})^5-20\cdot(\sqrt{2})^3=-28\sqrt{2}.$$

Demak, $(-\sqrt{2};\ 28\sqrt{2})$, $(0;\ 0)$ va $(\sqrt{2};\ -28\sqrt{2})$ nuqtalar funksiyaning egilish nuqtalari ekan.



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2-vazifa. $f(x)=-6x^5-10x^3$ funksiyaning egilish nuqtalarini toping.

3.2.4. Funksiya grafigini chizish

1- va 2-tartibli hosilalar grafik chizish borasidagi bizning tasavvurlarimizni kengaytiradi. Funksiya grafigini chizish uchun biz quyidagi ketma-ketlikni qo‘llaymiz:

1) **Funksiyaning aniqlanish sohasi topish.**
2) **Funksiyaning kritik nuqtalarini topish**, ya’ni $f'(x)=0$ bo‘ladigan yoki $f'(x)$ mavjud bo‘lmaydigan nuqtalarni topamiz. Bu sonlar bizni maksimum va minimum qiymatlarga olib boradi. Funksiyaning shu nuqtalardagi qiymatlarini topamiz.
3) **O‘sish va kamayish oraliqlari, ekstremumlarini topish.** x_0 nuqta atrofida $f''(x_0)$ ni ishorasini tekshiramiz. Agar $f''(x_0)<0$ bo‘lsa, $f(x_0)$ maksimum nuqta, agar $f''(x_0)>0$ bo‘lsa, $f(x_0)$ minimum nuqta bo‘ladi.
4) **Egilish nuqtalarini topish.** $f''(x_0)=0$ bo‘ladigan yoki $f''(x_0)$ mavjud bo‘lmaydigan nuqtalarni aniqlaymiz. Bu nuqtalarda funksiya qiymatlarini hisoblaymiz.
5) **Qavariqlik va botiqlik oraliqlarini topish.** 4-shartdagi egilishga shubhali nuqtalardan foydalanamiz. Agar $f''(x_0)>0$ bo‘lsa, bu oraliqda funksiya botiq, $f''(x_0)<0$ bo‘lsa, funksiya qavariq bo‘ladi.
6) **Grafikni yasash.** Yuqoridagi topilgan ma’lumotlar asosida funksiya grafigini chizamiz.

   Agar zarurat bo‘lsa, qo‘shimcha ma’lumotlarni ham hisoblab topamiz.

4-misol. $f(x)=x^3-3x+2$ funksiyani tekshiring va grafigini chizing.

Yechilishi: ►

1) Funksiyaning aniqlanish sohasini topamiz: Berilgan funksiya ko‘phaddan iborat, shuning uchun butun son o‘qida ma’noga ega, ya’ni $D(f)=(-\infty,\infty)$.

2) Funksiyaning kritik nuqtalarini topamiz: Buning uchun funksiyadan hosila olamiz va nolga tenglab, yechamiz.

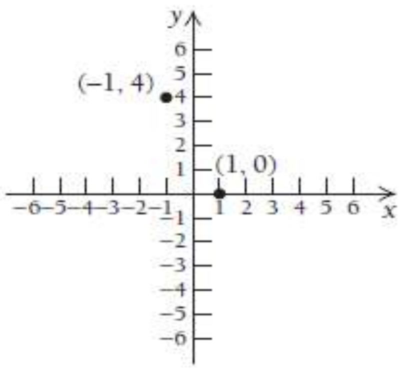
$$f'(x)=\left(x^3-3x+2\right)'=3x^2-3$$

$$3x^2-3=0,\quad x=-1 \text{ va } \quad x=1$$

$$f(-1)=(-1)^3-3\cdot(-1)+2=4,$$

$$f(1)=1^3-3\cdot 1+2=0,$$

$(-1;\ 4)$ $(1;\ 0)$ - kritik nuqtalar.



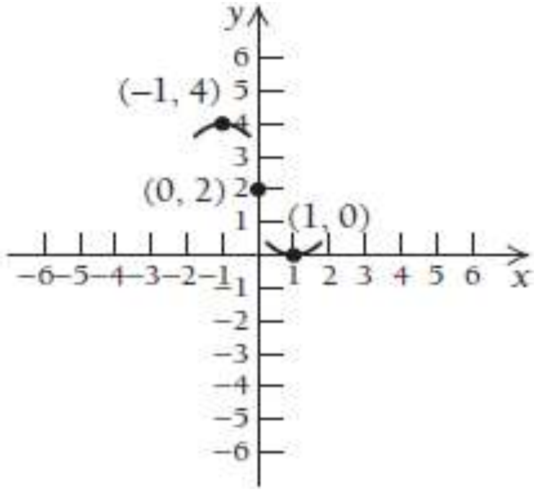
3) O‘sish va kamayish oraliqlari, ekstremumlarini topish.

2-tartibli hosilani olamiz: $f''(x)=\left(3x^2-3\right)'=6x$ va $x=-1$ va $x=1$ nuqtalarda 2-tartibli hosilani ishorasini tekshiramiz.

$$f''(-1)=6\cdot(-1)=-6<0,\qquad f''(1)=6\cdot 1=6>0.$$

Demak, $f(-1)=4$ maksimum nuqta va funksiya $(-\infty;\ -1)$ da o‘sadi, $(-1;\ 1)$ da kamayadi. $f(1)=0$ minimum nuqta va funksiya $(-1;\ 1)$ da kamayadi, $(1;\ \infty)$ da o‘sadi.

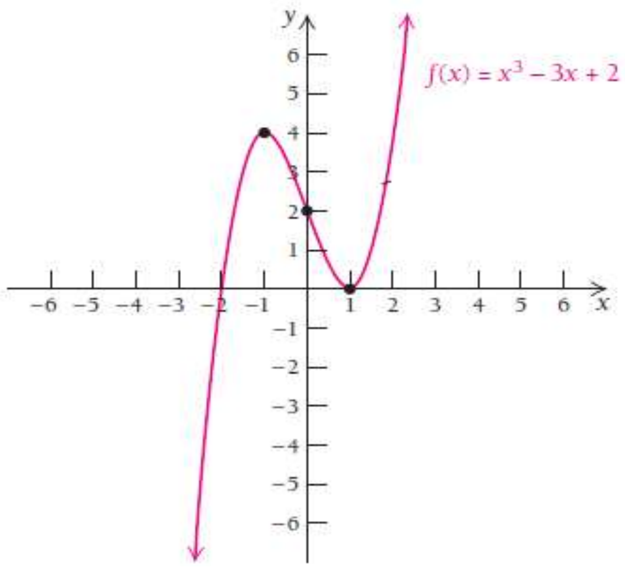
4) Egilish nuqtalarini topish. $f''(x)=0$ yoki $f''(x)$ mavjud bo‘lmagan nuqtalarni topamiz. $f''(x)=6x=0$. Bundan $x=0$. $f(0)=0^3-3\cdot 0+2=2$, demak $(0;\ 2)$ nuqtada grafik buriladi.



5) Qavariqlik va botiqlik oraliqlarini topish. $(0;\ 2)$ egilish nuqtasi bo‘lsa, u holda funksiya $(-\infty;\ 0)$ da qavariq, $(0;\ \infty)$ da botiq bo‘ladi.

	$x=-1$	$x=1$
$f''(x)$	$f''(-1)<0$	$f''(1)>0$

6) Grafikni yasash.



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5-misol. $f(x)=x^4-2x^2$ funksiyani tekshiring va grafigini chizing.

Yechilishi: ►

1) Funksiyaning aniqlanish sohasini topamiz: Berilgan funksiya ko‘phaddan iborat, shuning uchun butun son o‘qida ma’noga ega, ya’ni $D(f)=(-\infty,\infty)$.

2) Funksiyaning kritik nuqtalarini topamiz: Buning uchun funksiyadan hosila olamiz va nolga tenglab, yechamiz.

$$f'(x)=(x^4-2x^2)'=4x^3-4x$$

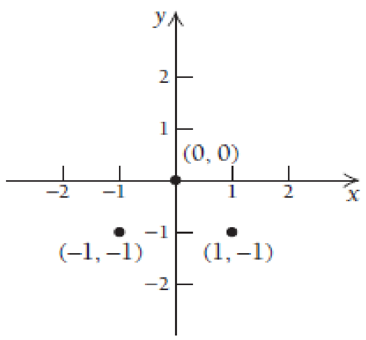
$4x^3-4x=0$, $\quad 4x(x^2-1)=0 \quad$ $x=0$, $x=-1$ va $x=1$

$f(0)=0^4-2\cdot 0^2=0$

$f(-1)=(-1)^4-2\cdot(-1)^2=-1$

$f(1)=1^4-2\cdot 1^2=-1$,

demak $(-1;\ -1)$, $(0;\ 0)$, $(1;\ -1)$ nuqtalarni topdik.



3) O‘sish va kamayish oraliqlari, ekstremumlarini topish.

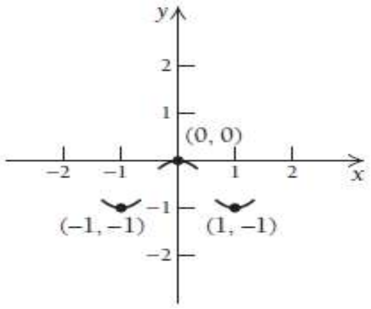
2-tartibli hosilani olamiz: $f''(x)=(4x^3-4x)'=12x^2-4$

$x=0$, $x=-1$ va $x=1$ nuqtalarda 2-tartibli hosila ishorasini tekshiramiz.

$f''(-1)=12\cdot(-1)^2-4=8>0$, $\quad (-\infty;\ -1)$ da kamayadi, $(-1;\ 0)$ da o‘sadi.

$f''(0)=12\cdot 0^2-4=-4<0$, $\quad (-1;\ 0)$ da o‘sadi, $(0;\ 1)$ da kamayadi.

$f''(1)=12\cdot 1^2-4=8>0$, $\quad (0;\ 1)$ da kamayadi, $(1;\ \infty)$ da o‘sadi.

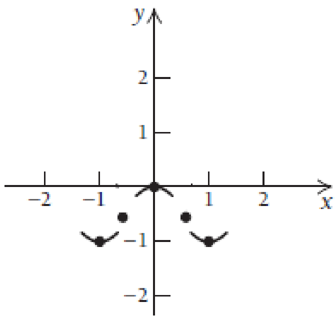


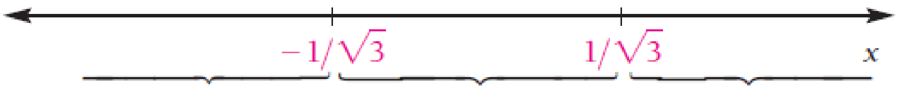
4) Egilish nuqtalarini topish. $f''(x) = (4x^3 - 4x)' = 12x^2 - 4 = 0$,

$$3x^2 - 1 = 0, \qquad x = \pm\frac{1}{\sqrt{3}}.$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = \left(-\frac{1}{\sqrt{3}}\right)^4 - 2\cdot\left(-\frac{1}{\sqrt{3}}\right)^2 = -\frac{5}{9}, \quad f\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^4 - 2\cdot\left(\frac{1}{\sqrt{3}}\right)^2 = -\frac{5}{9}.$$

Shunday qilib, $\left(-\frac{1}{\sqrt{3}};-\frac{5}{9}\right)$ va $\left(\frac{1}{\sqrt{3}};-\frac{5}{9}\right)$ nuqtalar egilish nuqtalari ekanini topdik.





	$x = -1$	$x = 0$	$x = 1$
$f''(x)$	$f''(-1) > 0$	$f''(0) < 0$	$f''(1) > 0$

5) Qavariqlik va botiqlik oraliqlarini topish.

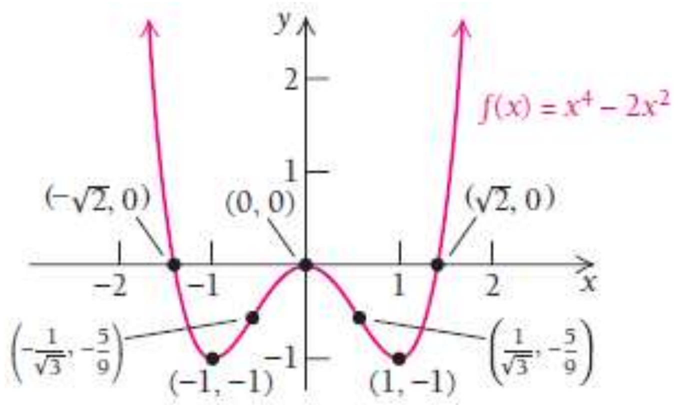
$\left(-\frac{1}{\sqrt{3}};-\frac{5}{9}\right)$ va $\left(\frac{1}{\sqrt{3}};-\frac{5}{9}\right)$ egilish nuqtalari bo‘lsa, u holda

$\left(-\infty;-\frac{1}{\sqrt{3}}\right)$ va $\left(\frac{1}{\sqrt{3}};\ \infty\right)$ da funksiya botiq,

$\left(-\frac{1}{\sqrt{3}};\ \frac{1}{\sqrt{3}}\right)$ oraliqda funksiya qavariq bo‘ladi.

6) Grafikni yasash. Grafik uchun funksiya nollarini ham topib qo‘yamiz: $f(x) = x^4 - 2x^2 = 0$, $x = 0$, $x = -\sqrt{2}$ va $x = \sqrt{2}$.

Bundan $(-\sqrt{2};0)$, $(0;0)$, $(\sqrt{2};0)$ nuqtalarda koordinata o‘qlarini kesib o‘tishini bilib olamiz.



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MUSTAQIL YECHISH UCHUN MASALALAR

1-8 misollarda funksiyaning ekstremumlarini toping va ularning maksimum yoki minimum ekanligini ko‘rsating. Zarurat bo‘lsa, 2-tartibli hosiladan foydalaning.

1. $f(x) = 4 - x^2$ **2.** $f(x) = 7 - x^2$

3. $f(x) = x^2 - x$ **4.** $f(x) = x^2 + x - 1$

5. $f(x) = -5x^2 + 8x - 7$ **6.** $f(x) = -4x^2 + 3x - 1$

7. $f(x) = 8x^3 - 6x + 1$ **8.** $f(x) = x^3 - 12x - 1$

9-46 misollarda funksiyani to‘liq tekshiring va grafigini chizing.

9. $f(x) = x^3 - 15x$ **10.** $f(x) = x^3 - 27x$

11. $f(x) = 3x^3 - 36x - 3$ **12.** $f(x) = 2x^3 - 3x^2 - 36x + 28$

13. $f(x) = \frac{8}{3}x^3 - 2x - \frac{7}{5}$ **14.** $f(x) = -x^3 - 9x^2 + 100$

15. $f(x) = 4 + 3x^2 - x^3$ **16.** $f(x) = 1 + 3x - x^3$

17. $f(x) = 3x^4 - 16x^3 + 18x^2$ **18.** $f(x) = 3x^4 + 4x^3 - 12x^2 + 5$

19. $f(x) = x^4 - 6x^2$ **20.** $f(x) = 2x^2 - x^4$

21. $f(x)=x^3-2x^2-4x+3$

22. $f(x)=x^3-6x^2+9x+1$

23. $f(x)=3x^4+4x^3$

24. $f(x)=x^4-2x^3$

25. $f(x)=x^3-6x^2-135x$

26. $f(x)=x^3-3x^2-144x-140$

27. $f(x)=x^4-4x^3+10$

28. $f(x)=\frac{4}{3}x^3-2x^2+x$

29. $f(x)=x^3-6x^2+12x-6$

30. $f(x)=x^3+3x+1$

31. $f(x)=5x^3-3x^5$

32. $f(x)=20x^3-3x^5$

33. $f(x)=x^2(3-x)^2$

34. $f(x)=x^2(1-x)^2$

35. $f(x)=(x+1)^{2/3}$

36. $f(x)=(x-1)^{2/3}$

37. $f(x)=5-2(x-4)^{2/3}$

38. $f(x)=3-3(x-2)^{2/3}$

39. $f(x)=(x-3)^{1/3}-1$

40. $f(x)=(x-2)^{1/3}+3$

41. $f(x)=x\sqrt{4-x^2}$

42. $f(x)=-x\sqrt{1-x^2}$

43. $f(x)=\frac{x}{x^2+1}$

44. $f(x)=\frac{8x}{x^2+1}$

45. $f(x)=\frac{3}{x^2+1}$

46. $f(x)=\frac{-4}{x^2+1}$

47-57 misollarda berilgan ma’lumotlar asosida funksiya grafigini yasang. Chizmalar yagona bo‘lmasligi mumkin.

47. $f(x)$: $(-\infty;\ 4)$ oraliqda qavariq, $(4;\ \infty)$ oraliqda botiq;

48. $G(x)$ funksiya $(-\infty;\ 2)$ oraliqda botiq, $(2;\ \infty)$ oraliqda qavariq;

49. $f(x)$: $(-\infty;\ -1)$ oraliqda qavariq, $(-1;\ \infty)$ oraliqda botiq;

50. $G(x)$ funksiya $(-\infty;\ -3)$ oraliqda botiq, $(-3;\ \infty)$ oraliqda qavariq;

51. $G(x)$ funksiya $(1;\ 5)$ nuqtada botiq, $(7;\ -2)$ nuqtada qavariq va $(4;\ 1)$ nuqta egilish nuqtasi;

52. $G(x)$ funksiya $(1;\ -3)$ nuqtada qavariq, $(8;\ 6)$ nuqtada botiq va $(3;\ 5)$ nuqta egilish nuqtasi;

53. $f'(-1)=0$, $f''(-1)>0$, $f(-1)=-5$,

$f'(7)=0$, $f''(7)<0$, $f(7)=10$, $f''(3)=0$, $f(3)=2$.

54. $f'(-3)=0$, $f''(-3)<0$, $f(-3)=8$,

$f'(9)=0$, $f''(9)>0$, $f(9)=-6$, $f''(2)=0$, $f(2)=1$.

55. $f'(-1)=0$, $f''(-1)>0$, $f(-1)=-2$,

$f'(1)=0$, $f''(1)>0$, $f(1)=-2$, $f''(0)=0$, $f''(0)<0$, $f(0)=0$.

56. $f'(0)=0$, $f''(0)<0$, $f(0)=5$,

$f'(2)=0$, $f''(2)>0$, $f(2)=2$, $f'(4)=0$, $f''(4)<0$, $f(4)=3$.

57. $f'(-2)=0$, $f''(-2)<0$, $f(-2)=-5$,

$f'(2)=0$, $f''(2)>0$, $f(2)=-5$, $f'(1)=0$, $f''(1)<0$, $f(1)=3$.

MAVZUNING AMALIYOTGA TATBIQLARI

Iqtisodiyot: Tannarx, daromad, foyda masalalari

58. Daromad $D(x)=50x-0.5x^2$ va tannarx $T(x)=4x+10$ bo‘lsa, umumiy daromad, umuminy tannarx funksiyalari grafiklarini chizing.

59. Daromad $D(x)=50x-0.5x^2$ va tannarx $T(x)=3x+10$ bo‘lsa, umumiy daromad, umuminy tannarx funksiyalari grafiklarini chizing.

60. Kichik tadbirkorlik. Qishloq xo‘jaligiga ixtisoslashmagan tadbirkorning foydasi $f(x)=\dfrac{13x^3-240x^2-2460x+585000}{75000}$ funksiya bilan approksimatsiyalanadi, bunda 1980 yildan buyon $x-$yillar soni bo‘lsa, $0\le x\le 40$ oraliqda funksiya grafigini chizing.

61. Ishchi kuchi. Qaysidir davlatda 45-54 yoshdagi ishchi kuchini $f(x)=0.025x^2-0.71x+20.44$ funksiya bilan modellashtirish mumkin, bunda 1980 yildan buyon $x-$yillar soni bo‘lsa, $0\le x\le 30$ oraliqda funksiya grafigini chizing.

Fizika: Tezlik, harorat.

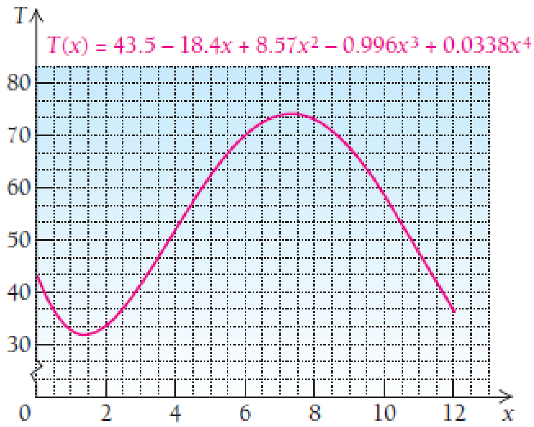
62. Yo‘tal tezligi. Odam traxeyasiga yot jism o‘tib qolsa, u yo‘taladi.

Yo‘tal tezligi yot jismning o‘lchamiga bog‘liq. Aytaylik, odamning traxeyasi 20 mm radiusga ega. Agar yot jism r (mm) radiusga ega bo‘lsa, u holda v tezlikdagi yo‘tal bu jismni chiqarib tashlashi mumkin: $v(r) = k(20r^2 - r^3)$, $0 \le r \le 20$. Bunda $k = const$ va $k > 0$. Talab qilinadigan tezlikning maksimal qiymati qanday bo‘ladi?

63. Toshkentda havo harorati. Toshkentda o‘rtacha harorat

$$T(x) = 45.3 - 18.4x - 8.57x^2 - 0.996x^3 + 0.0338x^4$$

funksiya bilan approksimatsiyalanadi (Farengeyt birligida), bunda $x = 1$ yanvar oyining o‘rtalaridagi, $x = 2$ fevral oyining o‘rtalaridagi va h.k. havo harorati ko‘rsatkichlari.



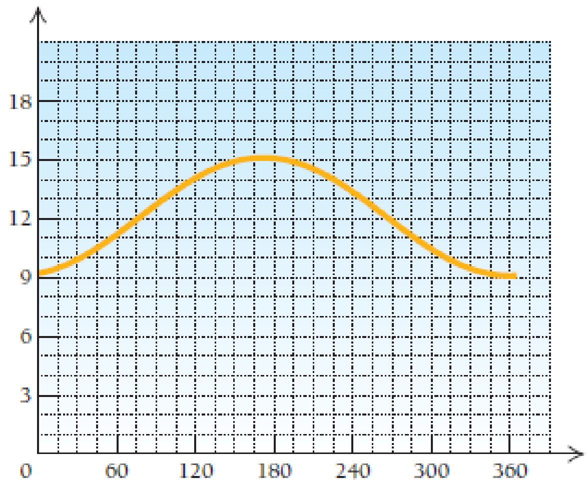
Diagrammadan foydalanib,

a) eng issiq havo qachon bo‘lishi,

b) eng sovuq havo qachon bo‘lishi;

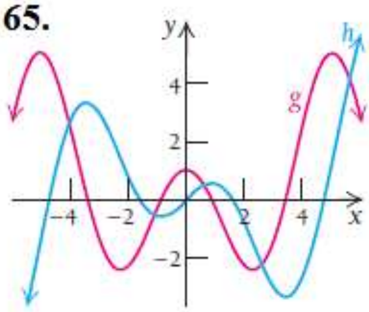
v) 2-tartibli hosila yordamida funksiyaning botiq, qavariqlik va egilish nuqtalarini toping.

64. Kun uzunligi. Toshkentda kun uzunligi diagrammasi keltirilgan, bunda vertikal o‘q kun uzunligi (soat hisobida), gorizontal o‘qda yilning 365 kuni (yil boshidan hisoblangan). Qaysi kunda kun eng uzun bo‘ladi? Buni qanday aniqlaysiz?

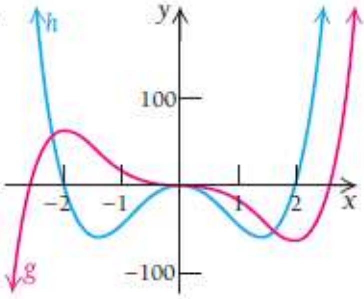


65-66 misollarda keltirilgan grafiklarning qaysi biri boshqasining hosilasidan iborat, javobingizni tushuntiring.

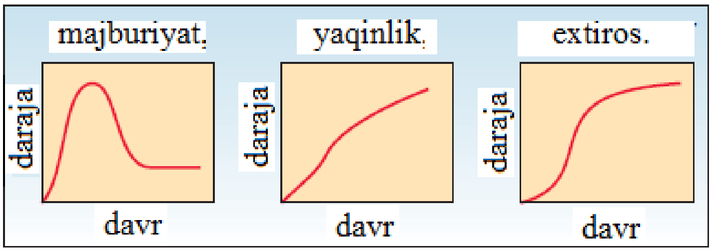
65.



66.



67. Psixologiya. Olimlarning aniqlashicha odamlar orasida sevgining 3 turi uchraydi: majburiyat, yaqinlik, extiros.



Diagrammalarni tahlil qilib, ekstremum, o‘sish, kamayish, egilish va boshqa kattaliklarni aniqlang. Diagrammalar borasida tadqiqotchilar bilan bir xil fikrdamisiz? Nima uchun?

68. $f(x)=ax^2+bx+c$, $a\neq 0$ funksiyalarning barchasida maksimum yoki minimum $x=-\dfrac{b}{2a}$ nuqtada bo‘lishini isbotlang.

69. $f(x)=ax^3+bx^2+cx+d$, $a\neq 0$ funksiyalarning barchasida $x=-\dfrac{b}{3a}$ egilish nuqtasi bo‘lishini isbotlang.

70. $y=x^3-5x^2+3x-5$ funksiya grafigining qavariq, botiqlik oraliqlari va egilish nuqtalarini toping.

71. $y=\ln(1+x^2)$ funksiya grafigining qavariq, botiqlik oraliqlari va egilish nuqtalarini toping.

3.3. Funksiyaning asimptotalari va ularni aniqlash

3.3.1. Ratsional funksiyalar

Shu paytgacha biz hosila yordamida tekshirgan va grafiklarini chizgan funksiyalarning barchasi uzluksiz funksiyalar edi. Endi uzilishga ega boʻlgan funksiyalarni qarab chiqamiz. Ularning koʻpchiligi ratsional funksiyalardan iborat.

Ta’rif. $f(x)=\dfrac{P(x)}{Q(x)}$ koʻrinishdagi funksiyaga **ratsional funksiya** deyiladi, bunda $P(x)$ va $Q(x)$ koʻphadlar, $Q(x)$ noldan farqli koʻphad. $f(x)$ funksiyaning aniqlanish sohasi $Q(x)\neq 0$ munosabatni qanoatlantiradigan barcha x lardan iborat.

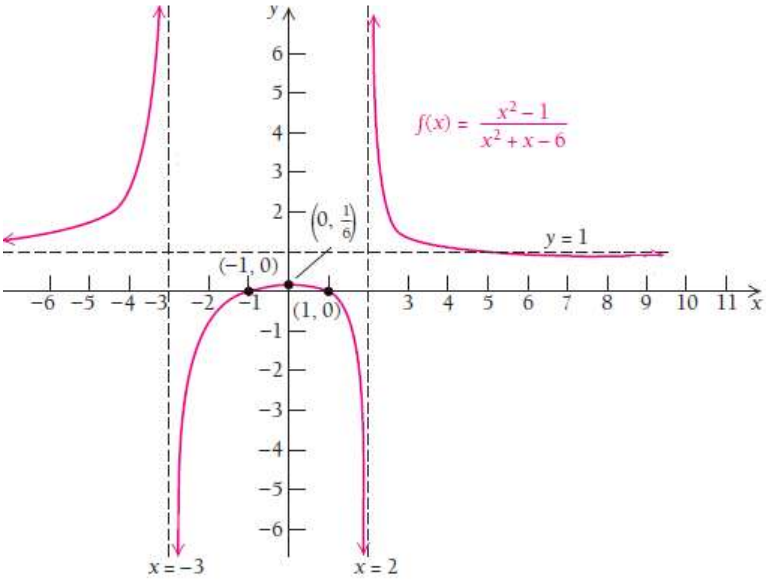
Koʻphadlar – ratsional funksiyaning $Q(x)=1$ boʻlgandagi alohida maxsus koʻrinishi hisoblanadi. Ushbu boʻlimda biz ratsional funksiyaning maxraji oʻzgarmas son boʻlmaydigan holatlarini oʻrganamiz va grafigini yasaymiz. Dastlab, bunday funksiyalarning asimptotalari nima va ular qanday topiladi, degan savollarga javob izlaymiz.

3.3.2. Funksiyalarning asimptotalari

1-misol. $f(x)=\dfrac{x^2-1}{x^2+x-6}=\dfrac{(x-1)(x+1)}{(x+2)(x-3)}$ funksiya grafigini qaraylik.

► Grafikdan koʻrinadiki, argument $x\to 2$ ga chap tomondan yaqinlashgani sari funksiyaning qiymati cheksiz kichiklashib, $-\infty$ ga intilib boraveradi. Argument $x\to 2$ ga oʻngdan yaqinlashib borgani sari funksiyaning qiymati cheksiz kattalashib, ∞ ga intilib boraveradi. Buni quyidagicha yozamiz: $\lim_{x\to 2^-} f(x) = -\infty$ va $\lim_{x\to 2^+} f(x) = \infty$.

$x = 2$ chiziqqa vertikal asimptota deyiladi.



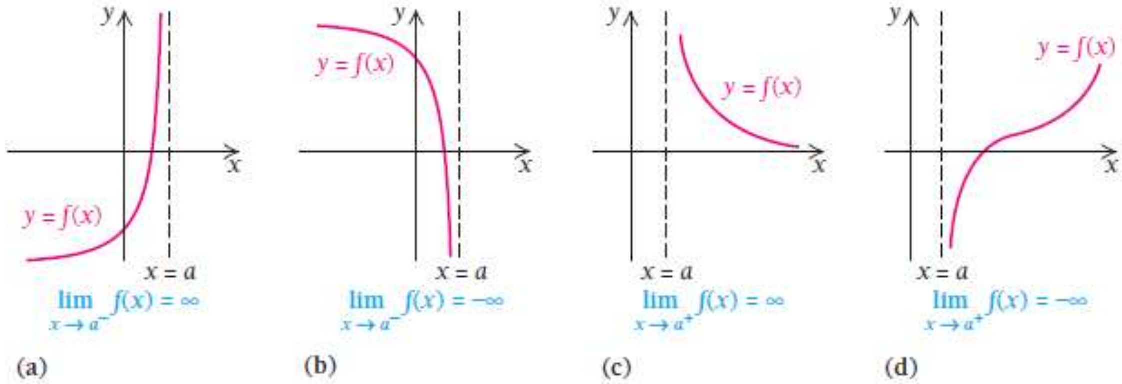
◀

Ta'rif. Agar $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^-} f(x) = -\infty$, $\lim_{x\to a^+} f(x) = \infty$ yoki $\lim_{x\to a^+} f(x) = -\infty$ limit tengliklardan birortasi o'rinli bo'lsa, $x = a$ to'g'ri chiziqqa **vertikal asimptota** deyiladi.

Ratsional funksiyaning grafigi hech qachon vertikal asimptota bilan kesishmaydi. Agar a soni ratsional funksiyaning mahrajini nolga aylantirsa, u holda $x = a$ chiziq vertikal asimptota bo'ladi.

Misol uchun, $f(x) = \frac{x^2 - 16}{x - 4} = \frac{(x-4)(x+4)}{x-4}$ funksiyada $x = 4$ nuqtada mahraj nol bo'lsa ham, $x = 4$ chiziq vertikal asimptota bo'la olmaydi, chunki $x - 4$ ifoda kasrning surat va mahraji uchun umumiy ko'paytuvchisi bo'ladi va ular qisqarib ketadi. $g(x)=\dfrac{x^2-4}{x^2+x-12}=\dfrac{(x-2)(x+2)}{(x-3)(x+4)}$ funksiyada esa $x=-4$ va $x=3$ vertikal asimptota chiziqlari bo‘ladi.

Quyida turli ko‘rinishdagi vertikal asimptotalar ko‘rsatilgan:



1-vazifa. $f(x)=\dfrac{1}{x(x^2-25)}$ **funksiyaning vertikal asimptotalarini toping.**

2-misol. $g(x)=\dfrac{2x-1}{x(x-3)(x+5)}$ funksiyaning vertikal asimptotalarini toping.

Yechilishi: ► Mahrajni nolga aylantiradigan qiymatlarni aniqlaymiz. Bular $x=0$, $x=3$ va $x=-5$ chiziqlardir. ◄

3-misol. $f(x)=\dfrac{x^2-2x}{x^3-x}$ funksiyaning vertikal asimptotalarini toping.

Yechilishi: ► Mahrajni nolga aylantiradigan qiymatlarni aniqlaymiz.

$$f(x)=\frac{x^2-2x}{x^3-x}=\frac{x(x-2)}{x(x-1)(x+1)}=\frac{x-2}{(x-1)(x+1)},\qquad x\neq 0$$

Shunday qilib, $x=1$ va $x=-1$ vertikal asimptotalarni hosil qilamiz. ◄

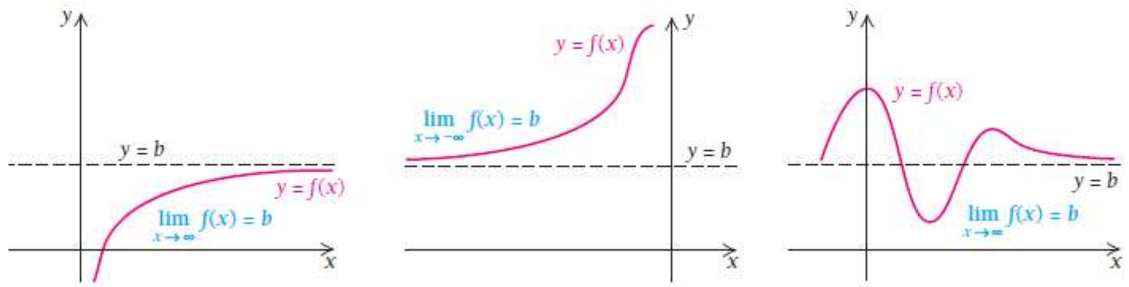
2-vazifa. $f(x)=\dfrac{x^2-2x}{x^3-x}$ **nima uchun** $x=0$ **vertikal asimptota emas, bu nuqtada qanday uzilish ro‘y beradi?**

1-misoldagi funksiya grafigiga qarang. $x \to -\infty$ yaqinlashib borganda, funksiya qiymati $f(x) \to 1$ ga yaqinlashib bordi. Xuddi shuningdek, argument $x \to \infty$ yaqinlashib borganda ham funksiya qiymati $f(x) \to 1$ ga yaqinlashib bordi. Bu mulohazalarni limit ko‘rinishida yozamiz: $\lim\limits_{x \to -\infty} f(x) = 1$ va $\lim\limits_{x \to \infty} f(x) = 1$.

U holda ushbu $y = 1$ to‘g‘ri chiziqqa gorizontal asimptota deymiz.

Ta’rif. Agar $\lim\limits_{x \to -\infty} f(x) = b$ yoki $\lim\limits_{x \to \infty} f(x) = b$ limitlarning bittasi yoki ikkalasi ham o‘rinli bo‘lsa, $y = b$ to‘g‘ri chiziqqa **gorizontal asimptota** deyiladi.

Quyida turli ko‘rinishdagi gorizontal asimptotalar keltirilgan:



4-misol. $f(x) = \dfrac{3x - 4}{x}$ funksiyaning gorizontal asimptotasini toping.

Yechilishi:

► Gorizontal asimptotani topish uchun $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \dfrac{3x - 4}{x}$ ni hisoblash kerak. Limitni hisoblash esingizdan chiqdimi? U holda jadval tuzamiz.

x	1	10	50	100	2000
$\dfrac{3x-4}{x}$	−1	2.6	2.92	2.96	2.998

Jadvaldan ko‘rish mumkinki, $\lim\limits_{x\to\infty}\frac{3x-4}{x}=3$ ekan.

Limit xossalaridan foydalanib, hisoblasak:

$$\lim_{x\to\infty}\frac{3x-4}{x}=\lim_{x\to\infty}\frac{3x}{x}-\lim_{x\to\infty}\frac{4}{x}=\lim_{x\to\infty}(3-0)=3.$$

Shunday qilib, $y=3$ gorizontal asimptota ekaniga ishonch hosil qilamiz.

5-misol. $f(x)=\frac{5x^2+3x-4}{2x^2-x-1}$ funksiyaning gorizontal asimptotasini toping.

Yechilishi: ► $\lim\limits_{x\to\infty}\frac{5x^2+3x-4}{2x^2-x-1}$ limitni hisoblash uchun kasrning surat va mahrajidagi x ning darajalarini qaraymiz. Argument cheksizlikka intilganda, limitni hisoblash uchun argumentning eng yuqori daraja ko‘rsatkichiga ega bo‘lgan hadga bo‘lamiz.

$$\lim_{x\to\infty}\frac{5x^2+3x-4}{2x^2-x-1}=\lim_{x\to\infty}\frac{\frac{5x^2}{x^2}+\frac{3x}{x^2}-\frac{4}{x^2}}{\frac{2x^2}{x^2}-\frac{x}{x^2}-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{5}{2}=\frac{5}{2}$$

$y=\frac{5}{2}$ to‘g‘ri chiziq gorizontal asimptota bo‘ladi. ◄

3-vazifa. $f(x)=\frac{(2x-1)(x+1)}{(3x+2)(5x-6)}$ **funksiyaning gorizontal asimptotasini toping.**

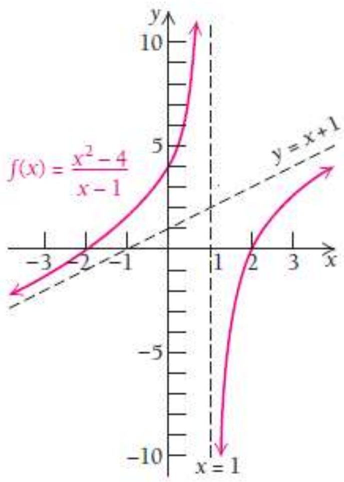
6-misol. $f(x)=\frac{3x+1}{x^3-2x^2+4}$ funksiyaning gorizontal asimptotasini toping.

Yechilishi: ► $$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{3x+1}{x^3-2x^2+4}=\lim_{x\to\infty}\frac{\frac{3x}{x^3}+\frac{1}{x^3}}{\frac{x^3}{x^3}-\frac{2x^2}{x^3}+\frac{4}{x^3}}=\lim_{x\to\infty}\frac{0}{1}=0$$

$y=0$ to‘g‘ri chiziq gorizontal asimptota bo‘ladi. ◄

Xulosa: Agar ratsional funksiyaning suratidagi ko‘phadning daraja ko‘rsatkichi maxrajdagi ko‘phad daraja ko‘rsatkichidan kichik bo‘lsa, u holda $y = 0$ **to‘g‘ri chiziq yoki** X **o‘qi gorizontal asimptota** bo‘ladi.

Ba’zi funksiyalarda vertikal va gorizontal asimptotalardan tashqari og‘ma asismptota ham mavjud. $f(x) = \frac{x^2 - 4}{x - 1}$ funksiyaning asimptotasini qaraylik.



$|x|$ ning qiymatlari cheksizlikka intilgan sari, funksiya grafigi $y = x + 1$ to‘g‘ri chiziqqa intiladi. Bu to‘g‘ri chiziq og‘ma asimptota deyiladi.

Ta’rif. Vertikal ham, gorizontal ham bo‘lmagan chiziqli asimptotaga **og‘ma asimptota** deyiladi.

Agar $f(x) = \frac{p(x)}{q(x)}$ ratsional funksiyaning suratidagi ko‘phadning darajasi mahrajdagi ko‘phad darajasidan 1 birlik yuqori bo‘lsa, bu funksiya og‘ma asimptotaga ega bo‘ladi.

Og‘ma asimptotani qanday aniqlaymiz?

7-misol. $f(x)=\dfrac{x^2-4}{x-1}$ funksiyaning asimptotasini topamiz.

Yechilishi: ► Ratsional ifodaning suratini mahrajiga bo‘lib, butun qismini ajratib olamiz: $f(x)=\dfrac{x^2-4}{x-1}=(x+1)-\dfrac{3}{x-1}$.

$|x|$ ning qiymatlari cheksizlikka intilganda funksiyaning $-\dfrac{3}{x-1}$ qismi nolga intiladi. Funksiyaning butun qismidan tuzilgan $y=x+1$ to‘g‘ri chiziq og‘ma asimptota bo‘ladi. ◄

Og‘ma asimptotani topishning boshqa usuli ham mavjud.

Ratsional funksiyaning og‘ma asimptotasi $y=kx+b$ tenglama bilan aniqlanadi. k burchak koeffitsiyentini $k=\lim\limits_{x\to\infty}\dfrac{f(x)}{x}$ limitdan, b ozod hadni esa $k=\lim\limits_{x\to\infty}(f(x)-kx)$ limit yordamida aniqlaymiz.

8-misol. $y=\dfrac{x^3}{x^2-4}$ funksiya grafigining asimptotalarini toping.

Yechilishi: ► Funksiya $x\neq\pm2$ da aniqlangan.

$\lim\limits_{x\to\pm2}\dfrac{x^3}{x^2-4}=\pm\infty$ bo‘lgani uchun, $x=-2$ va $x=2$ to‘g‘ri chiziqlar funksiya grafigining **vertikal asimptotalari** bo‘ladi.

Endi og‘ma asimptotalarni izlaymiz: $k=\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to\infty}\dfrac{x^2}{x^2-4}=1$,

$$b=\lim_{x\to\infty}(f(x)-kx)=\lim_{x\to\infty}\left(\frac{x^3}{x^2-4}-x\right)=\lim_{x\to\infty}\frac{4x}{x^2-4}=0.$$

Demak, $y=x$ to‘g‘ri chiziq og‘ma asimptota bo‘ladi.◄

4-vazifa. $f(x)=\dfrac{2x^2+x-1}{x-3}$ **funksiyaning og‘ma asimptotasini toping.**

3.3.3. Funksiya grafigining koordinata o‘qlari bilan kesishish nuqtalari

Agar funksiyaning $f(x)=0$ tenglamaning yechimini topsak, bu yechimlar bizga funksiyaning X o‘qi bilan kesishish nuqtalarini beradi. Agar funksiya ifodasida barcha x lar o‘rniga nol qiymat, ya’ni $x=0$ desak, u holda funksiyaning Y o‘qi bilan kesishish nuqtalarini topamiz.

9-misol. $f(x)=\dfrac{x^3-x^2-6x}{x^2-3x+2}$ funksiyaning koordinata o‘qlari bilan kesishish nuqtalarini toping.

Yechilishi: ► Kasrning surat va mahrajidagi ko‘phadlarni ko‘paytuvchilarga ajratamiz: $f(x)=\dfrac{x(x+2)(x-3)}{(x-1)(x-2)}$.

Endi hosil bo‘lgan ifodani nolga tenglaymiz: $f(x)=0$, buning uchun faqat kasrning surati nolga teng bo‘lishi kerak: $x(x+2)(x-3)=0$.

$x=0$, $x=-2$, $x=3$ qiymatlarni olamiz.

Bu nuqtalarning 2-koordinatalari 0 ga teng. Demak, funksiya $(0;\ 0)$, $(-2;\ 0)$ va $(3;\ 0)$ nuqtalarda X o‘qini kesib o‘tadi. Y o‘qini ham (0; 0) nuqtada kesadi. ◄

5-vazifa. $f(x)=\dfrac{x^3-x}{x^2-4}$ **funksiyaning qavariqlik va botiqlik oraliqlarini toping.**

3.3.4. Funksiyani to'liq tekshirish va grafigini chizish

3.2.4-mavzuda 1- va 2- tartibli hosilalar yordamida funksiya grafigini chizish algoritmini bayon qilgan edik. Endi funksiyani to'liq tekshirish uchun nimalar qilish va qanday ketma-ketlikda bajarish algoritmini keltiramiz:

1) **Funksiyaning koordinata o'qlari bilan kesishish nuqtalarini aniqlash;**
2) **Funksiyaning vertikal, gorizontal, og'ma asimptotalarini topish;**
3) **Funksiyaning aniqlanish sohasi topish;**
4) **Funksiyaning kritik nuqtalarini topish**, ya'ni $f'(x)=0$ bo'ladigan yoki $f'(x)$ mavjud bo'lmaydigan nuqtalarini topish. Ulardan foydalanib, finksiyaning maksimum va minimum qiymatlarini topish; funksiyaning shu nuqtalardagi qiymatlarini topish;
5) **O'sish va kamayish oraliqlari, ekstremumlarini topish.** x_0 nuqta atrofida $f''(x_0)$ ni ishorasini tekshiramiz. Agar $f''(x_0)<0$ bo'lsa, $f(x_0)$ maksimum nuqta, agar $f''(x_0)>0$ bo'lsa, $f(x_0)$ minimum nuqta bo'ladi.
6) **Egilish nuqtalarini topish.** $f''(x_0)=0$ bo'ladigan yoki $f''(x_0)$ mavjud bo'lmaydigan nuqtalarni aniqlaymiz. Bu nuqtalarda funksiya qiymatlarini hisoblaymiz.
7) **Qavariqlik va botiqlik oraliqlarini topish.** 4-shartdagi egilishga shubhali nuqtalardan foydalanamiz. Agar $f''(x_0)>0$ bo'lsa, bu oraliqda funksiya botiq, $f''(x_0)<0$ bo'lsa, funksiya qavariq bo'ladi.

8) Grafikni yasash. Yuqoridagi topilganlarga ko‘ra funksiya grafigini chizamiz. Agar zarurat bo‘lsa, qo‘shimcha ma’lumotlarni ham hisoblab topamiz.

10-misol. $f(x)=\dfrac{8}{x^2-4}$ funksiyani to‘liq tekshiring va grafigini chizing.

Yechilishi: ►

1) Funksiyaning koordinata o‘qlari bilan kesishish nuqtalarini aniqlaymiz: $f(0)=\dfrac{8}{0^2-4}=-2$, funksiya $(0;-2)$ nuqtada Y o‘qini kesadi.

Funksiya X o‘qi bilan kesishmaydi. Chunki, kasrning suratini nolga aylantiradigan qiymat yo‘q.

2)Funksiyaning vertikal, gorizontal, og‘ma asimptotalarini topamiz: Vertikal asimptotalar: $x^2-4=0$ ekanligidan $x=-2$ va $x=2$ to‘g‘ri chiziqlar bo‘lishini aniqladik.

Gorizontal asimptota: funksiyaning butun qismi, ya’ni $y=0$.

Og‘ma asimptotasi mavjud emas, chunki kasrning surati maxrajidan bir birlikka katta emas.

3) Funksiyaning aniqlanish sohasi topamiz: $x^2-4\neq 0$, $x\neq\pm 2$, shu sabali $D(f)=(-\infty;\ -2)\cup(-2;\ 2)\cup(2;\ \infty)$

4) Funksiyaning kritik nuqtalarini topamiz: Funksiyaning $f'(x)=0$ bo‘ladigan yoki $f'(x)$ mavjud bo‘lmaydigan nuqtalarini topamiz.

$$f'(x)=\left(\frac{8}{x^2-4}\right)'=8\cdot\left((x^2-4)^{-1}\right)'=-8\cdot 2x\cdot(x^2-4)^{-2}=-\frac{16x}{(x^2-4)^2}$$

Funksiya $x=0$ da hosilaga ega. $x=-2$ va $x=2$ nuqtalarda esa hosilasi mavjud emas. $x=-2$ va $x=2$ nuqtalar funksiyaning aniqlanish sohasiga kirmaydi, shuning uchun ularni kritik nuqtalar deb qaramaymiz. Shunda $x=0$ kritik nuqta bo‘ladi.

5) O‘sish va kamayish oraliqlari, ekstremumlarini topamiz:

$(-\infty;-2)$, $(-2;2)$ va $(2;\infty)$ oraliqlarda $f'(x)$ ning ishoralarini tekshiramiz.

$x=0$ nuqta atrofida $f''(0)$ ni ishorasini tekshiramiz. Agar $f''(0)<0$ bo‘lsa, $f(0)$ maksimum nuqta, agar $f''(0)>0$ bo‘lsa, $f(0)$ minimum nuqta bo‘ladi.

$$f''(x)=\left(-\frac{16x}{(x^2-4)^2}\right)'=\frac{-16(x^2-4)^2+16x\cdot 2(x^2-4)\cdot 2x}{(x^2-4)^4}=\frac{16(3x^2+4)}{(x^2-4)^3}$$

$f''(0)=\dfrac{16(3\cdot 0+4)}{(0-4)^3}=\dfrac{64}{-64}=-1<0$, demak $(0;f(0))$ maksimum nuqta, aniqrog‘i $(0;-2)$ nuqta maksimum nuqta bo‘ladi.

	$x=-3$	$x=0$	$x=3$
$f'(x)$	$f'(-3)>0$	$(0;-2)$	$f'(3)<0$
	f o'suvchi	maksimum	f kamayuvchi

6) Egilish nuqtalarini topamiz. $f''(x_0)=0$ bo‘ladigan yoki $f''(x_0)$ mavjud bo‘lmaydigan nuqtalarni aniqlaymiz. Bu nuqtalarda funksiya qiymatlarini hisoblaymiz.

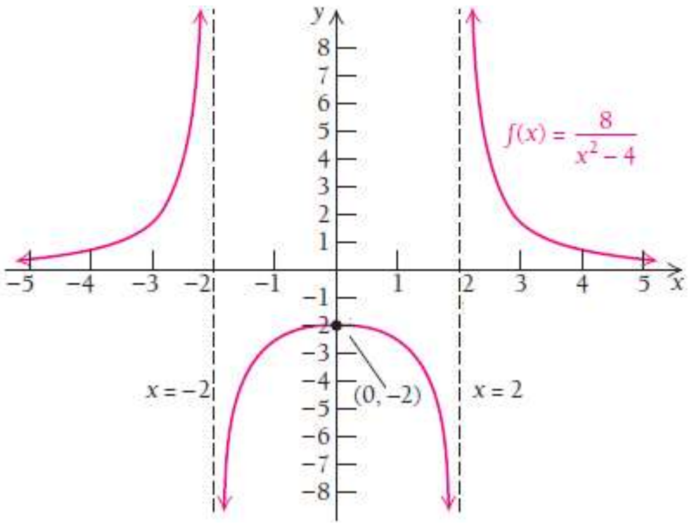
$x=-2$ va $x=2$ nuqtalarda 1-tartibli hosila mavjud emas, shuning uchun 2-tartibli hosila ham mavjud bo‘lmaydi.

$$f''(x)=\frac{16(3x^2+4)}{(x^2-4)^3}=0$$

7) Qavariqlik va botiqlik oraliqlarini topish. 4-shartdagi egilishga shubhali nuqtalardan foydalanamiz. Agar $f''(x_0) > 0$ bo‘lsa, bu oraliqda funksiya botiq, $f''(x_0) < 0$ bo‘lsa, funksiya qavariq bo‘ladi.

	$x < -2$	$-2 < x < 2$	$x > 2$
	$x = -3$	$x = 0$	$x = 3$
$f''(x)$	$f''(-3) > 0$	$f''(0) < 0$	$f''(3) > 0$
	f **botiq**	f **Qavariq**	f **botiq**

8) Grafikni yasaymiz. Yuqoridagi topilganlarga ko‘ra funksiya grafigini chizamiz.



◄

11-misol. $f(x) = \frac{x^2 + 4}{x}$ funksiyani to‘liq tekshiring va grafigini chizing.

Yechilishi: ► 1) **Funksiyaning koordinata o‘qlari bilan kesishish nuqtalarini aniqlaymiz:** $f(0) = \frac{0^2 + 4}{0}$, funksiya Y o‘qi bilan kesishmaydi. $f(x) = 0$ qiymatlari ham mavjud emas, ya’ni funksiya X o‘qi bilan ham kesishmaydi.

2) Funksiyaning vertikal, gorizontal, og'ma asimptotalarini topamiz: Vertikal asimptota mavjud emas, chunki $x^2+4=0$ bo'ladigan haqiqiy qiymatlar yo'q. Gorizontal asimptota ham yo'q, chunki funksiyada kasrning surati va mahrajidagi ko'phadlarning daraja ko'rsatkichlari teng emas. Ular teng bo'lgandagina kasrning butun qismini ajratar edik.

Og'ma asimptota: $f(x)=\dfrac{x^2+4}{x}=x+\dfrac{4}{x}$, bundan $y=x$ to'g'ri chiziqni hosil qilamiz.

1) Funksiyaning aniqlanish sohasi topamiz: $x\neq 0$ shu sabali

$$D(f)=(-\infty;\ 0)\cup(0;\ \infty)$$

2) Funksiyaning kritik nuqtalarini topamiz: Funksiyaning $f'(x)=0$ bo'ladigan yoki $f'(x)$ mavjud bo'lmaydigan nuqtalarini topamiz.

$$f'(x)=\left(\frac{x^2+4}{x}\right)'=\frac{x^2-4}{x^2}\ ,\qquad \frac{x^2-4}{x^2}=0\ ,$$

Funksiya hosilasi $x=-2$ va $x=2$ nuqtalarda nolga teng, shuning uchun ular kritik nuqtalar bo'ladi.

3) O'sish va kamayish oraliqlari, ekstremumlarini topamiz:

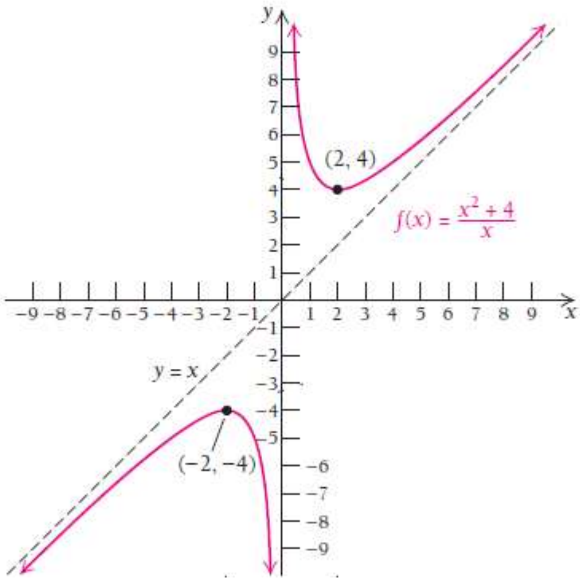
Funksiya $x=-2$ da maksimumga erishadi: $f(-2)=\dfrac{(-2)^2+4}{-2}=-4$, $(-2;-4)$.

$x=2$ da minimumga erishadi: $f(2)=\dfrac{2^2+4}{2}=4$, minimum nuqta $(2;4)$.

4) Egilish nuqtalarini topamiz. $f''(x_0)=0$ bo'ladigan yoki $f''(x_0)$ mavjud bo'lmaydigan nuqtalarni aniqlaymiz. Bu nuqtalarda funksiya qiymatlarini hisoblaymiz. $f''(x)=\left(\dfrac{x^2-4}{x^2}\right)'=\dfrac{2x^3-2x^3+8x}{x^4}=\dfrac{8}{x^3}$

5) **Qavariqlik va botiqlik oraliqlarini topamiz va grafigini chizamiz:**

	$x = -2$	$x = 2$
$f''(x)$	$f''(-2) < 0$	$f''(2) > 0$
	Qavariq	**botiq**



4-vazifa. $f(x) = \dfrac{x^2 - 9}{x - 1}$ **funksiyani to‘liq tekshiring va grafigini yasang.**

12-misol. Vertikal asimptotasi $x = -5$ va $x = 2$ to‘g‘ri chiziqlar, gorizontal asimptotasi esa $y = 2$ to‘g‘ri chiziq, shu bilan birga $f(1) = 3$ qiymat qabul qiluvchi qisqarmaydigan ratsional funksiyani aniqlang va grafigini chizing.

Yechilishi: Vertikal asimptotasi $x = -5$ va $x = 2$ bo‘lgani uchun kasrning mahrajida $(x+5)(x-2)$ ko‘paytma bo‘ladi.

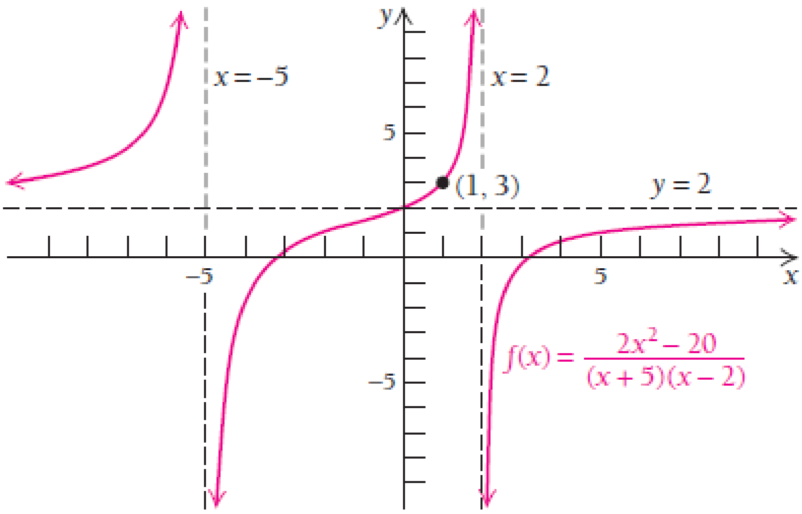
Gorizontal asimptotasi $y = 2$ ekanligidan va mahrajdagi qavslarni ko‘paytirganimizda ko‘phadning kvadrati hosil bo‘lishidan, kasrning butun qismini ajratganda 2 kelib chiqishi uchun suratida $2x^2 + C$ ko‘phad bo‘lishi kerak deb faraz qilamiz, bu yerda C qandaydir o‘zgarmas son.

Shunday qilib, bizning ratsional funksiya quyidagi ko‘rinishda bo‘ladi: $f(x)=\dfrac{2x^2+C}{(x+5)(x-2)}$

Endi C nimaga teng ekanligini aniqlaymiz. Buning uchun $f(1)=3$ tenglikdan foydalanamiz.

$$3=\frac{2\cdot1^2+C}{(1+5)(1-2)}, \quad \rightarrow \quad 3=\frac{2+C}{-6}, \quad \rightarrow \quad -18=2+C \quad \rightarrow \quad C=-20.$$

U holda $f(x)=\dfrac{2x^2-20}{(x+5)(x-2)}$ ratsional funksiyani hosil qilamiz.Ushbu funksiyaning grafigi quyidagicha bo‘ladi:



◀

5-vazifa. Vertikal asimptotasi $x=-2$ va $x=2$ to‘g‘ri chiziqlar, gorizontal asimptotasi esa $y=3$ to‘g‘ri chiziq, shu bilan birga $f(1)=-3$ qiymat qabul qiluvchi qisqarmaydigan ratsional funksiyani aniqlang va grafigini chizing.

MUSTAQIL YECHISH UCHUN MASALALAR

1-12 misollarda funksiyaning vertikal asimptotasini toping.

1. $f(x)=\dfrac{2x-3}{x-5}$

2. $f(x)=\dfrac{2x+3}{x-1}$

3. $f(x)=\dfrac{x+4}{x-2}$

4. $f(x)=\dfrac{3x}{x^2-25}$

5. $f(x)=\dfrac{12}{x^2-16}$

6. $f(x)=\dfrac{x+1}{x^2-9}$

7. $f(x)=\dfrac{7x+5}{x^2+7x+6}$

8. $f(x)=\dfrac{x+1}{x^2+6x+8}$

9. $f(x)=\dfrac{7x+5}{x^3-6x^2+8x}$

10. $f(x)=\dfrac{x+1}{x^3-3x}$

11. $f(x)=\dfrac{4}{x^2+8}$

12. $f(x)=\dfrac{3}{x^2+9}$

13-24 misollarda funksiyaning gorizontal asimptotasini toping.

13. $f(x)=\dfrac{6x}{8x+3}$

14. $f(x)=\dfrac{3x^2}{6x^2+x}$

15. $f(x)=\dfrac{4x}{x^2-3x}$

16. $f(x)=\dfrac{2x}{3x^3-x^2}$

17. $f(x)=5-\dfrac{3}{x}$

18. $f(x)=6+\dfrac{1}{x}$

19. $f(x)=\dfrac{8x^4-5x^2}{2x^3+x^2}$

20. $f(x)=\dfrac{6x^3+4x}{3x^2-x}$

21. $f(x)=\dfrac{6x^4+4x^2-7}{2x^5-x+3}$

22. $f(x)=\dfrac{4x^3-3x+2}{x^3+2x-4}$

23. $g(x)=\dfrac{x^2-3x+2}{-3x^2-x+4}$

24. $g(x)=\dfrac{2x^2-x-10}{x^2+3x+2}$

25- 40 misollarda funksiyani to‘la tekshiring

va grafigini yasang.

25. $f(x) = -\frac{3}{x}$ **26.** $f(x) = \frac{5}{x}$

27. $f(x) = -\frac{2}{x-5}$ **28.** $f(x) = \frac{1}{x-7}$

29. $f(x) = \frac{1}{x+2}$ **30.** $f(x) = -\frac{3}{x-3}$

31. $g(x) = \frac{3x-1}{x}$ **32.** $g(x) = \frac{5x+1}{x}$

33. $g(x) = x + \frac{3}{x}$ **34.** $g(x) = x - \frac{6}{x}$

35. $g(x) = \frac{2}{x^2}$ **36.** $g(x) = \frac{-3}{x^2}$

37. $g(x) = \frac{x}{x-1}$ **38.** $g(x) = \frac{x}{x+3}$

39. $g(x) = -\frac{1}{x^2+1}$ **40.** $g(x) = \frac{1}{x^2+4}$.

41-46 misollarda ma’lumotlar asosida qisqarmaydigan $f(x)$ **ratsional funksiya ko‘rinishini tiklang va grafigini yasang.**

41. Vertikal asimptotasi $x = 2$ to‘g‘ri chiziq, gorizontal asimptotasi esa

$y = -2$ to‘g‘ri chiziq, shu bilan birga $f(0) = 0$ qiymat qabul qiladi;

42. Vertikal asimptotasi $x = 0$ to‘g‘ri chiziq, gorizontal asimptotasi esa

$y = 3$ to‘g‘ri chiziq, shu bilan birga $f(1) = 2$ qiymat qabul qiladi;

43. Vertikal asimptotasi $x = -1$ va $x = 1$ to‘g‘ri chiziqlar, gorizontal asimptotasi esa $y = 1$ to‘g‘ri chiziq, shu bilan birga $f(0) = 2$ qiymat qabul qiladi;

44. Vertikal asimptotasi $x = -2$ va $x = 0$ to‘g‘ri chiziqlar, gorizontal asimptotasi esa $y = -3$ to‘g‘ri chiziq, shu bilan birga $f(1) = 4$ qiymat qabul qiladi;

45. Vertikal asimptotasi $x = -3$ va $x = 2$ to‘g‘ri chiziqlar, gorizontal asimptotasi esa $y = 0$ to‘g‘ri chiziq, shu bilan birga $f(1) = 2$ qiymat qabul qiladi;

46. Vertikal asimptotasi $x = -\frac{1}{2}$ va $x = \frac{1}{2}$ to‘g‘ri chiziqlar, gorizontal asimptotasi esa $y = 0$ to‘g‘ri chiziq, shu bilan birga $f(0) = -3$ qiymat qabul qiladi;

AMALIY TATBIQLARI (Iqtisodiyot)

47. Amortizatsiya. Aytaylik, “XROMTEX” MChJ ning mavjud inventarlarining qiymati t vaqt (oy hisobida) o‘tishi bilan pasayadi:

$$F(t) = 50 - \frac{25t^2}{(t+2)^2}.$$

a) $F(0)$, $F(5)$, $F(10)$, $F(70)$ ni hisoblang;

b) $[0;\infty)$ oraliqdagi inventarlarning maksimum qiymatini toping;

c) $F(t)$ funksiyaning grafigini chizing;

d) $F(t)$ ning qiymati hech qachon pasaymasligining yo‘li bormi?

48. O‘rtacha harajat. “Xorazm ipagi” korxonasining x mahsulot ishlab chiqarish uchun ketadigan umumiy harajatlari $H(x) = 3x^2 + 80$ funksiya bilan approksimatsiyalanadi.

a) O‘rtacha harajat $A(x) = \frac{H(x)}{x}$ ga teng bo‘lsa, uni toping;

b) O‘rtacha tannarx grafigini chizing;

v) Grafik asiptotalarini aniqlang va u nimani bildirishini tushuntiring.

49. Atrof-muhit ifloslanishini nazorat qilishga sarflanadigan harajatlar. Kimyoviy chiqindi chiqaradigan korxonalar atrof-muhit ifloslanishini nazorat qiluvchilar uchun p % ajratadi:

$$H(p)=\frac{48000}{100-p}$$

a) $H(0),\ H(20),\ H(80),\ H(90)$ ni hisoblang;

b) $H(p)$ aniqlanish sohasini toping;

v) Grafigini chizing.

50. Sarf-harajat va foyda. “SAG” korxonasining x dona gilam ishlab chiqarish uchun ketadigan umumiy harajatlari (dollar hisovbida) $H(x)=600x+5000$ va daromadi $D(x)=-\frac{1}{2}x^2+1000x$ funksiyalar bilan approksimatsiyalanadi.

a) Umumiy foyda $F(x)$ ni toping;

b) O‘rtacha foyda $A(x)=\frac{F(x)}{x}$ ni toping va grafigini chizing;

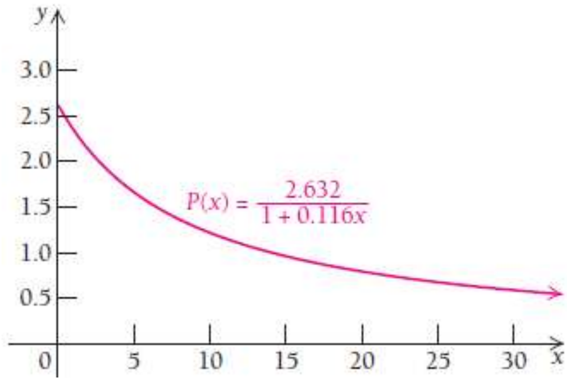
v) $y=A(x)$ funksiya asiptotalarini aniqlang.

51. Talab funksiyasi. 1970 yildan buyon bug‘doyga bo‘lgan talabni x yillarga nisbatan $T(x)=\frac{2.632}{1+0.116x}$ (dollar hisobida) funksiya bilan modellashtirish mumkin.

a) $T(10),\ T(20),\ T(40)$ ni hisoblang;

b) Qaysi yilda talab 0.5$ bo‘lgan?

v) $\lim\limits_{x\to\infty}T(x)$ ni aniqlang.



52. Qon ketishini to‘xtatish. A (sm^3) dori in’yeksiyasidan t soat o‘tib, qon ketishi kamaydi : $A(t)=\frac{A_0}{1+t^2}$, bunda A_0 dorining boshlang‘ich dozasi. $A_0=100$ sm^3 deb qabul qiliamiz.

a) $A(0)$, $A(1)$, $A(2)$, $A(7)$, $A(10)$ ni hisoblang.

b) $[0;\infty)$ oraliqda qon ketishini davolashning maksimal dozasini hisoblang;

v) Funksiya grafigini chizing;

g) Ushbu davolash tartibiga ko‘ra, qachon qon ketishi butunlay to‘xtaydi? Javobingizni tushuntiring.

53-60 misollarda funksiyalar grafiklarini “MathCad” dasturida chizing:

53. $f(x)=\frac{x}{\sqrt{x^2+1}}$;

54. $f(x)=\left|\frac{1}{x}-2\right|$;

55. $f(x)=\frac{x^3+2x^2-15x}{x^2-5x-14}$;

56. $f(x)=\frac{x^3+4x^2+x-6}{x^2-x-2}$;

57. $f(x)=x^2+\frac{1}{x^2}$;

58. $f(x)=\frac{x^3+2x^2-3x}{x^2-25}$;

59. $f(x)=\frac{x^2-3}{2x-4}$;

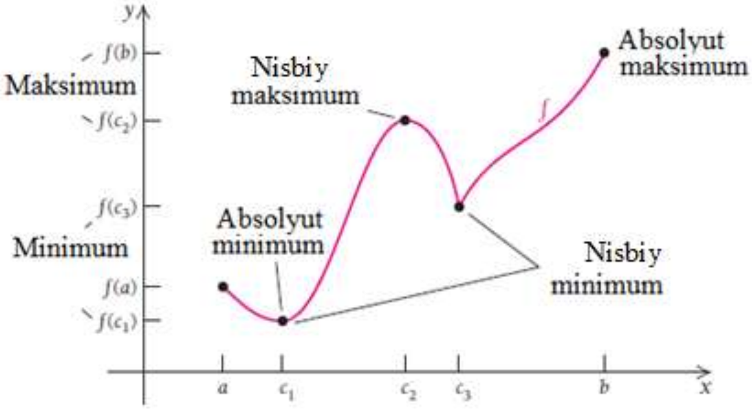
60. $f(x)=\frac{x^5+x-9}{x^3+6x}$.

3.4. Funksiyaning kesmadagi eng katta va eng kichik qiymatlarini topish

Ekstremum funksiyaning butun diagrammasi bo‘yicha olib qaralganda eng tepadagi yoki eng pastdagi nuqtasi bo‘lishi mumkin, bunday nuqtalarni **absolyut ekstremum** deyiladi. Misol uchun $y = x^2$ parabolada (0; 0) nuqta nisbiy minimum hisoblanadi. Grafikning to‘liq olib qaralganda ham u eng quyidagi nuqta hisoblanadi, shuning uchun uni absolyut minimum deb ham aytish mumkin.

3.4.1. Funksiyaning kesmadagi eng katta va eng kichik qiymatlari

Nisbiy minimum funksiyaning biror kesmasida eng kichik qiymat bo‘lishiga qaramay absolyut minimum bo‘lmasligi mumkin. Xuddi shuningdek, kesmadagi nisbiy maksimum butun aniqlanish sohasi uchun absolyut maksimum bo‘lmasligi mumkin. Shuning uchun kesmadagi eng katta va eng kichik qiymatlar tushunchasini kiritamiz.



$[a;b]$ kesmaning c_1 va c_3 nuqtalarida absolyut va nisbiy minimumga ega. Biz eng kichik nuqta deb, c_1 nuqtadagi qiymatni qabul qilamiz. Xuddi shuningdek, $[a;b]$ kesmaning c_2 va b nuqtalarida nisbiy va absolyut maksimumga ega. Biz eng katta nuqta deb, b nuqtadagi qiymatni qabul qilamiz.

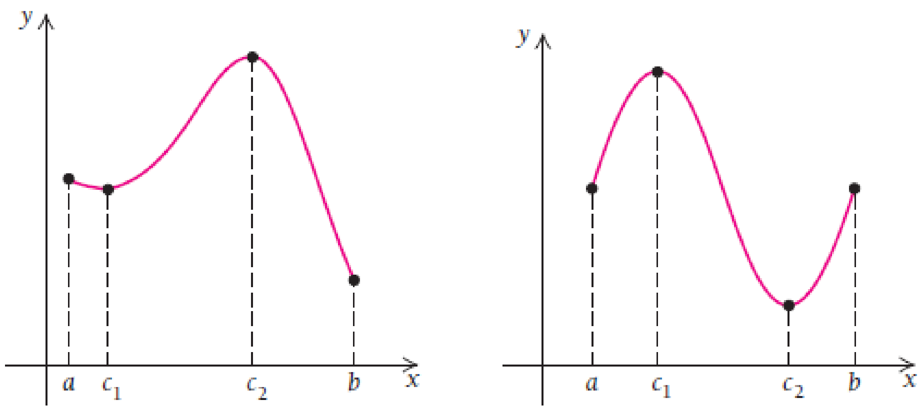
Ta'rif. $f(x)$ funksiya biror $[a;b]$ kesmada aniqlangan bo'lsin.

Agar $[a;b]$ kesmada $f(c) \le f(x)$ tengsizlik o'rinli bo'lsa, $f(c)$ qiymat shu **kesmada funksiyaning eng kichik qiymati** deyiladi;

Agar $[a;b]$ kesmada $f(c) \ge f(x)$ tengsizlik o'rinli bo'lsa, $f(c)$ qiymat shu **kesmada funksiyaning eng katta qiymati** deyiladi.

3.4.2. Funksiyaning kesmadagi eng katta va eng kichik qiymatlarini topish

$[a;b]$ kesmada aniqlangan funksiyani qaraymiz. Quyidagi teorema o'rinli:



Teorema. $[a;b]$ kesmada aniqlangan $f(x)$ funksiya shu oraliqda ham eng kichik qiymatga, ham eng katta qiymatga ega bo'ladi.

8-teorema. $f(x)$ uzluksiz funksiya $[a;b]$ kesmada aniqlangan bo‘lsin. $[a;b]$ kesmada funksiyaning eng kichik va eng katta qiymatlarini topish uchun

1) $f'(x)$ hosila topiladi;

2) $[a;b]$ kesmada kritik nuqtalar aniqlanadi, ya’ni $f'(x)=0$ bo‘lgan yoki $f'(x)$ hosila mavjud bo‘lmagan nuqtalar aniqlanadi;

3) bu nuqtalar va kesma oxirlari $a,\ c_1,\ c_2,...,c_n,\ b$ bo‘lsin.

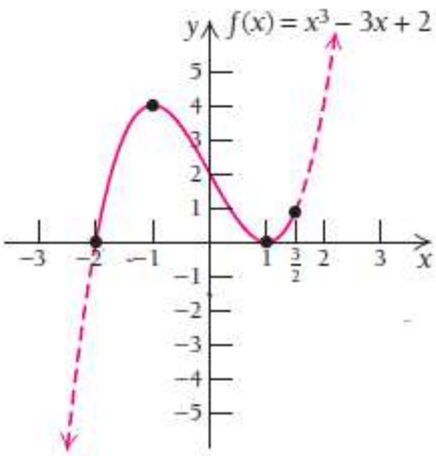
4) $f(a),\ f(c_1),\ f(c_2),...,f(c_n),\ f(b)$ funksiya qiymatlari hisoblanadi. Ularning eng kattasi kesmadagi eng katta qiymat, eng kichigi esa kesmadagi eng kichik qiymat deb qabul qilinadi.

1-misol. $f(x)=x^3-3x+2$ funksiyaning $\left[-2;\frac{3}{2}\right]$ kesmadagi eng katta va eng kichik qiymatlarini toping.

Yechilishi: ► 1) $f'(x)=\left(x^3-3x+2\right)'=3x^2-3$

$f'(x)=0$

$3x^2-3=0$, $x=\pm 1$ qiymatlarni aniqladik.



Endi -2, -1, 1, 3/2 nuqtalardagi funksiya qiymatlarini hisoblaymiz:

$f(-2)=(-2)^3-3\cdot(-2)+2=0$ - eng kichik qiymat;

$f(-1)=(-1)^3-3\cdot(-1)+2=4$ - eng katta qiymat;

$f(1)=1^3-3\cdot 1+2=0$ - eng kichik qiymat;

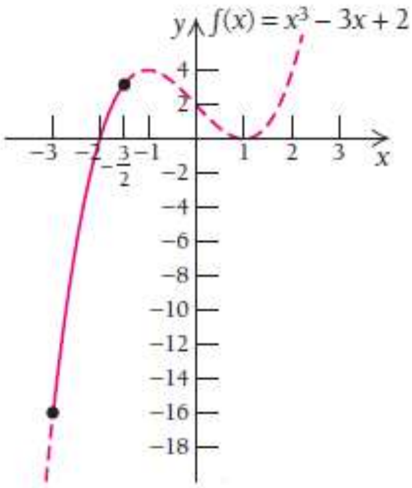
$f\left(\frac{3}{2}\right)=\left(\frac{3}{2}\right)^3-3\cdot\left(\frac{3}{2}\right)+2=\frac{7}{8}$. ◀

2-misol. $f(x)=x^3-3x+2$ funksiyaning $\left[-3;-\frac{3}{2}\right]$ kesmadagi eng katta va eng kichik qiymatlarini toping.

Yechilishi: -1 va 1 qiymatlar kesmaga tegishli bo‘lmaganligi sababli, faqat -3, -3/2 kesmaning chetlaridagi nuqtalarni tekshiramiz:

$f(-3)=(-3)^3-3\cdot(-3)+2=-16$ - eng kichik qiymat;

$f\left(-\frac{3}{2}\right)=\left(-\frac{3}{2}\right)^3-3\cdot\left(-\frac{3}{2}\right)+2=3\frac{1}{8}$ - eng katta qiymat bo‘ladi.



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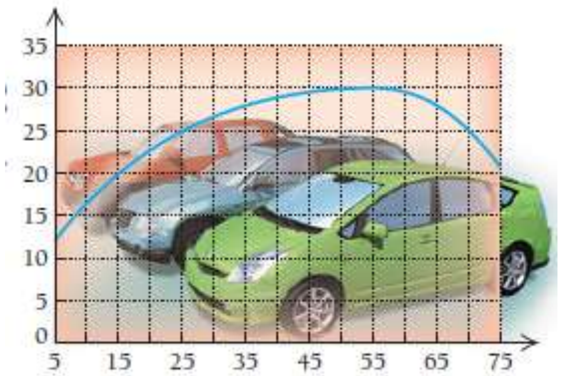
1-vazifa. $f(x)=x^2-20x$ funksiyaning $[0;\ 6]$ va $[4;\ 10]$ kesmalardagi eng katta va eng kichik qiymatlarini toping

MUSTAQIL YECHISH UCHUN MASALALAR

1. **Yoqilg‘i sarfini tejash.** “CTAG” MChJ ma’lumotiga ko‘ra, transport vositasi yoqilg‘i sarfini kamaytirishi uchun **(**gallon/mil) 60 mil/soat dan kam tezlikda yurishi kerak.



a) Agar avtomobil benzinda yursa, maksimum masofani (mil) toping;

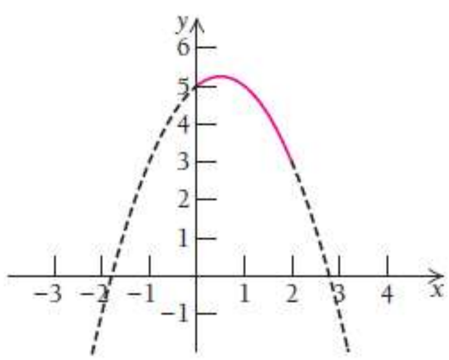
b) Agar avtomobil benzinda yursa, minimum masofani (mil) toping;

c) 70 mil/soat da yursa, yoqilg‘i qancha masofaga yetadi?

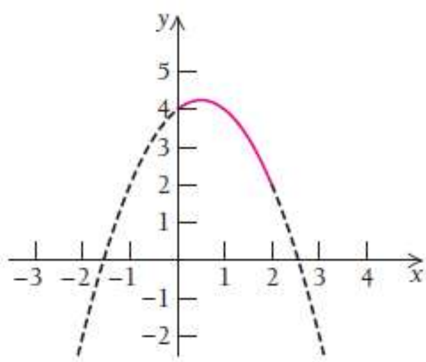
2. Yoqilg‘i sarfini tejash. 1-misol diagrammasidan foydalanib, tezlikning [30; 70] oraliqda eng katta va eng kichik qiymatini toping.

3- 42 misollarda funksiyaning kesmalardagi eng katta va eng kichik qiymatlarini toping:

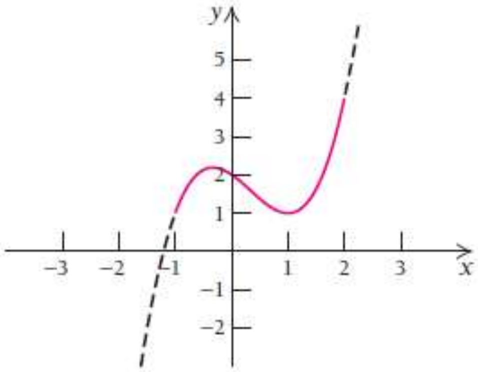
3. $f(x) = 5 + x - x^2$; $[0, 2]$



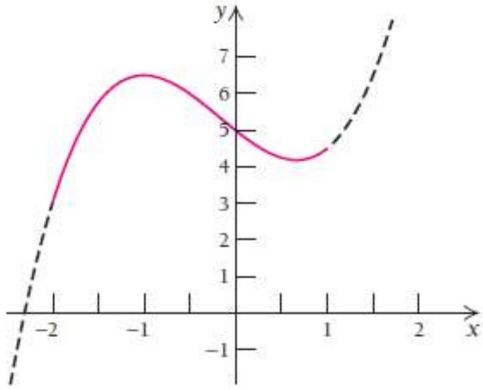
4. $f(x) = 4 + x - x^2$; $[0, 2]$



5. $f(x) = x^3 - x^2 - x + 2;\ [-1, 2]$



6. $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 5;\ [-2, 1]$



7. $f(x) = x^3 - x^2 - x + 3;\quad [-1, 0]$

8. $f(x) = x^3 + \frac{1}{2}x^2 - 2x + 4;\quad [-2, 0]$

9. $f(x) = 5x - 7;\quad [-2, 3]$

10. $f(x) = 2x + 4;\quad [-1, 1]$

11. $f(x) = 7 - 4x;\quad [-2, 5]$

12. $f(x) = -2 - 3x;\quad [-10, 10]$

13. $f(x) = -5;\quad [-1, 1]$

14. $g(x) = 24;\quad [4, 13]$

15. $f(x) = x^2 - 6x - 3;\quad [-1, 5]$

16. $f(x) = x^2 - 4x + 5;\quad [-1, 3]$

17. $f(x) = 3 - 2x - 5x^2;\quad [-3, 3]$

18. $f(x) = 1 + 6x - 3x^2;\quad [0, 4]$

19. $f(x) = x^3 - 3x^2;\quad [0, 5]$

20. $f(x) = x^3 - 3x + 6;\quad [-1, 3]$

21. $f(x) = x^3 - 3x;\quad [-5, 1]$

22. $f(x) = 3x^2 - 2x^3;\quad [-5, 1]$

23. $f(x) = 1 - x^3;\quad [-8, 8]$

24. $f(x) = 2x^3;\quad [-10, 10]$

25. $f(x) = 12 + 9x - 3x^2 - x^3;\quad [-3, 1]$

26. $f(x) = x^3 - 6x^2 + 10;\quad [0, 4]$

27. $f(x) = x^4 - 2x^3;\quad [-2, 2]$

28. $f(x) = x^3 - x^4;\quad [-1, 1]$

29. $f(x) = x^4 - 2x^2 + 5;\quad [-2, 2]$

30. $f(x) = x^4 - 8x^2 + 3;\quad [-3, 3]$

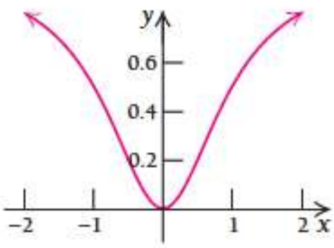
31. $f(x) = (x + 3)^{2/3} - 5;\quad [-4, 5]$

32. $f(x) = 1 - x^{2/3};\quad [-8, 8]$

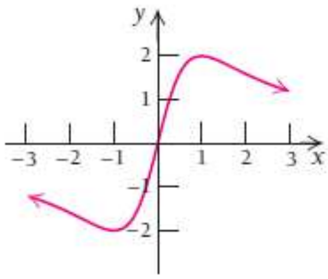
33. $f(x) = x + \dfrac{1}{x};\quad [1, 20]$

34. $f(x) = x + \dfrac{4}{x};\quad [-8, -1]$

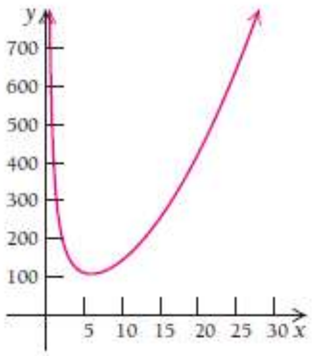
35. $f(x) = \dfrac{x^2}{x^2 + 1};\quad [-2, 2]$



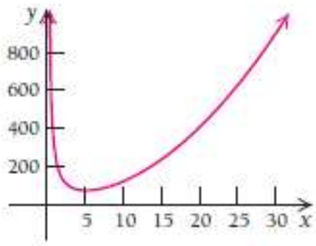
36. $f(x) = \dfrac{4x}{x^2 + 1},\quad [-3, 3]$



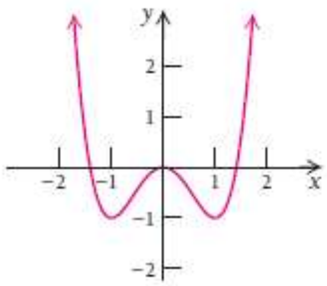
37. $f(x) = x^2 + \frac{432}{x}$; $(0, \infty)$



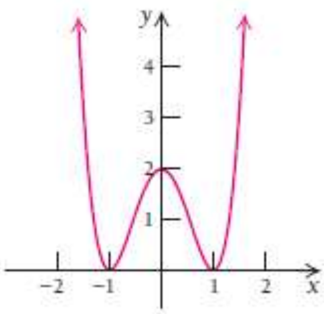
38. $f(x) = x^2 + \frac{250}{x}$; $(0, \infty)$



39. $t(x) = x^4 - 2x^2$



40. $f(x) = 2x^4 - 4x^2 + 2$



41. $f(x) = x\sqrt{x+3}$, $[-3;\ 3]$

42. $g(x) = x\sqrt{1-x}$, $[0;\ 1]$.

AMALIY TATBIQLARI

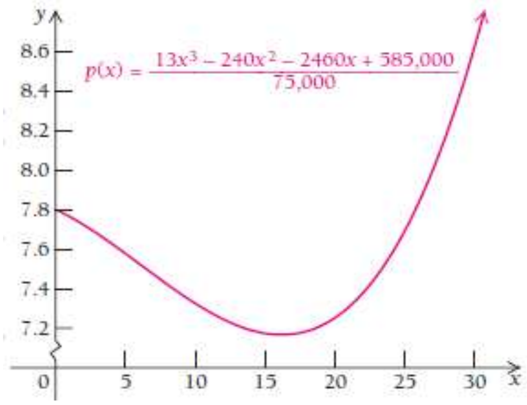
41. Pensiya jamg‘armasi. Xizmatchi M. ning oylik maoshlari funksiyasi $M(t) = -2t^2 + 100t + 180$, $0 \le t \le 40$

bunda $t-$ hizmatchining oylik olgan yillari. Xizmatchining maksimal oyligi qanday bo‘lgan va uni qaysi yilda olgan.

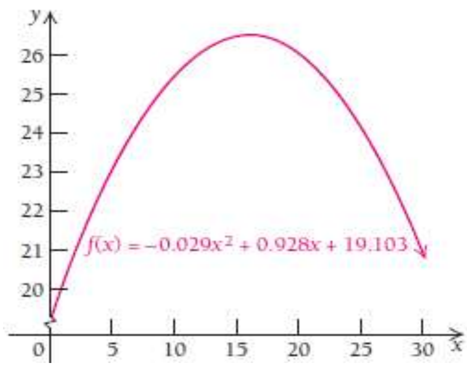
42. Reklama. Ovozli dasturiy vosita reklamasiga x dollar sarflangandan keyin N donasi sotildi: $N(x) = -x^2 + 300x + 6$, $0 \le x \le 300$.

Reklamaga maksimal qancha sarflangan va maksimal nechta dasturiy vosita sotilgan?

43. Kichik biznes. Sanoat korxonasiga ega bo‘lgan amerikalik hamkasbimizni daromadini $p(x)=\dfrac{13x^3-240x^2-2460x+585000}{75000}$ funksiya bilan modellashtirish mumkin, bunda $x-$ 1980 yildan buyon yillar soni. 1980 – 2000 yillarda ushbu daromad funksiyasi haqiqatan ham minimumga ega bo‘lganmi? Hisoblang va javobibgizni grafik bilan tekshirib ko‘ring.

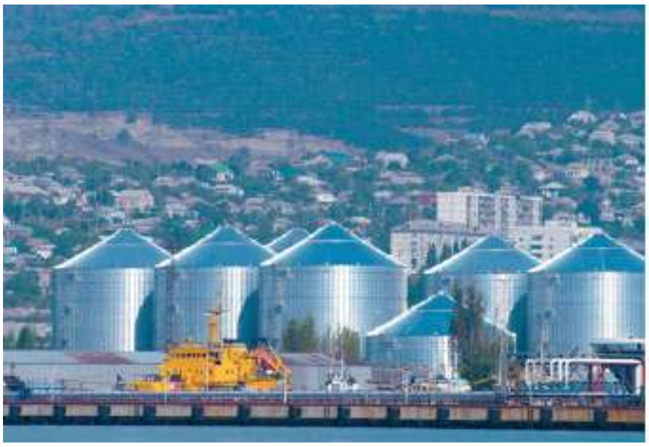


44. O‘zbekiston fuqarolarining 35-44 yoshdagi ishchi kuchi foizini $f(x)=-0.029x^2+0.928x+19.103$ funksiya bilan modellashtirish mumkin (O‘zb.stat.ma’lumorlari). Bunda $x-$ 1980 yildan buyon yillar soni. Bu modelga ko‘ra, 1980 – 2010 yillarda haqiqatan ham foiz maksimum bo‘lganmi? Hisoblang va javobibgizni grafik bilan tekshirib ko‘ring.



45. Xalqaro neft qazib olish. Butun duyo bo‘yicha neft qazib olish $P(t)=0.000008533\ t^4-0.001685t^3+0.09t^2-0.687t+4$, $0\le t\le 90$

funksiya bilan modellashtiriladi. Bunda $P(t)$ milliard barrel birlikda va 1950 yildan keyingi yillar hisobida olingan. ***(Manba***: *Beyond Oil*, by Kenneth S. Deffeyes, p. xii, Hill and Wang, New York, 2005.)
Modelga asosan qaysi yilda neft qazib olish cho'qqisiga chiqqan?



46. Maksimal foyda. Ovoz kuchaytirgich ishlab chiqadigan "Elektronika" korxonasi har haftada ularning x donasini sotib, $F(x)=\frac{1500}{x^2-6x+10}$ foyda ko'radi. Haftalik foyda maksimum bo'lgan ovoz kuchaytirgich x sonini toping.

47. Qon bosimi. Ma'lum dori turini x sm^3 dozasini qabul qilgan bemorning qon bosimi $B(x)=305x^2-1830x^3, \quad 0\le x\le 0.16$ funksiya bilan approksimatsiyalanadi. Maksimal qon bosimini va unda qo'llaniladigan dori dozasini aniqlang.

48-51 misollarda funksiyalarning maksimum va minimum qiymatlarini toping va grafik eskizini chizing:

48. $f(x)=\begin{cases} 2x+1 & -3\le x\le 1, \\ 4-x^2, & 1<x\le 2 \end{cases}$

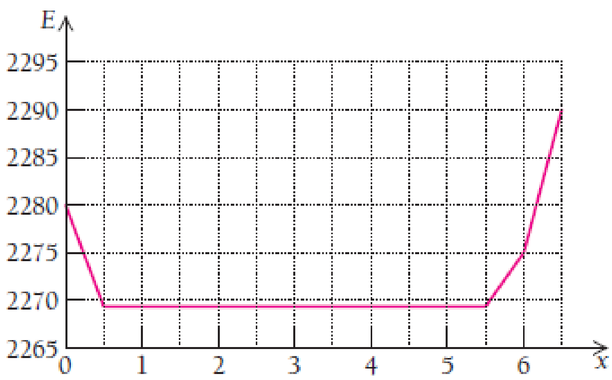
49. $g(x)=\begin{cases} x^2, & -2\le x\le 0, \\ 5x, & 0<x\le 2 \end{cases}$

50. $h(x) = \begin{cases} 1 - x^2, & -4 \le x < 0, \\ 1 - x, & 0 \le x < 1, \\ x - 1, & 1 \le x \le 2 \end{cases}$

51. $F(x) = \begin{cases} x^2 + 4, & -2 \le x < 0, \\ 4 - x, & 0 \le x < 3, \\ \sqrt{x - 2}, & 3 \le x \le 67 \end{cases}$

52. Fizika. Ko‘l suvining kamayishi natijasida hosil bo‘lgan tepalik.

AQSh ning g‘arbiy sahrolarida suvi kamayib borayotgan ko‘llar mavjud. So‘nggi yillarda Kaliforniyaning Rojers ko‘lida paydo bo‘lgan tepalikdan qo‘nish maydonchasi sifatida foydalanilmoqda. Diagrammada *E* o‘qi tepaliklar balandligini, *x* esa ko‘lning g‘arbiy qismidan sharqiy qismigacha bo‘lgan masofani (mill hisobida) bildiradi. (Ma’lumot www.mytopo.com. dan olindi.)



a) Tepalikning maksimal balandligini toping.

b) Tepalikning minimal balandligi qanday bo‘ladi?

3.5. Optimallashtirish masalalari

3.5.1. Iqtisodiyot va tadbirkorlikda optimallashtirish masalalari

Funksiyaning maksimum yoki minimumini topish masalasi iqtisodiy sohada juda ham muhim hisoblanadi.

Tadbirkorlar o‘z faoliyati davomida biror ish sarf-harajat bilan bog‘liq bo‘lsa, ularni kamaytirishni, daromad bilan bog‘liq bo‘lsa, daromadni ko‘paytirishni maqsad qilishadi. Masala shartidan kelib chiqib, sarf-harajat yoki daromadning **maqsad funksiyasi** tuziladi. So‘ngra maqsad funksiyasining eng kichik yoki eng katta qiymati topiladi. Bunday masalalarga **optimallashtirish masalalari** deyiladi.

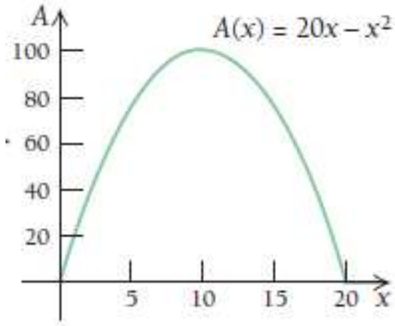
1-misol. Yuzani maksimallashtirish. Bolalar o‘yin maydonchasiga elektropoyezd joylashtirishmoqchi bo‘lishdi. U holda maydon atrofini 20 metr panjara bilan o‘rash kerak. Ikki tomoni g‘isht devor bilan to‘silganligi uchun hech qanday panjara kerak emas. Maydonni to‘g‘ri to‘rtburchak shaklida shunday o‘rash kerakki, yuzasi maksimal bo‘lsin. Maksimal yuza qanday bo‘ladi?

Yechilishi: ► Bir qarashda perimeter bir xil bo‘lgandan keyin yuza ham bir xil bo‘ladi, deb o‘ylashingiz mumkin. Yo‘q, unday emas. Keling dastlab chizmasini chizib ko‘ramiz. Agar bir tomonini x deb, 2-tomonni y deb belgilasak, u holda

$$x + y = 20, \qquad y = 20 - x \quad \text{bo‘ladi.}$$

Yuza esa $A(x) = x \cdot y$ ga teng. Bundan $A(x) = x \cdot y = x(20 - x) = 20x - x^2$.

Biz $A(x) = 20x - x^2$ ning (0;20) oraliqdagi maksimal qiymatini topishimiz kerak.



Oraliqni (0;20) deb olishimizga sabab, x uzunlik, shuning uchun u 0 ham manfiy ham bo‘la olmaydi. Faqat 20 m panjara bo‘lganligi uchun x soni 20 dan katta bo‘la olmaydi, shuningdek, 20 ga ham teng bo‘la olmaydi. Aks holda y nolga teng bo‘lib qoladi.

1) $A(x)$ dan hosila olamiz; $A'(x) = \left(20x - x^2\right)' = 20 - 2x$.

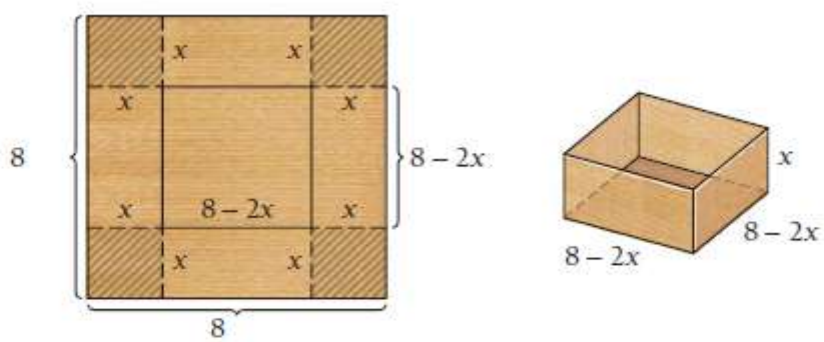
2) Hosil bo‘lgan ifodani 0 ga tenglaymiz: $20 - 2x = 0$, $x = 10$. Bittagina kritik nuqta hosil bo‘ldi. Endi bu nuqta maksimum yoki minimum ekanligini bilish uchun 2-tartibli hosila olamiz: $A''(x) = \left(20 - 2x\right)' = -2 < 0$, 2-tartibli hosila manfiy chiqdi, demak, $A(10)$ maksimum qiymat bo‘lar ekan. Shunda maksimal yuza

$$A(10) = 20x - x^2 = 20 \cdot 10 - 10^2 = 100\,\text{m}^2 \text{ bo‘ladi.}$$

Agar $x = 5$ bo‘lsa, yuza $A(5) = 75$; $x = 16$ bo‘lsa, $A(16) = 64$; $x = 9$ bo‘lsa, $A(12) = 96$ yuzaga ega bo‘lar edik. Ular maksimal yuza bo‘lmasdi.◄

2-misol. **Hajmni maksimallashtirish.** Tomonining uzunligi 8 bo‘lgan kvadrat shaklidagi kartondan uchidagi ortiqcha kvadratchalarni ichki tomonga bukib, eng katta hajmga ega quti yasash kerak bo‘lsin. Uning hajmi nimaga teng bo‘ladi?

Yechilishi: ► Qutining hajmi maksimal bo‘lishi uchun x balandligini shunday olish kerakki, hosil bo‘lgan kvadratchalar minimal bo‘lsin. Chunki uni ichki tomonga bukib qo‘yamiz.



Prizmaning hajmi: $V = abc$ ga teng. Chizmaga ko‘ra,

$$V = abc = x(8-2x)(8-2x) = x(64-32x+4x^2) = 4x^3-32x^2+64x.$$

$8-2x>0$ bo‘lishi kerak, bundan $x<4$ kelib chiqadi. Demak biz x ni (0, 4) oraliqda olishimiz kerak. $V = 4x^3-32x^2+64x$ hajm funksiyasidan hosila olamiz va nolga tenglab, yechamiz:

$$V'(x) = \left(4x^3-32x^2+64x\right)' = 12x^2-64x+64$$

$$12x^2-64x+64=0, \qquad 3x^2-16x+16=0$$

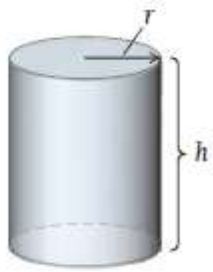
$$(3x-4)(x-4)=0, \qquad x=\frac{4}{3} \text{ yoki } x=4.$$

Oraliq (0; 4) bo‘lganligi uchun $x=4$ yechim bo‘la olmaydi. Shunda yechim $x = 4/3$ ekan. Bu qiymatda hajm maksimum bo‘ladimi? Tekshirish uchun hajm funksiyasidan 2-tartibli hosila olamiz va ishorasini tekshiramiz: $V''(x) = 24x-64$, $V''(\frac{4}{3}) = 24\cdot\frac{4}{3}-64 = 32-64 = -32 < 0$.

$V\left(\frac{4}{3}\right)$ maksimum bo‘larkan. $V\left(\frac{4}{3}\right) = 4\cdot\left(\frac{4}{3}\right)^3 - 32\cdot\left(\frac{4}{3}\right)^2 + 64\cdot\frac{4}{3} = 37\frac{25}{27}$. ◄

3-misol. Mahsulot sarfini kamaytirish. Ishlab chiqaruvchi 500 ml hajmdagi konserva bankalari chiqaradi. Tadbirkor silindrik qutining balandlik va asos radiusi o‘lchamlarini qanday olganda, material sarfi kamayadi (1 ml= 1 sm^3)?

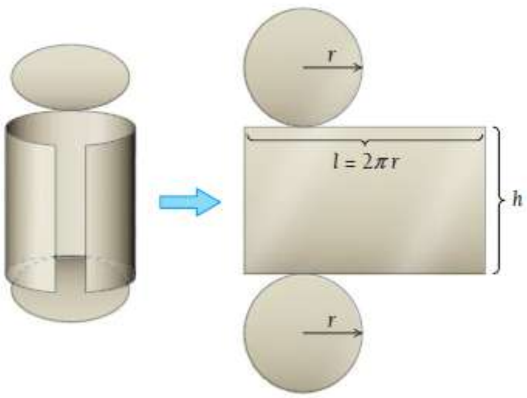
Yechilishi: ► Tadbirkor material sarfini kamaytirishi uchun u silindrning to‘la sirtini minimallashtirishi kerak.

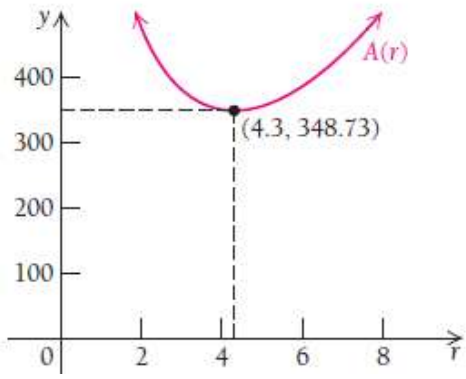


Silindr hajmi $V_s = \pi r^2 h$ ga teng. To‘la sirti esa $S_t = 2\pi r^2 + 2\pi r h$.

Bizda $\pi r^2 h = 500$ ga teng. Bundan $h = \dfrac{500}{\pi r^2}$ ni topib, to‘la sirt formulasiga qo‘yamiz: $S_t = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \cdot \dfrac{500}{\pi r^2}$

$$S_t(r) = 2\pi r^2 + \frac{1000}{r} \text{ kelib chiqadi.}$$





Funksiya hosilasini olamiz va nolga tenglaymiz: $S_t'(r) = 4\pi r - \dfrac{1000}{r^2}$.

$$4\pi r - \frac{1000}{r^2} = 0, \quad 4\pi r = \frac{1000}{r^2}, \quad 4\pi r^3 = 1000, \quad r = \sqrt[3]{\frac{250}{\pi}} \approx 4.3 \text{ sm}^3.$$

$(0;\infty)$ oraliqda $r \approx 4.3\,\text{sm}^3$ kritik nuqtani topdik. Endi u minimum bo'ladimi? Tekshirish uchun to'la sirt formulasidan 2-tartibli hosila olamiz: $S_t''(r) = \left(4\pi r - \frac{1000}{r^2}\right)' = 4\pi + \frac{2000}{r^3}$ va $r \approx 4.3$ nuqtada ishorasini tekshiramiz: $S_t''(4.3) = 4\pi + \frac{2000}{(4.3)^3} > 0$ manfiy. Bundan $r \approx 4.3$ da funksiya minimum ekanligini bilib olamiz. Demak, qutining asosi radiusini $r \approx 4.3\,\text{sm}$ qilib, balandligini esa $h = \frac{500}{\pi r^2} = 8.6$ sm deb olish kerak ekan. Shunda minimal to'la sirt 348.73 sm^2 ga yaqin bo'ladi.◄

4-misol. Daromadni maksimallashtirish. "Artel" korxonasi ma'lum muddatda x dona sovutgich ishlab chiqarish va sotishdan tushadigan mablag'i (dollar hisobida) $p(x) = 1000 - x$ bo'lishi, tannarxi esa $T(x) = 3000 + 20x$ bo'lishi kerak deb hisoblaydi.

a) Umumiy daromad $D(x) = ?$

b) Umumiy foyda $F(x) = ?$

c) Foydani maksimallashtirish uchun korxona qancha sovutgich sotishi kerak?

d) Maksimal foyda nimaga teng?

e) Maksimal foyda olish uchun sovutgich narxi qancha bo'lishi kerak?

Yechilishi: ► a) Umumiy daromad mahsulot sonini uning narxiga ko'paytmasiga teng; $D(x) = x \cdot p = x(1000 - x) = 1000x - x^2$.

b) Umumiy foyda daromaddan tannarxni ayirganimizga teng:

$F(x) = D(x) - T(x) = (1000x - x^2) - (3000 + 20x) = -x^2 + 980x - 3000$.

c) Korxonaning maksimal foydasini topamiz. Buning uchun foyda funksiyasidan hosila olib, nolga tenglab yechamiz:

$$F'(x) = \left(-x^2 + 980x - 3000\right)' = -2x + 980$$

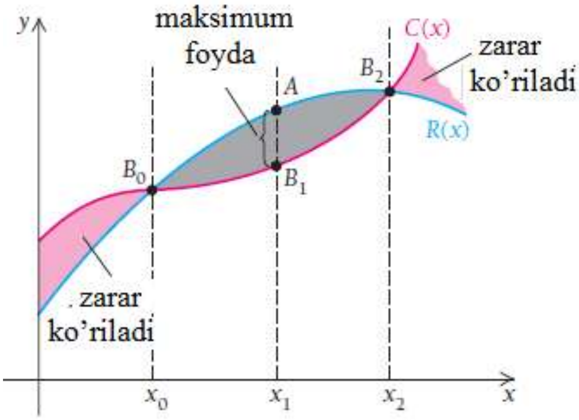
$$-2x + 980 = 0, \qquad x = 490.$$

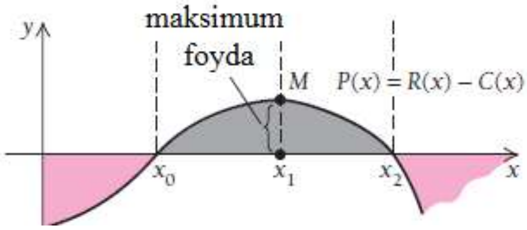
Bitta kritik nuqta topildi. Bu nuqta maksimum bo‘lishi yoki bo‘lmasligini 2-tartibli hosilaning ishorasi yordamida tekshirib ko‘ramiz: $F''(x) = \left(-2x + 980\right)' = -2 < 0$, 2-tartibli hosila manfiy, demakki $x = 490$ maksimum nuqta ekan. Korxona maksimum foyda ko‘rishi uchun 490 ta sovutgich sotishi kerak.

d) Maksimal foyda $F(490) = -490^2 + 980 \cdot 490 - 3000 = 237\,100\,\$$.

e) Maksimal foyda olish uchun sovutgich narxi qancha bo‘lishi kerakligini aniqlaymiz: $p(x) = 1000 - 490 = 510\,\$$ ◄

Keling umumiy foyda funksiyasiga va unga bog‘liq bo‘lgan funksiyalarga nazar tashlaymiz.





Chapdagi chizmada umumiy tannarx va umumiy daromad funksiyalarini ko‘rib turibmiz. Chizmadan $D(x)$ va $T(x)$ orasidagi farqni kattaligiga qarab, agar $D(x) > T(x)$ bo‘lsa, maksimal foydani baholashimiz mumkin.

O‘ngdagi chizma foyda funksiyasiga bog‘liq bo‘lgan funksiyalarni ko‘rsatadi. E’tibor bering, agar ishlab chiqarish (x_0 dan kichik) ko'rsatkichi past bo'lsa, korxona zarar ko'radi. Bu balki sarf harajat va kam daromad evazigadir. Agar ishlab chiqarish (x_2 dan katta) ko'rsatkichi yuqori bo'lsa ham, korxona zarar ko'radi, balki ishlab chiqarishni kengaytirish sabablidir.

Biznes mahsulot soni x_0 va x_2 orasida bo'lsagina, foyda beradi. Maksimal foydani esa x_1 dona mahsulot chiqarganda olish mumkin.

Aytaylik, mahsulot soni $x \in (0;\infty)$ bo'lsin. Maksimal foyda olish uchun nimaga e'tibor berish kerak? Maksimal foyda olish uchun umumiy foyda funksiyasi $F'(x) = 0$ va $F''(x) < 0$ shartlarni qanoatlantirishi kerak.

$F(x) = D(x) - T(x)$ tenglikdan

$F'(x) = D'(x) - T'(x) = 0$ va $F''(x) = D''(x) - T''(x) < 0$ yoki

$D'(x) = T'(x)$ va $D''(x) < T''(x)$ bo'lishi kerak.

Quyidagi teorema maksimal foyda olish shartlari haqida:

Teorema. Maksimal foyda x ning shunday qiymatlarida olinadiki, bunda umumiy daromad funksiyasidan olingan hosila tannarx funksiyasidan olingan hosilaga teng bo'lishi va daromad funksiyasidan olingan 2-tartibli hosila tannarx funksiyasidan olingan 2-tartibli hosiladan kichik bo'lishi zarur, ya'ni

$$D'(x) = T'(x) \text{ va } D''(x) < T''(x).$$

3.5.2. Oraliqni teng ikkiga bo'lish usulidan foydalanib, funksiyaning maksimum va minimumini topish

Optimallash masalalari bilan inson faoliyatining istalgan doirasida, shaxsiy ishlardan tortib umumdavlat ishlarigacha bo'lgan darajada oshkor yoki oshkormas shaklda duch kelamiz.

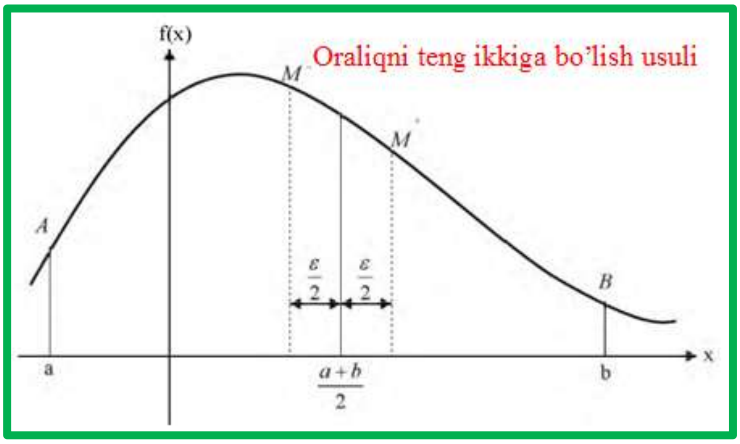
Iqtisodiy rejalashtirish, boshqarish, chegaralangan resurslarni taqsimlash, ishlab chiqarish jarayonini tahlil qilish, murakkab ob'yektlarni loyihalash doim mo'ljallangan maqsad nuqtai nazaridan eng yaxshi variantni izlashga qaratilgan bo'lishi lozim.

Matematik nuqtai nazardan maqsad funksiyasi oshkor formula bilan berilgan va differensiallanuvchi funksiyadan iborat bo'lgan hol **eng sodda optimallashtirish masalasidir.** Bu holda funksiyaning xossalarini tekshirish, uning o'sish va kamayish yo'nalishlarini aniqlash, lokal ekstremum nuqtalarini izlashda hosiladan foydalanish mumkin. Optimallashtirish masalalarini hal qilishning 3 ta usuli keng qo'llaniladi. Bular oraliqni teng ikkiga bo'lish usuli, oltin kesim usuli, Nyuton usullaridir.

Oraliqni teng ikkiga bo'lish usuli: Bizga $[a,b]$ oraliqda berilgan $y = f(x)$ funksiyaning maksimumini aniqlash kerak bo'lsin. Faraz qilaylik, funksiya o'zining maksimum(minimumi) qiymatiga ε oraliqda erishadi.

Dastlab oraliq o'rtasini aniqlab olamiz: $x_0 = \frac{a+b}{2}$. So'ngra funksiyaning $f\left(\frac{a+b}{2} + \frac{\varepsilon}{2}\right)$ va $f\left(\frac{a+b}{2} - \frac{\varepsilon}{2}\right)$ qiymatlarini hisoblaymiz.

Agar $f\left(\frac{a+b}{2}+\frac{\varepsilon}{2}\right) \geq f\left(\frac{a+b}{2}-\frac{\varepsilon}{2}\right)$ tengsizlik o‘rinli bo‘lsa, funksiyaning maksimumi (minimumi) $\left[\frac{a+b}{2}-\frac{\varepsilon}{2};b\right]$ oraliqda yotadi.



Agar $f\left(\frac{a+b}{2}+\frac{\varepsilon}{2}\right) \leq f\left(\frac{a+b}{2}-\frac{\varepsilon}{2}\right)$ tengsizlik o‘rinli bo‘lsa, funksiyaning maksimumi (minimumi) $\left[a;\frac{a+b}{2}+\frac{\varepsilon}{2}\right]$ oraliqda yotgan bo‘ladi.

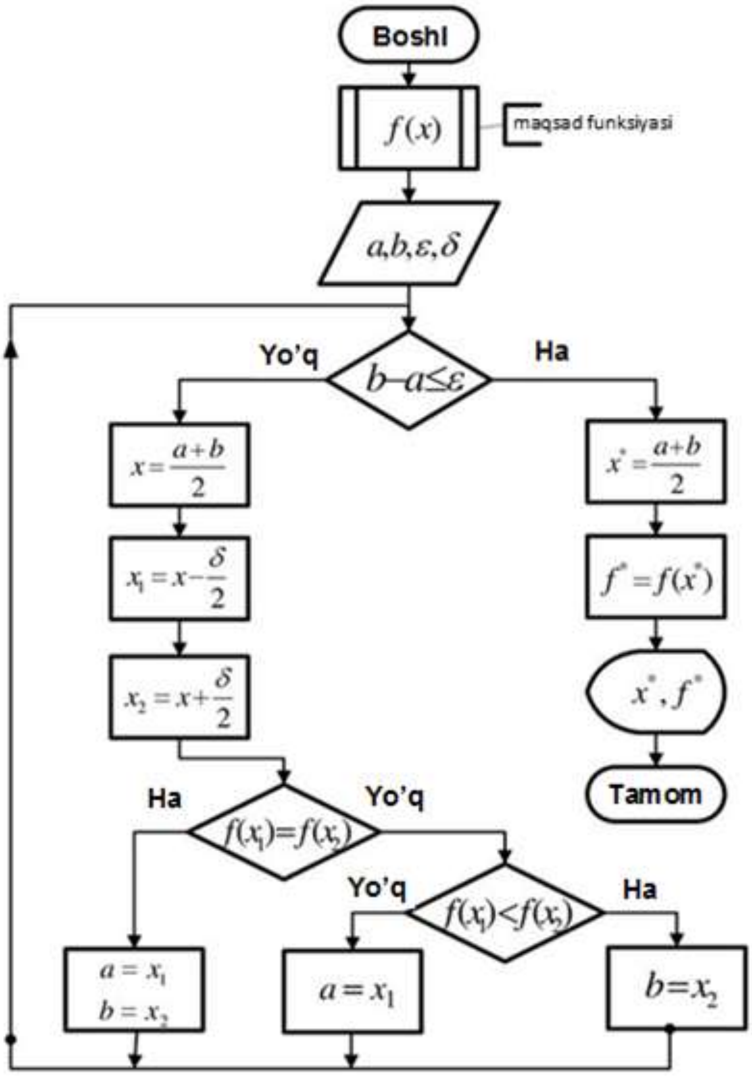
Hosil bo‘lgan oraliqni o‘rtasini topib olib, shu tariqa davom qilib, oraliqlarni kichraytirib boraveramiz va lokal maksimum (minimumi)ga erishamiz.

Oraliqni teng ikkiga bo‘lish usulida optimal yechim yotgan oraliqning chegaralari berilgan bo‘lishi kerak.

Nyuton usulida $y = f(x)$ funksiya berilishining o‘zi yetarli, unda yechim izlanishi kerak bo‘lgan oraliq talab qilinmaydi. Boshlang‘ich optimal qiymat faraz qilinadi.

Nyuton usulida boshlang‘ich qiymat noto‘g‘ri tanlansa, optimal yechimga yaqinlashmasligi ham mumkin.

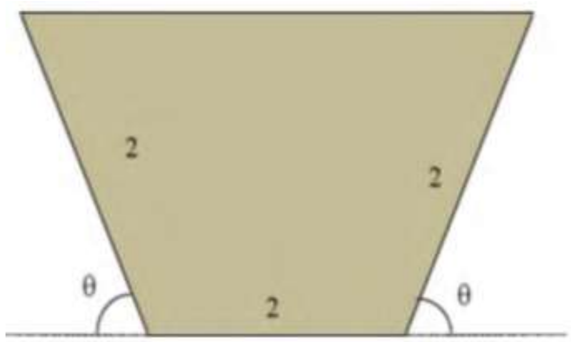
Quyida oraliqni teng ikkiga bo‘lish usulida optimal qiymatni topish algoritmi keltirilgan:



5-misol. Ma’lumot uzatish tezligi optik tolaning ko‘ndalang kesimiga bog‘liq. Optik tola ko‘ndalang kesimining yuzasi

$$S = 4\sin\theta(1+\cos\theta)$$

ga teng bo‘lib, asosi va yon tomonlarining uzunliklari 2 birlikka teng. θ burchak qanday bo‘lganda kesim yuzasi maksimal bo‘ladi?



Yechilishi: ► Oraliqni teng ikkiga bo'lish usulidan foydalanib yechamiz:

Burchak $0\le\theta\le 90^0$ oralig'ida bo'lishi mumkin, ya'ni

$\left[0;\frac{\pi}{2}\right]=[0;1.5708]$, funksiya esa berilgan $f(\theta)=4\sin\theta(1+\cos\theta)$.

1-iteratsiya: $f\left(\frac{a+b}{2}+\frac{\varepsilon}{2}\right)=f\left[\frac{0+1.5708}{2}+\frac{0.2}{2}\right]=f(0.8854)=5.0568$

$$f\left(\frac{a+b}{2}-\frac{\varepsilon}{2}\right)=f\left(\frac{0+1,5708}{2}-\frac{0,2}{2}\right)=f(0,6854)=4,4921$$

$$f(0,8854)>f(0,6854)$$

$$\left[\frac{a+b}{2}-\frac{\varepsilon}{2};b\right]=[0,6854;\ 1,5708\ ]$$

2-iteratsiya: $f\left(\frac{a+b}{2}+\frac{\varepsilon}{2}\right)=f\left[\frac{0.6854+1.5708}{2}+\frac{0.2}{2}\right]=f(1.2281)=5.0334$

$$f\left(\frac{a+b}{2}-\frac{\varepsilon}{2}\right)=f\left(\frac{0,6854+1,5708}{2}-\frac{0,2}{2}\right)=f(1,0281)=5,1942$$

$$f(1,2281)<f(1,0281)$$

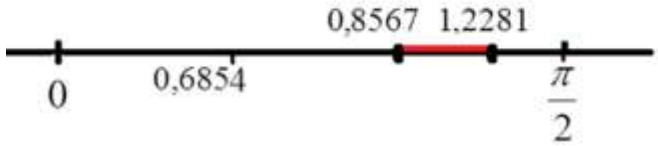
$$\left[a;\frac{a+b}{2}+\frac{\varepsilon}{2}\right]=[0,6854;\ 1,2281]$$

3-iteratsiya: $f\left(\frac{a+b}{2}+\frac{\varepsilon}{2}\right)=f\left[\frac{0.6854+1.2281}{2}+\frac{0.2}{2}\right]=f(1.0567)=5.1957$

$$f\left(\frac{a+b}{2}-\frac{\varepsilon}{2}\right)=f\left(\frac{0,6854+1,2281}{2}-\frac{0,2}{2}\right)=f(0,8567)=5,0025$$

$$f(1,0567)>f(1,8567)$$

$$\left[\frac{a+b}{2}-\frac{\varepsilon}{2};b\right]=[0,8567;\ 1,2281]$$



16-iteratsiyadan keyin: $f(\theta)=5.1962$, $\theta=1.0472$, $\theta=60^0$

Demak, θ burchak 60^0 bo‘lganda optik tolaning kesim yuzasi maksimal bo‘lar ekan. ◄

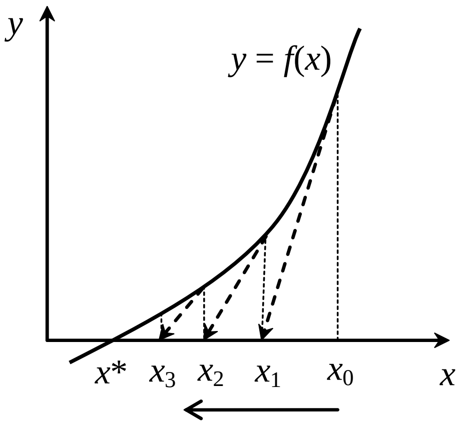
3.5.3. Nyuton usulidan foydalanib, funksiyaning maksimum va minimumini topish

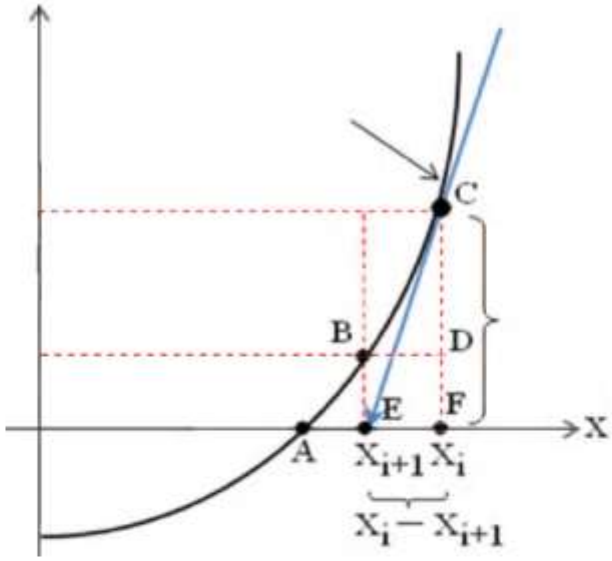
Nyuton usuli – $y=f(x)$ funksiyaning ekstremum qiymatlarini topishda muhim ahamiyatga ega. Bu usulda $[a,b]$ oraliqda funksiyaning silliq bo‘lishi talab qilinadi, ya’ni $f(x)$ funksiyaning ixtiyoriy $x\in[a,b]$ qiymatlari uchun $f'(x)$ va $f''(x)$ hosilalari mavjud bo‘lishi va ular noldan farqli bo‘lishi kerak.

Nyuton usuli $f(x)=0$ tenglamani yechishda qo‘llaniladigan urinmalar (Nyuton-Rapson) usuliga o‘xshaydi.

Urinmalar usuli: Argument orttirmasi nolga intilganda C nuqtadan o‘tuvchi CE urinmaning burchak koeffitsiyentini aniqlaymiz:

$$k \approx \frac{F(X_i)-F(X_{i+1})}{X_i - X_{i+1}}.$$





Agar keyingi X_{i+1} yaqinlashish yechim bo‘lsa yoki $F(X_{i+1})=0$ tenglik o‘rinli bo‘lsa, u holda

$$k=\frac{F(X_i)-0}{X_i-X_{i+1}}; \qquad k=F'(X_i);$$

$F'(X_i)=\dfrac{F(X_i)}{X_i-X_{i+1}}$ tenglikni hosil qilamiz. Bundan $X_i - X_{i+1} = \dfrac{F(X_i)}{F'(X_i)}$

kelib chiqadi. X_{i+1} ni topib olsak, hosil bo‘lgan tenglik urinmalar usuli formulasi bo‘ladi:

$$X_{i+1}=X_i-\frac{F(X_i)}{F'(X_i)}$$

Agar ushbu tenglamada $F(X)\equiv f'(X)$ deb qabul qilsak, u holda $F'(X)\equiv f''(X)$ ham o‘rinli bo‘lib,

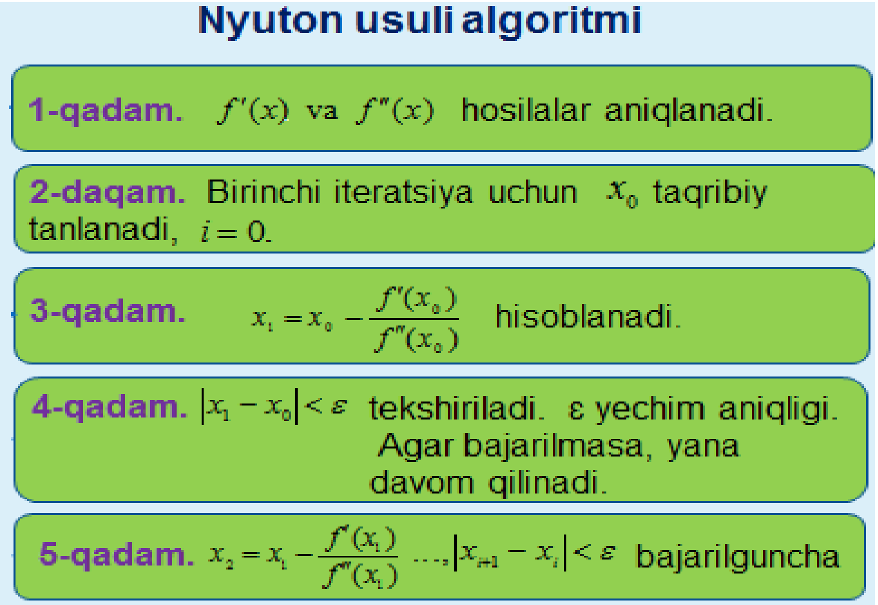
$$X_{i+1} = X_i - \frac{f'(X_i)}{f''(X_i)}$$

optimal yechimga yaqinlashishning Nyuton usuli formulasi hosil bo‘ladi. Bu yerda x_i - maksimum (minimum) nuqtaga $i-$yaqinlashish bo‘lsa, x_{i+1} - yaqinlashish quyidagi formuladan aniqlanadi:

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)}.$$

Nyuton usulida boshqa usullarga qaraganda ekstremum qiymatga tezroq yaqinlashiladi.

Qachon hisoblash to‘xtatiladi? Hisoblash ishlari $|x_{i+1} - x_i| < \varepsilon$ tengsizlik bajarilguncha davom ettiriladi.

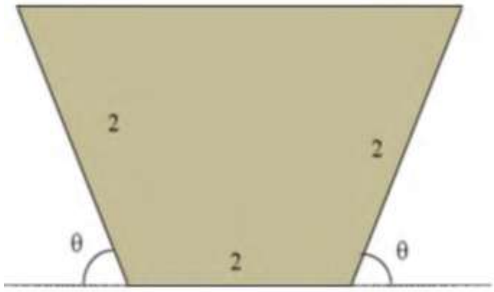


6-misol. Ma’lumot uzatish tezligi optik tolaning ko‘ndalang kesimiga bog‘liq. Optik tola ko‘ndalang kesimining yuzasi

$$S = 4\sin\theta(1+\cos\theta)$$

ga teng bo‘lib, asosi va yon tomonlarining uzunliklari 2 birlikka teng. θ burchak qanday bo‘lganda kesim yuzasi maksimal bo‘ladi?

Yechilishi: ► 5-misolni Nyuton usulidan foydalanib yechamiz: Ko‘ndalang kesim yuzasini funksiya deb faraz qilsak, $f(\theta) = 4\sin\theta(1+\cos\theta)$ ga ega bo‘lamiz.



Yuza funksiyasidan 1- va 2-tartibli hosilalar olamiz: $f'(\theta) = 4\cos\theta(1+\cos\theta) + 4\sin\theta \cdot (-\sin\theta) = 4(\cos\theta + \cos^2\theta - \sin^2\theta)$

$f''(\theta) = -4\sin\theta \cdot (1+4\cos\theta)$. Endi Nyuton formulasiga qo‘yamiz:

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)} \qquad \theta_{i+1} = \theta_i - \frac{f'(\theta_i)}{f''(\theta_i)}$$

1-iteratsiya: $\theta_1 = \theta_0 - \dfrac{f'(\theta_0)}{f''(\theta_0)}$. Oraliq $\left[0; \dfrac{\pi}{2}\right]$ ga, oraliq o‘rtasi $\theta_0 = \dfrac{\pi}{4}$.

$$\theta_1 = \frac{\pi}{4} - \frac{f'\left(\frac{\pi}{4}\right)}{f''\left(\frac{\pi}{4}\right)} = \frac{\pi}{4} - \frac{4(\cos\frac{\pi}{4} + \cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{4})}{-4\sin\frac{\pi}{4}(1+4\cos\frac{\pi}{4})} = 1.0466$$

$$\theta_1 = 1.0466$$

$$f(\theta_1) = f(1.0466) = 4\sin 1.0466 \cdot (1+\cos 1.0466) = 5.1962$$

2-iteratsiya: $$\theta_2 = \theta_1 - \frac{f'(\theta_1)}{f''(\theta_1)} = 1.0466 - \frac{f'(1.0466)}{f''(1.0466)} =$$

$$= 1.0466 - \frac{4(\cos 1.0466 + \cos^2 1.0466 - \sin^2 1.0466)}{-4\sin 1.0466(1+4\cos 1.0466)} = 1.0472$$

$$f(\theta_2) = f(1.0472) = 4\sin 1.0472 \cdot (1+\cos 1.0472) = 5.1962$$

Agar 1-tartibli hosila nolga teng chiqsa, ko‘zlangan natijaga erishgan bo‘lamiz. Ish shu joyda to‘xtatiladi.

$\theta = 1.0472$ radianni gradusga o‘tkazamiz:

$$\theta = 1.0472\,\text{rad} = 1.0472 \cdot \frac{180^0}{\pi} = 60^0$$

Demak, θ burchak 60^0 bo‘lganda optik tolaning kesim yuzasi maksimal bo‘ladi va $S(1.0472) = 5.1962$ ga teng bo‘ladi.

Iteratsiya	θ_i	$f'(\theta_i)$	$f''(\theta_i)$	θ_{i+1}	$f(\theta_{i+1})$
1	0.78540	2.8284	-10.828	1.0466	5.1962
2	1.0466	0.0061898	-10.396	1.0472	5.1962
3	1.0472	1.0613E-06	-10.392	1.0472	5.1962
4	1.0472	3.0642E-14	-10.392	1.0472	5.1962
5	1.0472	1.3323E-15	-10.392	1.0472	5.1962

◄

MUSTAQIL YECHISH UCHUN MASALALAR

1-12 misollar nazariy optimallashtirishga doir

1. Yig‘indisi 50 ga teng bo‘lgan qanday x va y sonlarning ko‘paytmasi maksimal bo‘ladi? Ya’ni $x + y = 50$ bo‘lsa, $P_{max} = xy$ ni toping.

2. Yig‘indisi 70 ga teng bo‘lgan qanday x va y sonlarning ko‘paytmasi maksimal bo‘ladi? Ya’ni $x + y = 70$ bo‘lsa, $P_{max} = xy$ ni toping.

3. $x + y = 50$ bo‘lsa, $P_{min} = xy$ bo‘ladigan x va y sonlar mavjudmi?

4. $x + y = 70$ bo‘lsa, $P_{min} = xy$ bo‘ladigan x va y sonlar mavjudmi?

5. Ayirmasi 4 ga teng bo‘lgan qanday x va y sonlarning ko‘paytmasi minimal bo‘ladi?

6. Ayirmasi 6 ga teng bo‘lgan qanday x va y sonlarning ko‘paytmasi minimal bo‘ladi?

7. $x+y^2=1$ bo‘ladigan musbat x va y sonlar orasidan ko‘paytmasi $P_{max}=xy^2$ maksimal bo‘ladiganlarini toping.

8. $x+y^2=4$ bo‘ladigan musbat x va y sonlarning ko‘paytmasi maksimal bo‘ladiganlarini toping.

9. $x+y=5$ bo‘lsa, $P_{min}=2x^2+3y^2$ bo‘ladigan x va y sonlarni toping.

10. $x+y=3$ bo‘lsa, $P_{min}=x^2+2y^2$ bo‘ladigan x va y sonlarni toping.

11. $\frac{4}{3}x^2+y=16$ bo‘lsa, $P_{max}=xy$ bo‘ladigan musbat x, y sonlarni toping.

12. $x+\frac{4}{3}y^2=1$ bo‘lsa, $P_{max}=xy$ bo‘ladigan musbat x, y sonlarni toping.

13-25 misollar optimallashtirishning amaliy tatbiqlariga doir

13. Maydon yuzasini maksimallashtirish. Qutqaruvchida cho‘milish maydonini to‘g‘ri to‘rtburchak shaklida o‘rab olish uchun 180 m arqon bor. Maydonning o‘lchamlari qanday bo‘lganda, maksimal yuzaga ega bo‘ladi? Maydonning bir tomoni qirg‘oq deb hisoblang.

14. Maydon yuzasini maksimallashtirish. Duradgor o‘ziga perimetri 54 m bo‘lgan to‘rtburchak shaklidagi ustaxona qurmoqchi. Bino keng bo‘lishi uchun u qanday o‘lchamlarni olishi kerak? Maksimal maydon nimaga teng?

15. Perimetri 42 m bo‘lgan to‘rtburchaklar ichida yuzasi eng katta bo‘ladigani qaysi?

16. Maydon yuzasini maksimallashtirish. Fermer daryo bo‘yida qo‘y va qoramollar uchun 2 ta to‘g‘ri to‘rtburchak shaklidagi maydonni o‘zida bor bo‘lgan 240 m panjaralar bilan o‘rab olmoqchi bo‘ldi. Tomonlarni qanday olsa, maydon yuzasi maksimal bo‘ladi?

17. Maydon yuzasini maksimallashtirish. Sug‘rish kanalining ko‘ndalang kesimi teng yonli trapetsiya shaklida bo‘lib, uning yon tomoni kichik asosiga teng. Bu trapetsiyaning yon tomoni nishablik burchagi qanday bo‘lganda, kanalning ko‘ndalang kesimi yuzi maksimal bo‘ladi?

18. Hajmni maksimallashtirish. 50x50 o‘lchamli alyuminiy listdan uchidagi ortiqcha kvadratchalarni ichki tomonga bukib, eng katta hajmdagi quti yasash kerak. Qutining o‘lchamlari qanday bo‘lishi kerak? Hajmi nimaga teng?

19. Hajmni maksimallashtirish. 20x20 o‘lchamli tunukadan uchidagi ortiqcha qismini ichki tomonga bukib, eng katta hajmdagi quti yasash uchun uning o‘lchamlarini qanday olish kerak? Hajmi nimaga teng bo‘ladi?

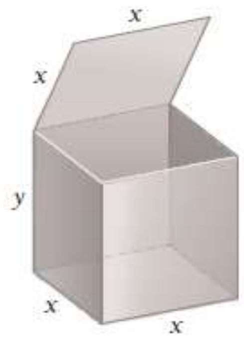
20. Hajmni maksimallashtirish. Yuzasi $S = 75\pi$ m^2 tunukadan asosining radiusi R va balandligi H bo‘lgan usti ochiq shunday silindr bak yasangki, uning hajmi maksimal bo‘lsin.

21. Mahsulot sarfini kamaytirish. “Aqua plast” MChJ asosi kvadrat, usti ochiq, hajmi 62.5 m^3 bo‘lgan idish ishlab chiqarmoqchi. O‘lchamlarni qanday olganda, material sarfi kamayadi? Minimal to‘la sirt nima teng bo‘ladi?

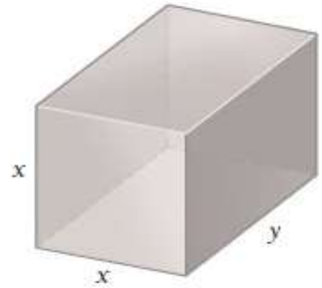
22. Mahsulot sarfini kamaytirish. Korxona asosi kvadrat, usti ochiq, hajmi 32 m^3 bo‘lgan rezervuar chiqarmoqchi. O‘lchamlarni qanday olganda, material sarfi kamayadi? Minimal to‘la sirt nima teng bo‘ladi?

23. Mahsulot sarfini kamaytirish. Chiqindilarni qayta ishlash korxonasi hajmi 12 m^3, asosi bo‘yi enidan 2 marta katta to‘rtburchak, usti ochiq konteyner loyihalamoqchi. O‘lchamlarni qanday olganda, konteynerning to‘la sirti minimal bo‘ladi?

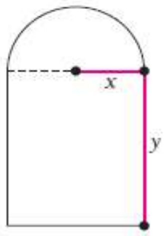
24. Mahsulot sarfini kamaytirish. Asosi kvadrat bo‘lgan prizma shaklidagi hajmi 320 m^3 rezervuarni loyihalash kerak. O‘lchamlarni qanday olganda, material sarfi kamayadi?



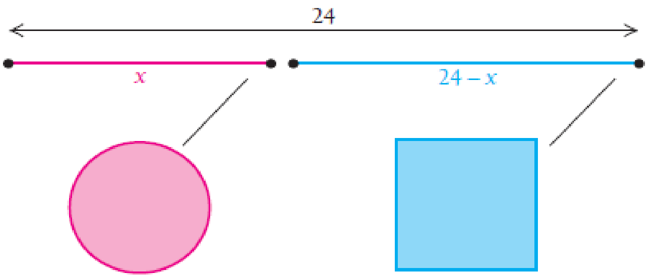
25. Hajmni maksimallashtirish. Pochta xizmati jo‘natma qutisining asoslari va balandligi o‘lchamlining yi’gindisi 82 sm bo‘lishiga ruxsat beradi. Eng katta hajmdagi jo‘natma qutisini yasash uchun uning o‘lchamlari qanday bo‘lish kerak? Hajmi nimaga teng bo‘ladi?



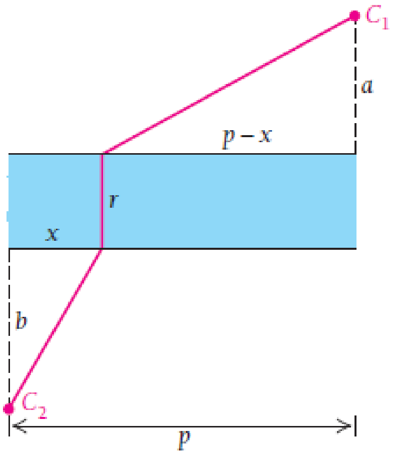
26. Tushayotgan yorug‘likni maksimallashtirish. Rasmda ko‘rsatilgan derazaning perimetri 24 m. Uning o‘lchamlari qanday bo‘lganda xonaga maksimal yorug‘lik tushadi?



27. Uzunligi 24 m bo‘lgan o‘tkazgich 2 qismga ajratildi. 1-bo‘lagidan aylana, 2- bo‘lagidan kvadrat yasaldi. O‘tkazgichni qanday bo‘laklarga ajratsak, doira va kvadrat yuzlarining yig‘indisi minimum bo‘ladi? Qachon maksimum bo‘ladi?

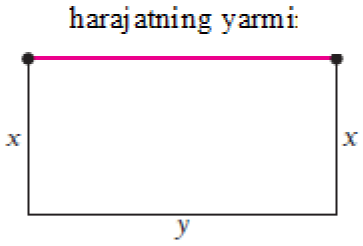


28. Masofani qisqartirish. Eni r bo‘lgan daryoning ikki tomonidagi C_1 va C_2 shaharlar orasiga yo‘l o‘tkazishmoqchi bo‘lishdi.



C_1 daryodan a masofada, C_2 esa b uzoqlikda joylashgan, bunda $a \le b$. Ko‘prik daryoning qayeridan qurilsa, to‘liq masofani minimallashtirish mumkin? a, b, p, r lar ishtirokida yechimning umumiy ko‘rinishini toping.

29. Yuzasi 48 m^2 bo‘lgan o‘yin maydonchasini mahalla aholisi panjara bilan o‘ramoqchi bo‘lishdi. Maydonning yonida yashaydigan qo‘shni mablag‘ning yarmini beradigan bo‘ldi. Maydonning o‘lchamlari qanday bo‘lganda kam mablag‘ sarflanadi?



30-35 misollar foydani maksimallashtirish haqida.

Maksimal foyda olish uchun ishlab chiqarib, sotiladigan mahsulot sonini va maksimal foydani toping (dollar hisobida). Bunda tannarxni $T(x)$, daromadni $D(x)$ deb hisoblang:

30. $D(x)=50x-0.5x^2$, $T(x)=4x+10$;

31. $D(x)=50x-0.5x^2$, $T(x)=10x+3$;

32. $D(x)=2x$, $T(x)=0.01x^2+0.6x+30$;

33. $D(x)=5x$, $T(x)=0.001x^2+1.2x+60$;

34. $D(x)=9x-x^2$, $T(x)=x^3-3x^2+4x+1$;

35. $D(x)=100x-x^2$, $T(x)=\frac{1}{3}x^3-6x^2+89x+100$.

36. Tannarxni kamaytirish. Ma’lum bir mahsulotni ishlab chiqarish umumiy tannarxi (dollar hisobida) $T(x)=\frac{x^3}{100}+8x+20$ bo‘lsa,

a) O‘rtacha tannarxni $T_{o'rt}(x)=\frac{T(x)}{x}$ ni toping;

b) $T'(x)$ va $T'_{o'rt}(x)$ ni hisoblang;

v) $T_{o'rt}(x)$ minimum qiymatini va uni minimumga olib keladigan x_0 qiymatni toping. $T'(x_0)$ ni hisoblang;

g) $T_{o'rt}(x)$ va $T'(x_0)$ ni taqqoslang.

37. Tannarxni kamaytirish. $T_{o'rt}(x)=\frac{T(x)}{x}$ tenglik ma’lum.

a) $T'(x)$ va $T(x)$ larga nisbatan $T'_{o'rt}(x)$ ni baholang;

b) agar $T_{o'rt}(x)$ minimumga ega bo‘lsa, u holda shunday x_0 nuqta mavjudki, $T'(x_0)=T_{o'rt}(x_0)=\frac{T(x_0)}{x_0}$ tenglik o‘rinli bo‘ladi.

Bu tenglik, agar o‘rtacha tannarx minimallashtirilsa, bu minimum eng chegaraviy tannarx o‘rtacha tannarxga teng bo‘lganda ro‘y berishini ko‘rsatadi.

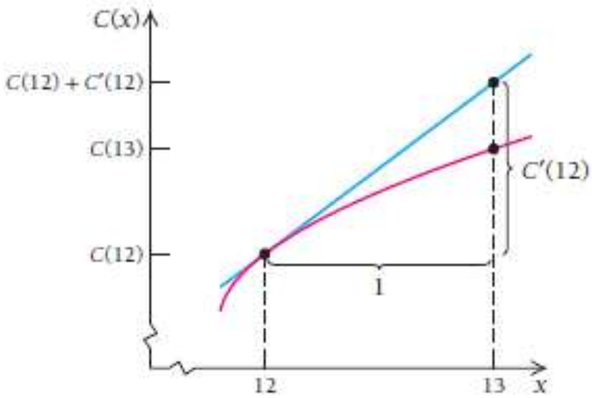
3.6. Funksiyalarni taqribiy hisoblash

Ushbu bo‘limda chiziqli approksimatsiyalashga differensiallash usullarini qanday tatbiq qilish mumkinligini o‘rganamiz. Faraz qilaylik, kompaniya ishlab chiqarishni kengaytirmoqchi. Buning uchun dastlab, kompaniya tannarx, daromad va foyda o‘zgarishini nazariy jihatdan taqribiy hisoblab ko‘rishi kerak.

3.6.1. Tannarx, daromad va foyda

Aytaylik, “Oila davrasida” ko‘rsatuvi ijodkorlari oyiga 12 ta chiqish tayyorlashadi. Endilikda ular 12 ta emas, 13 ta ko‘rsatuv tayyorlashmoqchi deylik. Bilingki, sarf-harajat (tannarx) ham o‘zgaradi. Diqqat qiling, 13 ta ko‘rsatuvni efirga uzatish xarajati (tannarxi) 12 ta ko‘rsatuvni efirga uzatish xarajati(tannarxi)ga shu 12 ta ko‘rsatuv tayyorlashga ketadigan chegaraviy xarajatni qo‘shganimizga teng bo‘ladi:

$$T(13) \approx T(12) + T'(12).$$



$T'(12)$ ga **chegaraviy tannarx** deyiladi. Urinma ta’rifiga ko‘ra, $T'(12)$ qiymat tannarx funksiyasiga $(12; T(12))$ nuqtada o‘tkazilgan urinmaning burchak koeffitsiyentidan iborat. Misol uchun, agar $T'(12) = \frac{3}{4}$ bo‘lsa, biz buni vertikal orttirma 3 ga teng, gorizontal orttirma 4 ga teng deb yoki vertikal orttirma $\frac{3}{4}$ ga, gorizontal orttirma 1 ga teng deb faraz qilishimiz mumkin. Aynan shu 2-farazni qabul qilamiz.

Grafikdan ko‘rish mumkinki, $T'(12)$ kattalik $T(13)$ va $T(12)$ orasidagi farqni bildiradi, ya’ni $T'(12) \approx T(13) - T(12)$ o‘rinli. 13-ko‘rsatuvni tayyorlash uchun ketadigan xarajat 12 ta ko‘rsatuv uchun ketadigan xarajatning chegaraviy qiymatiga teng.

Ta’rif: $T(x), D(x), F(x)$ mos ravishda x mahsulotni ishlab chiqarish va sotishdagi tannarx, daromad va foyda bo‘lsin.

x mahsulotni ishlab chiqarish $T'(x)$ chegaraviy tannarxi $(x+1)-$ mahsulot tannarxiga yaqinroq bo‘ladi:

$$T'(x) \approx T(x+1) - T(x) \text{ yoki } T(x+1) \approx T(x) + T'(x).$$

x mahsulotni ishlab chiqarish $D'(x)$ chegaraviy daromadi $(x+1)-$ mahsulot daromadiga yaqinroq bo‘ladi:

$$D'(x) \approx D(x+1) - D(x) \text{ yoki } D(x+1) \approx D(x) + D'(x).$$

x mahsulotni ishlab chiqarish $F'(x)$ chegaraviy foydasi $(x+1)-$ mahsulot faydasiga yaqinroq bo‘ladi:

$$F'(x) \approx F(x+1) - F(x) \text{ yoki } F(x+1) \approx F(x) + F'(x).$$

Bizga ma’lmki, $F'(x) = D'(x) - T'(x)$.

1-misol. Tadbirkorlik. Chegaraviy foyda, daromad va tannarx. Umumiy tannarx $T(x) = 62x^2 + 27500$ va umumiy daromad $D(x) = x^3 - 12x^2 + 40x + 10$ funksiyalari (dollar hisobida) ma’lum bo‘lsa, quyidagilarni toping:

a) Umumiy foyda $F(x)$ ni;

b) 50 ta tovarni ishlab chiqarish va sotishdagi umumiy tannarx, daromad va foydani;

c) 50 ta tovarni ishlab chiqarish va sotishdagi chegaraviy tannarx, daromad va foydani toping.

Yechilishi: ► a) Umumiy foyda

$$F(x) = D(x) - T(x) = (x^3 - 12x^2 + 40x + 10) - (62x^2 + 27500) =$$

$$= x^3 - 74x^2 + 40x - 27490;$$

b) 50 ta tovarni ishlab chiqarish va sotishdagi umumiy tannarx

$T(50) = 62 \cdot 50^2 + 27500 = 182500\,\$;$

Umumiy daromad $D(50) = 50^3 - 12 \cdot 50^2 + 40 \cdot 50 + 10 = 97010\ \$;$

Umumiy foyda $F(x) = D(x) - T(x) = 97010 - 182500 = -85490\ \$.$

Agarda 50 dona Tovar ishlab chiqarilsa, 85 490 $ zarar ko‘rilar ekan.

c) 50 ta tovarni ishlab chiqarish va sotishdagi chegaraviy tannarx, daromad va foydani topamiz:

$$T'(x) = 124x, \quad T'(50) = 124 \cdot 50 = 6200\,\$.$$

50 dona tovar ishlab chiqarilgandan keyin, to‘xtamasdan 51-tovarni ishlab chiqarish uchun 6200 $ kerak ekan.

$$D'(x) = 3x^2 - 24x + 40, \quad D'(50) = 3 \cdot 50^2 - 24 \cdot 50 + 40 = 6340\,\$.$$

50 dona tovar sotilgandan keyin, 51-tovarni sotganda 6340 $ daromad olinishi kerak.

$$F'(50) = D'(50) - T'(50) = 6340 - 6200 = 140\,\$\ .$$

50 dona tovar ishlab chiqarilib, sotgandan keyin, 51-tovardan 140 $ foyda olish kerak. ◄

Ko‘pincha amaliyotda $T(x), D(x), F(x)$ kattaliklarning formulalari noma’lum bo‘ladi, biroq $x=a$ dona tovar ishlab chiqarishga ketadigan xarajat, olinadigan daromad va qancha foyda qolishi haqidagi ma’lumotlar ma’lum bo‘lishi mumkin.

Misol uchun, $T(a)$ va $T'(a)$ ma’lum bo‘lsa, $T(a+1)$ haqida xulosa qilish mumkin. Xuddi shuningdek, $D(a+1)$ va $F(a+1)$ haqida xulosalar qila olamiz. Bizning farazlarimiz qanchalik haqiqatga yaqinligini 1-misolda formulalari bilan berilgan holatda ko‘rib chiqamiz.

$$T(51)-T(50)=62\cdot 51^2+27500-(62\cdot 50^2-27500)=6262\,\$$$

$$T'(x)=124x=124\cdot 50=6200\,\$$$

$$T'(x)\approx T(x+1)-T(x)$$

$T'(50)$ ning qiymati $T(51)-T(50)$ ga 1% xatolik bilan yaqinlashar - approksimatsiyalanar ekan.

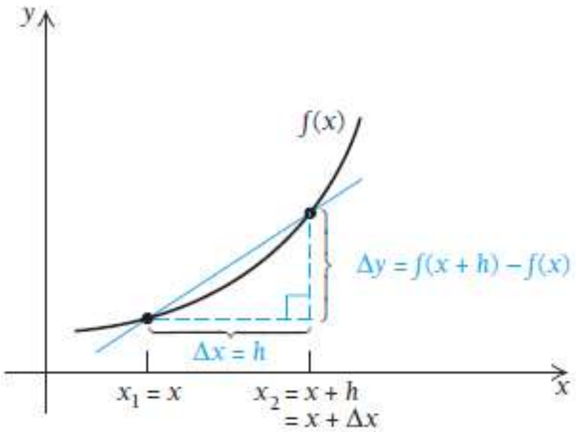
1-vazifa: $D(51)-D(50)$ va $F(51)-F(50)$ ni o‘zingiz tekshirib ko‘ring.

Yodda saqlang! Chegaraviy tannarx va o‘rtacha tannarx bir-biridan farq qiladi. O‘rtacha tannarx $T_{o'rt}(50)=\frac{T(50)}{50}=\frac{182500}{50}=3650\,\$$ ga teng, chegaraviy tannarx esa $T'(50)=6200\,\$$ taxminan 51- tovar narxiga teng bo‘ladi.

3.6.2. Funksiyalarni taqribiy hisoblash formulasi

Xuddi $T(x_0+1)$ qiymatni hisoblash uchun $T'(x_0)$ chegaraviy tannarxdan foydalanganimizdek, ixtiyoriy uzluksiz funksiyaning x_0 nuqta yonidagi nuqtada qiymatini topish uchun $f'(x_0)$ dan foydalanamiz.

Dastlab ishimiz uchun kerak boladigan belgilashlarni kiritib olamiz. Orttirmalar nisbatini eslang: $\dfrac{f(x+h)-f(x)}{h}$. Orttirmalar nisbati funksiyaning x nuqtadagi hosilasini aniqlash uchun ishlatiladi.



h- argumentning orttirmasi bo‘lib, uni boshqacha “delta iks” deb ham belgilanadi: $\Delta x=(x+h)-x=h$ yoki $\Delta x=x_2-x_1 \quad \rightarrow \quad x_2=x_1+\Delta x$.

Δx ning qiymati musbat bo‘lishi ham manfiy bo‘lishi ham mumkin.

Misol uchun,

agar $x_1=7$ va $\Delta x=0.1$ bo‘lsa, $x_2=x_1+\Delta x=7.1$ bo‘ladi;

agar $x_1=7$ va $\Delta x=-0.1$ bo‘lsa, $x_2=x_1+\Delta x=6.9$ bo‘ladi.

Umuman olganda x_1, x_2 deb yozmasdan x va $x+\Delta x$ deb belgilaymiz. Faraz qilaylik, $y=f(x)$ funksiya orttirma olgan bo‘lsin. Unda

$$\Delta y=f(x+\Delta x)-f(x)$$

tenglik o‘rinli bo‘ladi.

2-misol. $y=3x^2$ uchun $x=5$ va $\Delta x=0.1$ bo‘lsin. Δy ni toping.

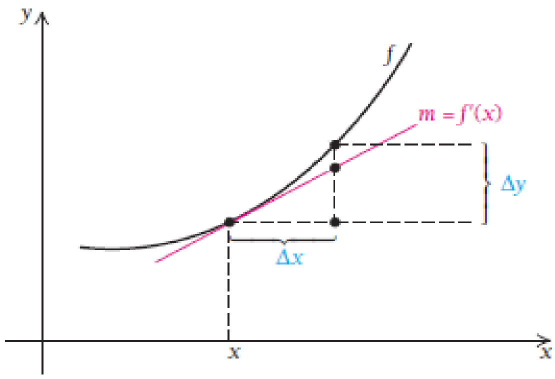
Yechilishi: ► $\Delta y=f(x+\Delta x)-f(x)=3(x+\Delta x)^2-3x^2=$

$$=3(5+0.1)^2-3\cdot 5^2=3\cdot(5.1^2-5^2)=3.03 \quad ◄$$

3-misol. $y=x^3$ uchun $x=2$ va $\Delta x=-0.1$ bo‘lsin. Δy ni toping.

Yechilishi: ► $\Delta y=(x+\Delta x)^3-x^3=(2-0.1)^3-2^3=6.859-8=-1.141$ ◄

Endi funksiya qiymatini hisoblashga harakat qilamiz. Yuqoridagi formulalarni hisobga olgan holda $\dfrac{f(x+h)-f(x)}{h}$ dan $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=\dfrac{\Delta y}{\Delta x}$ kelib chiqadi. Bu nisbatdan limitga o‘tsak, $\dfrac{dy}{dx}=\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$ tenglikni hosil qilish mumkin. Δ belgilash 2.5 bo‘limdagi Leybnits belgilashiga o‘xshaydi: $\Delta x\to 0$ da $\dfrac{dy}{dx}\approx\dfrac{\Delta y}{\Delta x}$ yoki $f'(x)\approx\dfrac{\Delta y}{\Delta x}$. Tenglikninh har ikki tomonini Δx ga ko‘paytiramiz: $\Delta y\approx f'(x)\Delta x$ tenglikni chizmadan ko‘rish mumkin.



Grafikdan ko‘rinadiki, Δx ning juda kichik qiymatida urinmaga qarab, egri chiziqni baholash mumkin.

Uzluksiz, differensiallanuvchi $y=f(x)$ funksiya va kichik Δx larda

$$f'(x)\approx\frac{\Delta y}{\Delta x} \quad \text{va} \quad \Delta y\approx f'(x)\Delta x \text{ tengliklar o‘rinli.}$$

Keling shu g‘oyani $f(x)=\sqrt{x}$ funksiyaga qo‘llaymiz.

4-misol. $\Delta y \approx f'(x)\Delta x$ dan foydalanib, $\sqrt{27}$ ni hisoblang.

Yechilishi: ► 27 ni kvadrat ildizdan chiqara olmaymiz. Shuning uchun 27 ga yaqin bo‘lgan biror sonning kvadrati bo‘lgan sonni izlaymiz. U 25. Shunda $x=\sqrt{25}$ va $\Delta x = 2$ ga teng.

$$\Delta y \approx f'(x)\Delta x = \left(\sqrt{x}\right)'\Delta x = \frac{1}{2\sqrt{x}}\Delta x = \frac{1}{2\sqrt{25}}\cdot 2 = \frac{1}{5} = 0.2$$

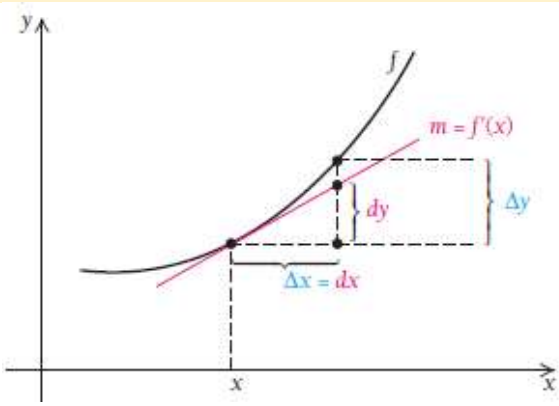
$$\sqrt{27} \approx \sqrt{25} + \Delta y = 5 + 0.2 = 5.2$$

Demak, $\sqrt{27} \approx 5.2$ qiymatni topdik. Endi kal’kulyatorda hisoblab ko‘ramiz: $\sqrt{27} = 5.19615$, ya’ni taqribiy hisobimiz yetarli aniqlikda bajarilgan. ◄

2-vazifa. $\Delta y \approx f'(x)\Delta x$ dan foydalanib, $\sqrt{98}$ ni hisoblang va natijani $\sqrt{98} \approx 9.89949$ bilan taqqoslang.

Biz hozirgacha dy va dx larni alohida deb qaraganimiz yo‘q, bu belgilarni bitta belgi deb hisoblab keldik. dy va dx belgilar differensiallardir.

Ta’rif. $y = f(x)$ funksiya uchun dx bu x **ning differensialini** bildiradi: $dx = \Delta x$, dy esa y **ning differensiali** $dy = f'(x)$ demakdir (chizmada ko‘rish mumkin).



5-misol. $y = x(2-x)^3$ funksiya berilgan.

a) dy ni hisoblang;

b) $x = 5$ va $dx = 0.01$ bo‘lganda dy ni hisoblang;

v) dy va Δy ni taqqoslang.

Yechilishi: ► a) Dastlab $\frac{dy}{dx}$ ni hisoblaymiz va dy ni topamiz:

$$\frac{dy}{dx} = (2-x)^3 - 3x(2-x)^2 = (2-x)^2(2-x-3x) = (2-x)^2(2-4x) = 2(2-x)^2(1-2x)$$

$$dy = 2(2-x)^2(1-2x)dx;$$

b) $x = 5$ va $dx = 0.01$ bo‘lganda dy ni hisoblaymiz:

$$dy = 2(2-5)^2(1-2\cdot 5)\cdot 0.01 = -1.62$$

v) $dy = -1.62$ qiymat y ning $x_1 = 5$ dan $x_2 = 5.01$ ga orttirma olgandagi qiymat bilan bir xil chiqadimi?

$$\Delta y = y_2 - y_1 = 5.01(2-5.01)^3 - 5(2-5)^3 = 5.01\cdot(-3.01)^3 + 5\cdot 27 = -1.627214$$

, bundan ko‘rinadiki dy va Δy lar bir-biriga ancha yaqin qiymatlar ekan.

Ushbu formuladan foydalanib, funksiyalar qiymatini taqribiy hisoblash mumkin. Yo‘l qo‘yiladigan xatolik esa ruxsat etilgan qiymatdan katta emas.

O‘tmishda amaliy masalalarni yechishda funksiyalarni taqribiy hisoblashning hosiladan foydalanish usuli juda muhim o‘rin tutar edi. Hozirgi kunda esa kal,kulyatorlar va grafik chizadigan dasturlar paydo bo‘lgandan keyin hosiladan foydalanish birmuncha kamaydi. Iqtisodiy masalalarni hal qilishda chegaraviy tannarx, chegaraviy daromad va chegaraviy foydani hisoblash formulalari hozir ham ahamiyatlidir.

6-misol. $\sqrt[3]{84}$ ni taqribiy hisoblang.

Yechilishi: ► Berilgan sonni $\sqrt[3]{84}=\sqrt[3]{4^3+20}$ yozamiz va $y=\sqrt[3]{x}$ funksiya ko`rinishida ifodalaymiz.

Bu yerda, $x=x_0+\Delta x$, $x_0=64$, $\Delta x=20$.

Funksiyalarni taqribiy hisoblash formulasidan foydalanamiz:

$$y(x_0+\Delta x)\approx y(x_0)+y'(x_0)\Delta x,$$

$$y(x_0)=\sqrt[3]{64}=4,\ y'=\frac{1}{\sqrt[3]{x^2}},\ y'(64)=\frac{1}{3\cdot 16}=\frac{1}{48}$$

ga ega bo`lamiz. Demak,

$$\sqrt[3]{84}\approx 4+20\cdot\frac{1}{48}=4+\frac{20}{48}=4.42.$$ ◄

7-misol. $\operatorname{arctg}0.98$ ni taqribiy hisoblang va nisbiy xatolikni baholang.

Yechilishi: $\operatorname{arctg}0.98$ ni taqribiy qiymatini topish uchun funksiya ko'rinishida yozib olamiz:

$$y=\operatorname{arctg}x,\ x_0=1,\ \Delta x=0.98-1=-0.02$$

$$y(x_0)=\operatorname{arctg}1=\frac{\pi}{4},$$

Funksiyalarni taqribiy hisoblash formulasiga qo'yamiz:

$$y(x_0+\Delta x)\approx y(x_0)+y'(x_0)\Delta x,$$

$$y'=\frac{1}{1+x^2},\ y'(1)=0.5,\ \operatorname{arctg}0.98\approx\frac{\pi}{4}-0.5\cdot 0.02=0.77.$$

Nisbiy hatolik δ ni topamiz:

$$\delta=\left|\frac{0.77-0.78}{0.77}\right|\cdot 100\%=13\%.$$ ◄

MUSTAQIL YECHISH UCHUN MASALALAR

1-4 misollar tadbirkorlikda hosiladan foydalanishga doir

1. Chegaraviy foyda, daromad va tannarx. Agar $D(x)=5x$ va $T(x)=0.001x^2+1.2x+60$ ma'lum bo'lsa, quyidagilarni toping:

a) Umumiy foyda $F(x)$ ni;

b) $T(100)$, $D(100)$ va $F(100)$ toping;

c) $T'(x)$, $D'(x)$ va $F'(x)$ ni hisoblang;

d) $T'(100)$, $D'(100)$ va $F'(100)$ toping;

e) b) va d) shartlardagi qiymarlar nimasi bilan farq qiladi?

2. Chegaraviy foyda, daromad va tannarx. Agar $D(x)=50x-0.5x^2$ va $T(x)=4x+10$ ma'lum bo'lsa, quyidagilarni toping:

a) Umumiy foyda $F(x)$ ni;

b) $T(20)$, $D(20)$ va $F(20)$ toping;

c) $T'(x)$, $D'(x)$ va $F'(x)$ ni hisoblang;

d) $T'(20)$, $D'(20)$ va $F'(20)$ toping.

3. Chegaraviy xarajat. Oyiga x stul ishlab chiqarish uchun $T(x)=0.001x^3+0.07x^2+19x+700$ xarajat qilinadi. Hozirgacha oyiga 25 ta stul chiqarilayotgan bo'lsin.

a) Oyiga qancha xarajat qilinmoqda?

b) Agar oyiga 26 stul chiqarila boshlansa, qo'shimcha qancha xarajat qilish kerak?

c) $x=25$ da chegaraviy xarajat qancha bo'ladi?

d) Chegaraviy xarajat formulasidan foydalanib, oyiga 25 va 27 stul chiqargandagi xarajatlarni baholang.

4. Chegaraviy xarajat. Kuniga x radio ishlab chiqarish uchun $T(x) = 0.002x^3 + 0.1x^2 + 42x + 300$ $ xarajat qilinadi. Hozirgacha kuniga 40 ta radio chiqarilayotgan bo‘lsin.

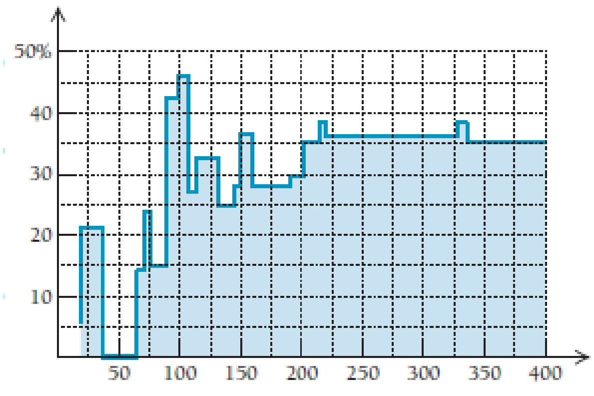
a) Kuniga qancha xarajat qilinmoqda?

b) Agar kuniga 41 rdio chiqarila boshlansa, qo‘shimcha qancha xarajat qilish kerak?

c) $x = 40$ da chegaraviy xarajat qancha bo‘ladi?

d) Chegaraviy xarajat formulasidan foydalanib, kuniga 42 radio chiqarganda xarajatlar qanchaga ko‘payishini baholang.

Chegaraviy soliq stavkasi. ***Firmalar va odamlarni ishlab topilgan har bir keyingi dollari soliqqa tortilishi masalasi qiynab keladi. Zamonaviy soliqchilikda 80 001- dollar 25001-dollarga qaraganda yuqoriroq, 140 001-dollarga qaraganda kamroq soliq stavkasi bilan hisoblanar ekan. Quyidagi 5-8 misollarni yechish uchun 2005 yildagi chegaraviy soliq stavkasi diagrammasidan foydalaning.***



100$ hisobida

5. 2005 yilda soliq stavkasi o‘suvchan bo‘lganmi? Nima uchun ha, nima uchun yo‘q?

6. Ali bilan Said marketing agentligida birga ishlashadi. Ali ishga yangi kelgan, shuning uchun uning daromadi yil oxiriga 95 000$, Saidning daromadi esa 150 000$ bo‘lishi kutilmoqda. Qaysi biri 5000 $ kam soliq to‘laydi? Nima uchun?

7. O‘qituvchi asosiy ishidan yiliga 25 000$ va o‘rindosh bo‘lib ishlagani uchun yiliga yana qo‘shimcha 2000 $ daromad qiladi. U qancha soliq to‘lashi kerak? Agar u yana qo‘shimcha ish olsa, bu soliq stavkasi o‘zgaradimi?

8. Shuhrat asosiy ishidan yiliga 50 000$ va qo‘shimcha ishidan yana 3000$ daromad qiladi. U qancha soliq to‘laydi? Agar u yana qo‘shimcha ish olsa, bu soliq stavkasi qanchaga oshadi?

9-16 misollarda Δy va $f'(x)\Delta x$ ni toping va 0.01, 0.0001 aniqlikda yaxlitlang.

9. $y = f(x) = x^2, x = 2, \quad \Delta x = 0.01$

10. $y = f(x) = x^3, x = 2, \quad \Delta x = 0.01$

11. $y = f(x) = x + x^2, x = 3, \quad \Delta x = 0.04$

12. $y = f(x) = x - x^2, x = 3, \quad \Delta x = 0.02$

13. $y = f(x) = 1/x^2, x = 1, \quad \Delta x = 0.5$

14. $y = f(x) = 1/x, x = 1, \quad \Delta x = 0.2$

15. $y = f(x) = 3x - 1, x = 4, \quad \Delta x = 2$

16. $y = f(x) = 2x - 3, x = 8, \quad \Delta x = 0.5$

17-46 misollarda differensial yordamida taqribiy hisoblang va nisbiy xatolikni baholang.

17. a) $\sqrt[5]{34}$; **b)** $\arcsin 0,6$;

18. a) $\sqrt[3]{26,19}$; **b)** $\mathrm{arctg}\, 0,95$;

19. **a)** $\sqrt[4]{15,8}$; **b)** $e^{0,2}$;

20. **a)** $\sqrt{8,76}$; **b)** $\lg 11$;

21. **a)** $\sqrt[5]{31}$; **b)** $\arcsin 0,54$;

22. **a)** $\sqrt[3]{70}$; **b)** $\cos 59^\circ$;

23. **a)** $(2,01)^3+(2,01)^2$; **b)** $e^{2,01}$;

24. **a)** $\sqrt[3]{65}$; **b)** $\lg \operatorname{tg} 46^\circ$;

25. **a)** $\dfrac{2,9}{\sqrt{(2,9)^2+16}}$; **b)** $\operatorname{arctg}\sqrt{1,02}$;

26. **a)** $\sqrt{\dfrac{4-3,02}{1+3,02}}$; **b)** $arctg\sqrt{0,97}$;

27. **a)** $\sqrt[4]{15,8}$; **b)** $\operatorname{arctg} 1,01$;

28. **a)** $\sqrt[3]{10}$; **b)** $\ln\left(e^2+0,2\right)$;

29. **a)** $\sqrt[5]{200}$; **b)** $\operatorname{arctg}\sqrt{0,97}$;

30. **a)** $(3,03)^5$; **b)** $\ln \operatorname{tg} 47^\circ 15'$;

31. **a)** $\sqrt{\dfrac{(2,037)^2-3}{(2,037)^2+5}}$; **b)** $\lg 9,5$;

32. **a)** $\sqrt[7]{130}$; **b)** $\operatorname{arctg}\sqrt{3,1}$;

33. **a)** $\sqrt[3]{27,5}$; **b)** $2^{2,1}$;

34. **a)** $\sqrt{17}$; **b)** $4^{1,2}$;

35. **a)** $\sqrt{640}$; **b)** $tg 59^\circ$;

36. **a)** $\sqrt{1,2}$; **b)** $\log_2 1,9$;

37. **a)** $\sqrt[10]{1025}$; **b)** $arctg\sqrt{3,2}$;

38. **a)** $(3,02)^4+(3,02)^3$; **b)** $\operatorname{ctg} 29^\circ$;

39. a) $(5,07)^3$; **b)** $\sin 93^\circ$;

40. a) $(4,01)^{1,5}$; **b)** $\lg 1,5$;

41. a) $\sqrt[3]{1,02}$; **b)** $\sin 29^\circ$;

42. a) $\cos 151^\circ$; **b)** $\lg 101$;

43. a) $\operatorname{arctg} 1,05$; **b)** $\sin 31^\circ$;

44. a) $\cos 61^\circ$; **b)** $\lg 0,9$;

45. a) $\operatorname{tg} 44^\circ$; **b)** $e^{0,25}$;

46. a) $\operatorname{arctg} 0,98$; **b)** $\sqrt{15}$;

47-58 misollar funksiyalarning birinchi tartibli differensiyalini topishga doir:

47. $y = x \operatorname{tg}^3 x$

48. $y = \sqrt{\operatorname{arctg} x} + (\arcsin x)^2$

49. $y = \dfrac{4x}{4 + x^2}$.

57. $y = \dfrac{4x^3}{x^3 - 1}$.

50. $y = \dfrac{x^2 - 1}{x^2 + 1}$.

58. $y = \dfrac{2 - 4x^2}{1 - 4x^2}$.

51. $y = \dfrac{(x+1)^2}{x^2 - 1}$.

52. $y = \dfrac{x^2}{x - 1}$.

53. $y = \dfrac{x^3}{x^2 + 1}$.

54. $y = \dfrac{4x^3 + 5}{x}$.

55. $y = \dfrac{x^2 - 5}{x - 3}$.

56. $y = \dfrac{x^4}{x^3 - 1}$.

3.7. Oshkormas funksiyaning hosilasi

3.7.1. Oshkormas funksiyaning ta'rifi

Ko'pincha biz funksiyani yozganda x erkli o'zgaruvchi bilan y erksiz o'zgaruvchini tenglikning turli tomonlarida ajratib yozamiz. Bunday $y = f(x)$ yoki $y = x^3 - 2x + 1$ ko'rinishdagi funksiyaga **oshkor funksiya** deyiladi. Shunday funksiyalar borki, ularni oshkor shaklda yozishning iloji yo'q yoki juda murakkab. Misol uchun $y^3 + y^2x^5 - x^3 = 15$ funksiyani $y = f(x)$ ko'rinishga keltirish qiyin.

$F(x, y) = 0$ ko'rinishdagi funksiyaga **oshkormas funksiya** deyiladi.

Funksiyani oshkor ko'rinishga o'tkazish uchun bu tenglamani y ga nisbatan yechish kerak.

1-misol. $x^3 + 2y^3 = 6$ funksiyani oshkor ko'rinishga o'tkazamiz. Buning uchun y ni topib olish kerak:

$$2y^3 = 6 - x^3$$

$$y = \sqrt[3]{\frac{6 - x^3}{2}}.$$

2-misol. $x^2 - 6xy + 9y^2 = 5$ funksiyani oshkor ko'rinishga keltiramiz:

$(x - 3y)^2 = 5$

$$x - 3y = \sqrt{5}$$

$$y = \frac{x - \sqrt{5}}{3}.$$

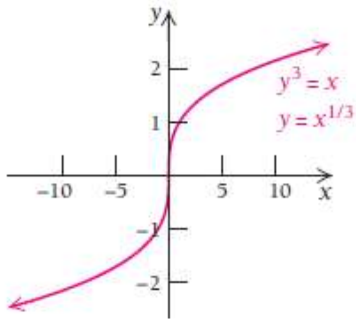
3.7.2. Oshkormas funksiyaning hosilasi

Oshkormas funksiyadan hosila olishning 2 usuli mavjud:

1) Funksiyani oshkor ko‘rinishga keltirib, so‘ngra hosila olish;

2) Oshkormas holida to‘g‘ridan – to‘g‘ri hosila olish.

I usul. $y^3 = x$ tenglamani qaraylik. Bu tenglamada y oshkormas ko‘rinishda, y ni oshkor ko‘rinishida x ning funksiyasi sifatida ifodalanadi.



Funksiyani oshkor ko‘rinishga keltiramiz: $y = \sqrt[3]{x}$ yoki $y = x^{\frac{1}{3}}$. Endi undan hosila olamiz: $y' = \left(x^{\frac{1}{3}}\right)' = \frac{1}{3}x^{-\frac{2}{3}}$.

II usul. Oshkormas funksiyadan hosila olish uchun tenglikning ikki tomoni ham differensiallanadi. Bunda y ni x ning funksiyasi deb, murakkab funksiya hosilasiga o‘xshash differensiallanadi.

$$y^3 = x$$

$$\left(y^3\right)' = x'$$

$$3y^2 \cdot y' = 1$$

$$y' = \frac{1}{3y^2} \quad \text{yoki} \quad y' = \frac{1}{3}y^{-2} = \frac{1}{3}\left(x^{\frac{1}{3}}\right)^{-2} = \frac{1}{3}x^{-\frac{2}{3}}.$$

Aytaylik, oshkormas $F(x, y) = 0$ funksiya berilgan bo‘lsin. Undan hosila olamiz:

$$F_x'(x, y) + F_y'(x, y) \cdot y_x' = 0;$$

$$F_y'(x, y) \cdot y_x' = -F_x'(x, y);$$

$$y_x' = -\frac{F_x'(x, y)}{F_y'(x, y)}$$

Ushbu formula oshkormas funksiyaning hosilasini topish formulasidir.

3-misol. $y^3 + y^2x^5 - x^3 = 15$ funksiya hosilasini toping.

Yechilishi:► Avvalo $y^3 + y^2x^5 - x^3 = 15$ tenglamani $F(x, y) = 0$ ko‘rinishga keltirib olamiz. $y^3 + y^2x^5 - x^3 - 15 = 0$. So‘ngra hosila olamiz:

$$y_x' = -\frac{F_x'(x, y)}{F_y'(x, y)} = \frac{(y^3 + y^2x^5 - x^3 - 15)_x'}{(y^3 + y^2x^5 - x^3 - 15)_y'} = \frac{5y^2x^4 - 3x^2}{3y^2 + 2yx^5}$$ ◄

Mahsulotga talab funksiyasi (1.5 bo‘lim) ko‘pincha oshkormas shaklda bo‘ladi.

4-misol. Mahsulotga bo‘lgan talab funksiyasi $x = \sqrt{200 - p^3}$ berilgan bo‘lsin. $\frac{dp}{dx}$ ni toping.

Yechilishi: ►Bizda p kirish (argument), x esa chiqish (funksiya) hisoblanadi. Shu sababli to‘g‘ridan-to‘g‘ri $\frac{dp}{dx}$ ni hisoblashning imkoni yo‘q. Dastlab $\frac{dx}{dp}$ ni hisoblaymiz va keyin uni $\frac{dp}{dx}$ ga o‘tkazamiz.

$$\frac{dx}{dp} = \left(\sqrt{200 - p^3}\right)' = \frac{-3p^2}{2\sqrt{200 - p^3}}$$

$$1=\frac{-3p^2}{2\sqrt{200-p^3}}\cdot\frac{dp}{dx}$$

$$\frac{dp}{dx}=-\frac{2\sqrt{200-p^3}}{3p^2}\quad ◀$$

5-misol. $e^y+y-x=0$ funksiyaning $M(1,0)$ nuqtadagi ikkinchi tartibli hosilasini toping.

Yechilishi: ► Berilgan oshkormas funksiyani birinchi tartibli hosilasini topamiz.

$$\left(e^y+y-x\right)'=0' \quad \text{va} \quad e^y\cdot y'+y'-1=0$$

tenglikdan y' ni topamiz: $y'=\frac{1}{e^y+1}$. y' hosilani M nuqtadagi hosilasi $y'(M)=\frac{1}{2}$ bo`ladi. Ikkinchi tartibli hosilasini topamiz:

$$y''=\left(\frac{1}{e^y+1}\right)'=-\frac{\left(e^y+1\right)'}{\left(e^y+1\right)^2}=-\frac{e^y\cdot y'}{\left(e^y+1\right)^2}$$

y'' hosilani M nuqtadagi qiymati $y''(M)=-\frac{e^0\cdot 1}{(1+1)^2}=-\frac{1}{4}$. ◀

3.7.3. Murakkab funksiyaning hosilasi

Biz 2-bobda (2.7.3 bo‘lim, 239-bet) murakkab funksiya hosilasini ozroq tushunib olgandik.

Faraz qilaylik, y chiqish x ning funksiyasi bo‘lsin: $y=f(x)$. x esa t vaqtning funksiyasi bo‘lsin. y funksiya x ga bog‘liq, x esa t ga bog‘liq, bundan y ning t ga bog‘liq ekanligi kelib chiqadi. Zanjir qoidasidan quyidagi kelib chiqadi:

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}.$$

Bundan y ning hosilasi x ning hosilasiga bog‘liq ekanini ko‘rish mumkin. Keling, bu qoida misolda qanday bo‘lishini ko‘rib chiqamiz. Bu istalgan funksiyani vaqtning funksiyasi sifatida qarash kerakligini bildiradi, agar funksiya vaqtga bogliq deb berilmasa ham yoki funksiyaning t bo‘yicha hosilasi 0 ga teng bo‘lsa ham.

5-misol. Tadbirkorlik. Daromad, xarajat, foyda hosilalari. Akmal landshaft bog‘bonligi bilan shug‘ullanadi. U x uy hovlisida landshaft yaratish uchun $T(x)=3000+20x$ \$ xarajat qilib, $D(x)=1000x-x^2$ \$ daromad topadi. Akmal 400 ta hovliga buyurtma olgan va har kuni 10 ta hovlida ishni tugatgan.

a) Shu momentdagi umumiy daromad hosilasini toping;

b) Shu momentdagi umumiy xarajat hosilasini toping;

c) Shu momentdagi umumiy foyda hosilasini toping.

Yechilishi: ► a) $D(x)=1000x-x^2$ tenglikda har ikki tomonni vaqtga nisbatan differensiallaymiz: $\frac{dD}{dt}=\left(1000x-x^2\right)'=1000\cdot\frac{dx}{dt}-2x\cdot\frac{dx}{dt}$

$\frac{dx}{dt}=10$, $x=400$ o‘rniga qo‘yamiz. Shunda kuniga

$$\frac{dD}{dt}=1000\cdot\frac{dx}{dt}-2x\cdot\frac{dx}{dt}=1000\cdot 10-2\cdot 400\cdot 10=2000\ \$$$

daromad qilishini aniqlaymiz.

b) $\frac{dT}{dt}=20\cdot\frac{dx}{dt}=20\cdot 10=200\,\$$ xarajat qiladi.

c) Foydani topish uchun daromaddan xarajatni ayiramiz: $F=D-T$.

Bundan $\frac{dF}{dt}=\frac{dD}{dt}-\frac{dT}{dt}=2000\$-200\$=1800\$$ foyda olishini topamiz.◄

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1-35 misollarda oshkormas funksiya hosilasini toping:

1. $x^3+2y^3=6;\quad (2,-1)$
2. $3x^3-y^2=8;\quad (2,4)$
3. $2x^2-3y^3=5;\quad (-2,1)$
4. $2x^3+4y^2=-12;\quad (-2,-1)$
5. $x^2-y^2=1;\quad (\sqrt{3},\sqrt{2})$
6. $x^2+y^2=1;\quad \left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
7. $3x^2y^4=12;\quad (2,-1)$
8. $2x^3y^2=-18;\quad (-1,3)$
9. $x^3-x^2y^2=-9;\quad (3,-2)$
10. $x^4-x^2y^3=12;\quad (-2,1)$
11. $xy-x+2y=3;\quad \left(-5,\frac{2}{3}\right)$
12. $xy+y^2-2x=0;\quad (1,-2)$
13. $x^2y-2x^3-y^3+1=0;\quad (2,-3)$
14. $4x^3-y^4-3y+5x+1=0;\quad (1,-2)$
15. $2xy+3=0$
16. $x^2+2xy=3y^2$
17. $x^2-y^2=16$
18. $x^2+y^2=25$
19. $y^5=x^3$
20. $y^3=x^5$
21. $x^2y^3+x^3y^4=11$
22. $x^3y^2+x^5y^3=-19$
23. $p^3+p-3x=50$
24. $p^2+p+2x=40$
25. $xp^3=24$
26. $x^3p^2=108$
27. $\frac{xp}{x+p}=2$
28. $\frac{x^2p+xp+1}{2x+p}=1$
29. $(p+4)(x+3)=48$
30. $1000-300p+25p^2=x$

34-57 misollarda oshkormas funksiyaning 2-tartibli hosilasini toping:

34. $e^{xy} - x^3 - y^3 = 0$

35. $xy - \text{arctg}\frac{x}{y} = 3$

36. $\text{tg}\, y = 4y - 5x$;

37. $y = 7x - \text{ctg}\, y$;

38. $xy - 6 = \cos y$;

39. $3y = 7 + xy^3$;

40. $y^2 = x + \ln\frac{y}{x}$;

41. $xy^2 - y^3 = 4x - 5$;

42. $y^2 = 8x$;

43. $\frac{x^2}{5} + \frac{y^2}{7} = 1$;

44. $y = x + \text{arctg}\, y$;

45. $\frac{x^2}{5} + \frac{y^2}{3} = 1$;

46. $y^2 = 25x - 4$;

47. $\text{arctg}\, y = 4x + 5y$;

48. $y^2 - x = \cos y$;

49. $3x + \sin y = 5y$;

50. $\text{tg}\, y = 3x + 5y$;

51. $xy = \text{ctg}\, y$;

52. $y = e^y + 4x$;

53. $\ln y - \frac{y}{x} = 7$;

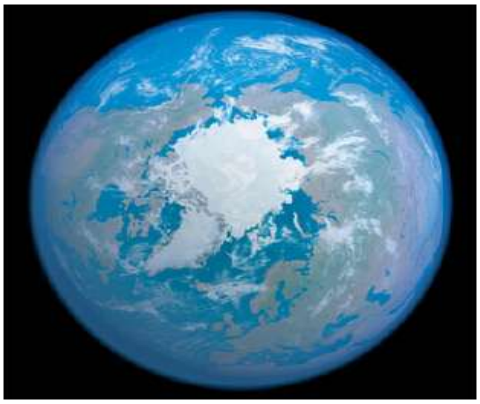
54. $y^2 + x^2 = \sin y$;

55. $e^y = 4x - 7y$;

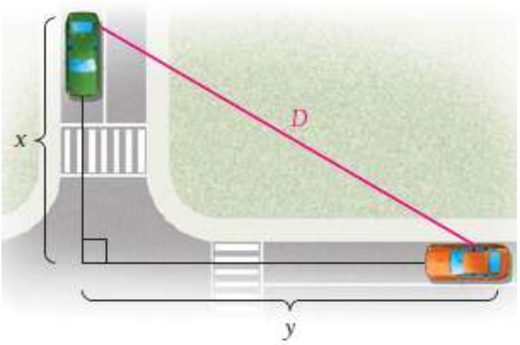
56. $4\sin^2(x + y) = x$;

57. $\sin y = 4x - 7y$;

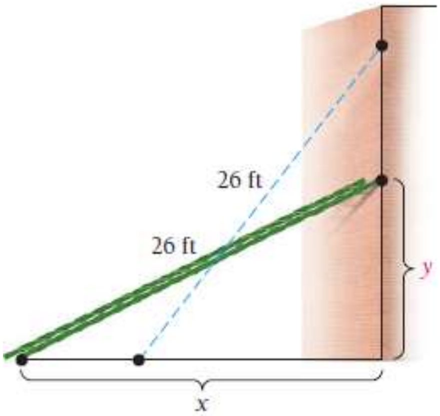
64. Arktika muzliklari. 1950 yildan olimlar shimoliy muz okeani muzliklarining ajralish jarayonini kuzatib, bu jarayon yer shari iqlimining qizib borishiga sabab deb ko‘rsatishadi. Muz maydoni yozda kichrayib, qishda yana kengayadi. Agar muz maydonining o‘rtacha o‘lchamini (kv. mill hisobida) $S = \pi R^2$ deb hisoblash mumkin bo‘lsa, 2015 yilda taxminan 4.3 mill/yil tezlikda maydon radiusi 808 millgacha kishraygan, bu vaqtda maydon qanchaga kichraygan?



65. Ikki avtomobil bir nuqtadan bir xil vaqtda yo‘lga chiqdi. Ulardan biri shimolga qarab 25 mil/soatda, ikkinchisi esa sharqqa qarab 60 mill/soat tezlik bilan harakatlanmoqda. 1 soatdan keyin ular orasidagi masofa qanday bo‘ladi? (Yo‘llanma: $D^2 = x^2 + y^2$ deb hisoblang.)



66. 26 fut uzunlikdagi zina devorga mahkamlangan. Agar zinaning asosi 10 fut devor tomonga surilsa, u qancha balandlikka siljiydi?



3.8. Lopital qoidasi

Agar limitlarni hisoblashda $\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ 0^0,\ 1^\infty,\ \infty^0$ ko‘rinishdagi natijalar hosil bo‘lsa, ularga **aniqmasliklar** deyiladi.

1-teorema. Agar $f(x)$ va $\varphi(x)$ funksiyalar $x=a$ nuqtaning biror atrofida uzluksiz, a nuqtaning o‘zidan tashqari shu atrofda differensiallanuvchi bo‘lib, $f(a)=0$, $\varphi(a)=0$ va $\varphi'(x)\neq 0$ hamda $\lim\limits_{x\to a}\frac{f'(x)}{\varphi'(x)}=A$ limit (chekli yoki cheksiz) mavjud bo‘lsa, u holda $\lim\limits_{x\to a}\frac{f(x)}{\varphi(x)}$ limit mavjud va ushbu $\lim\limits_{x\to a}\frac{f(x)}{\varphi(x)}=\lim\limits_{x\to a}\frac{f'(x)}{\varphi'(x)}=A$ tenglik o‘rinli bo‘ladi.

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\left(\frac{0}{0}\right)=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}=\left(\frac{0}{0}\right)=\lim_{x\to a}\frac{f''(x)}{\varphi''(x)}=\left(\frac{0}{0}\right)=\ldots=\lim_{x\to a}\frac{f^{(n)}(x)}{\varphi^{(n)}(x)}=A$$

1-misol. $\lim\limits_{x\to 0}\frac{sin4x}{3x}=\left(\frac{0}{0}\right)=\lim\limits_{x\to 0}\frac{(sin4x)'}{(3x)'}=\lim\limits_{x\to 0}\frac{4cos4x}{3}=\frac{4}{3}.$

2-teorema. Agar $f(x)$ va $\varphi(x)$ funksiyalar $x=a$ nuqtaning biror atrofida uzluksiz, shu oraliqda ($x=a$ nuqtaning o‘zidan tashqari) differensiallanuvchi bo‘lsa, hamda

$\lim\limits_{x\to a}f(x)=\infty$, $\lim\limits_{x\to a}\varphi(x)=\infty$, $\varphi'(x)\neq 0$ bo‘lsa va $\lim\limits_{x\to a}\frac{f'(x)}{\varphi'(x)}=A$ limit (chekli yoki cheksiz) mavjud bo‘lsa, u holda

$\lim\limits_{x\to a}\frac{f(x)}{\varphi(x)}$ limit mavjud va ushbu $\lim\limits_{x\to a}\frac{f(x)}{\varphi(x)}=\lim\limits_{x\to a}\frac{f'(x)}{\varphi'(x)}=A$

tenglik o‘rinli bo‘ladi.

2-misol. $\lim\limits_{x\to 0}\frac{lnx}{ctgx}=\left(\frac{\infty}{\infty}\right)=\lim\limits_{x\to 0}\frac{(lnx)'}{(ctgx)'}=\lim\limits_{x\to 0}\frac{-sin^2x}{x}=\left(\frac{0}{0}\right)=$

$\lim\limits_{x\to 0}\frac{-(sin^2x)'}{(x)'}=-\lim\limits_{x\to 0}2sinxcosx=0.$

3.8.1. Aniqmasliklarni ochish

1). $\mathbf{0\cdot\infty}$ ko'rinishdagi aniqmasliklar. Bunday aniqmaslikni ochish deganda $\lim\limits_{x\to a}f(x)=0,\quad \lim\limits_{x\to a}\varphi(x)=\infty$ bo'lganda $\lim\limits_{x\to a}f(x)\cdot\varphi(x)$ limitni topish tushuniladi.

Agar izlanayotgan ifoda

$$\lim_{x\to a}f(x)\cdot\varphi(x)=\lim_{x\to a}\frac{f(x)}{\frac{1}{\varphi(x)}}\quad \text{yoki}\quad \lim_{x\to a}f(x)\cdot\varphi(x)=\frac{\varphi(x)}{\frac{1}{f(x)}}$$

ko'rinishda yozilsa, u holda $x\to a$ da $0\cdot\infty$ ko'rinishdagi aniqmaslikka egamiz.

3-misol. $\lim\limits_{x\to 0}x^2\cdot lnx$ ni toping.

Yechilishi: ► Berilgan ifodani shakl almashtiramiz va yuqoridagiga ko'ra topamiz.

$$\lim_{x\to 0}x^2\cdot lnx=\lim_{x\to 0}\frac{lnx}{\frac{1}{x^2}}=\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{2}{x^3}}=-\lim_{x\to 0}\frac{x^2}{2}=0.◄$$

2). $\infty-\infty$ ko'rinishdagi aniqmasliklar. Bunday aniqmaslikni ochish deganda $\lim\limits_{x\to a}f(x)=\infty,\quad \lim\limits_{x\to a}\varphi(x)=\ \infty$ bir xil ishorali cheksizlik

bo'lganda $\lim\limits_{x\to a}\big(f(x)-\varphi(x)\big)$ limitni topish tushuniladi.

Bunday aniqmasliklar $\left(\frac{0}{0}\right)$ ko'rinishdagi aniqmaslikka keltiriladi.

4-misol. $\lim\limits_{x\to\frac{\pi}{2}-0}(secx-tgx)$ ni toping.

Yechilishi: ► $\lim\limits_{x\to\frac{\pi}{2}-0} secx=+\infty.$ $\operatorname{Lim}\limits_{x\to\frac{\pi}{2}-0} tgx=+\infty$ bo'lgani uchun $\infty-\infty$ ko'rinishdagi aniqmaslikka ega bo'lamiz. Eng sodda almashtirishlar $\left(\frac{0}{0}\right)$ ko'rinishga olib keladi:

$$\lim_{x\to\frac{\pi}{2}-0}(secx-tgx)=\lim_{x\to\frac{\pi}{2}-0}\frac{1-sinx}{\cos x}=\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{2}-0}\frac{-cosx}{-\sin x}=0.$$ ◄

3). 1^{∞}, 0^{0}, ∞^{0} ko'rinishdagi aniqmasliklar.

$\lim\limits_{x\to a}(f(x))^{\varphi(x)}$ limitni topish deganda

a) agar $f(x)\to 1,\ \varphi(x)\to\infty$ bo'lsa, 1^{∞} ko'rinishdagi aniqmaslikni ochishni;

b) agar $f(x)\to\infty,\ \varphi(x)\to 0$ bo'lsa, ∞^{0} ko'rinishdagi aniqmaslikni ochishni;

c) agar $f(x)\to 0,\ \varphi(x)\to 0$ bo'lsa, 0^{0} ko'rinishdagi aniqmaslikni ochishni tushinamiz.

Hamma hollarda ham funksiyani oldin logarifmlaymiz, bunda $0\cdot\infty$ ko'rinishdagi aniqmaslikka ega bo'lamiz, buni esa o'z navbatida $\frac{0}{0}$ yoki $\frac{\infty}{\infty}$ ko'rinishdagi aniqmaslikka keltiramiz. Shundan keyin logarifmning limiti bo'yicha berilgan funksiya limitini topamiz. Natijani potensirlaymiz.

5-misol. $\lim\limits_{x\to 0}(cosx)^{\frac{1}{x^2}}$ ni toping.

Yechilishi. ► $\lim\limits_{x\to 0} cosx=1,$ $\lim\limits_{x\to 0}\frac{1}{x^2}=\infty$ bo'lgani uchun 1^{∞} ko'rinishga egamiz.

$$A = \lim_{x\to 0}(cosx)^{\frac{1}{x^2}}$$

deb belgilaymiz. Bu ifodani e asos bo'yicha logarifmlaymiz:

$$lnA = \lim_{x\to 0}(cosx)^{\frac{1}{x^2}} = \lim_{x\to 0}\frac{\ln(cosx)}{x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{(\ln cosx)'}{(x^2)'}$$

$$= \lim_{x\to 0}\frac{\frac{1}{cosx}(-sinx)}{2x} = -\frac{1}{2}\cdot\lim_{x\to 0}\frac{sinx}{xcosx} = -\frac{1}{2}.$$

Shunday qilib $lnA = -\frac{1}{2}$, buni potensirlab,

$A = e^{-\frac{1}{2}}$ ni hosil qilamiz. ◄

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1-10 misollarda berilgan qiymatlarda Lopital qoidasidan foydalanib, limitlarni hisoblang:

1. $\lim_{x\to x_0}\frac{2x^2+x-1}{x^2-3x-4}$; a) $x_0=2$; b) $x_0=-1$; v) $x_0=\infty$

2. $\lim_{x\to x_0}\frac{x^2-3x+2}{-3x^2-x+4}$; a) $x_0=-1$; b) $x_0=1$; v) $x_0=\infty$

3. $\lim_{x\to x_0}\frac{2x^2-x-10}{x^2+3x+2}$; a) $x_0=2$; b) $x_0=-2$; v) $x_0=\infty$

4. $\lim_{x\to x_0}\frac{x^2-3x+2}{-3x^2-x+14}$; a) $x_0=1$; b) $x_0=2$; v) $x_0=\infty$

5. $\lim_{x\to x_0}\frac{x^2+5x+4}{2x^2-3x-5}$; a) $x_0=-2$; b) $x_0=-1$; v) $x_0=\infty$

6. $\lim_{x\to x_0}\frac{4x^2-5x+1}{-x^2+3x-2}$; a) $x_0=-1$; b) $x_0=1$; v) $x_0=\infty$

7. $\lim\limits_{x\to x_0}\dfrac{x^2+5x+6}{3x^2-x-14}$; a) $x_0=2$; b) $x_0=-2$; v) $x_0=\infty$

8. $\lim\limits_{x\to x_0}\dfrac{2x^2-7x+6}{-x^2-x+6}$; a) $x_0=1$; b) $x_0=2$; v) $x_0=\infty$

9. $\lim\limits_{x\to x_0}\dfrac{x^2-6x-7}{3x^2+x-2}$; a) $x_0=-2$; b) $x_0=-1$; v) $x_0=\infty$

10. $\lim\limits_{x\to x_0}\dfrac{3x^2+x-4}{-x^2+4x-3}$; a) $x_0=-1$; b) $x_0=1$; v) $x_0=\infty$

11-24 misollarda qulay usuldan foydalanib, limitlarni hisoblang:

11. $\lim\limits_{x\to 0}\dfrac{tg\,2x}{\sin 3x}$;

12. $\lim\limits_{x\to 0}\dfrac{\sin 4x}{2x\cos 3x}$.

13. $\lim\limits_{x\to 0}\dfrac{x\,tg\,3x}{\sin^2 2x}$;

14. $\lim\limits_{x\to 0}\dfrac{\sin 5x\cdot tg\,3x}{x^2}$.

15. $\lim\limits_{x\to 0}\dfrac{\sin 6x}{tg\,2x}$;

16. $\lim\limits_{x\to 0}\dfrac{3x\cos 5x}{\sin 3x}$.

17. $\lim\limits_{x\to 0}\dfrac{2x\cdot tg\,4x}{\sin^2 6x}$.

18. $\lim\limits_{x\to 0}\dfrac{\sin 2x\cdot tg\,4x}{x^2}$.

19. $\lim\limits_{x\to 0}\dfrac{\sin 8x}{tg\,5x}$.

20. $\lim\limits_{x\to 0}\dfrac{4x\cos 7x}{\sin 2x}$.

21. $\lim\limits_{x\to 0}\dfrac{3x\cdot tg\,2x}{\sin^2 3x}$.

22. $\lim\limits_{x\to 0}\dfrac{\sin 3x\cdot tg\,2x}{x^2}$.

23. $\lim\limits_{x\to 0}\dfrac{\sin 6x}{tg\,x}$.

24. $\lim\limits_{x\to 0}\dfrac{\sin 5x\cdot tg\,3x}{6x}$

IV BOB.

Ko‘rsatkichli va logarifmik funksiyalarning hosilalari

4.1. Ko‘rsatkichli funksiya va uning hosilasi

4.2. Logarifmik funksiya va uning hosilasi

4.3. Erkin va chegaralangan o‘sish modellari

4.4. a^x **va** $\log_a x$ **funksiyalarning hosilalari**

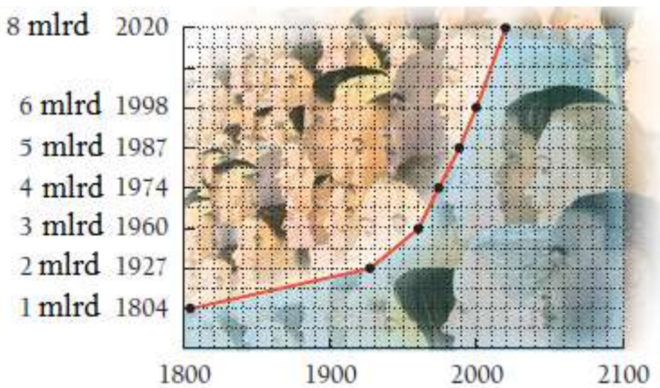
Ushbu bobda bir-biri bilan uzviy bog‘liq bo‘lgan ikki xil funksiyani o‘rganamiz, ular ko‘rsatkichli va logarifmik funksiyalardir. Shuningdek, funksiyalar hosilasini topish, ularning aholi sonining o‘sishi, talab funksiyasining uzluksizligi, kasalliklarning tarqalishi singari masalalarda amaliy tatbiqlarini ko‘rib chiqamiz.

Ko‘rsatkichli va logarifmik funksiyalar o‘zaro teskari funksiyalar hisoblanadi. Ulardan birining grafigini ikkinchisining grafigi yordamida chizish mumkin. Ko‘rsatkichli funksiyalar orasida eksponensial funksiya muhim ahamiyatga ega. Shuningdek, ushbu bo‘limda eksponensial o‘suvchi hodisa va jarayonlarni ham o‘rganamiz.

4.1. Ko‘rsatkichli funksiya va uning hosilasi

4.1.1. Ko‘rsatkichli funksiyaning grafigi

Yer shari aholisining o‘sish grafigini qaraylik. Nuqtalarni birlashtirishdan hosil bo‘lgan siniq chiziqlar ko‘rsatkichli funksiyaning taqribiy olingan qiymatlaridir.



Keling, a^x ning aniq qiymatini qaraymiz, ya’ni x ning o‘rniga qiymatlar berib ko‘ramiz, bu yerda $a = const$.

$$a^{1.72} \quad \text{yoki} \quad a^{\frac{172}{100}} = \sqrt[100]{a^{172}}$$

demak, “oldin 172- darajaga oshiramiz, so‘ng 100- ildizdan chiqaramiz”.

Irratsional ko‘rsatkichli ifodalar haqida nima deyish mumkin, masalan $2^{\sqrt{3}}$, 2^{π}, $2^{-\sqrt{2}}$ sonlarni qanday aniqlaymiz.

Irratsional son – davriy bo‘lmagan, cheksiz o‘nli kasr sondir.

2^{π} ni olaylik. Bilamizki, π sonining kengaytmasi davriy bo‘lmagan, cheksiz o‘nli kasr sondan iborat: 3.141592653… . Bu sonni ratsional songa yaxlitlab, yaqinlashtirish mumkin:

$$3, \quad 3.1, \quad 3.14, \quad 3.142, \quad 3.1416, \quad \ldots$$

Shunga ko‘ra, daraja ko‘rsatkichini ham ratsional songa yaqinlashtirish (approksimatsiyalash) mumkin:

$$2^{3},\ 2^{3.1},\ 2^{3.14},\ 2^{3.142},\ 2^{3.1416},\dots$$

Ushbu sonlarni kalkulyatorda darajaga oshirib, quyidagilarni hosil qilamiz:

$$8,\ \ 8.574188,\ \ 8.815241,\ \ 8.821353,\ \ 8.824411,\ \dots$$

Umuman olganda, a^x bu r ratsional son x ga intilgandagi a^r ning limitiga teng bo‘lgan son, ya’ni

$$\lim_{r\to x} a^r = a^x .$$

Shunga ko‘ra, $a>0$ bo‘lganda haqiqiy daraja ko‘rsatkichli sonlarning xossalari o‘rinli bo‘ladi:

$$a^x\cdot a^y = a^{x+y},\qquad \frac{a^x}{a^y}=a^{x-y},\qquad (a^x)^y = a^{xy},\qquad a^{-x}=\frac{1}{a^x}.$$

Bundan tashqari, $y=a^x$ uzluksiz funksiya.

1-vazifa. $7^{\sqrt{3}}$, 7^{π}, $4^{-\sqrt{3}}$, $16^{-\pi}$ kalkulyatorda hisoblang va yaxlitlang.

Ta’rif. $f(x)=a^x$ funksiyaga **ko‘rsatkichli funksiya** deyiladi, bunda x ixtiyoriy haqiqiy son, $a>0$ va $a\neq 1$ bo‘lishi kerak. a soniga **asos** deyiladi.

$f(x)=2^x$, $f(x)=\left(\frac{1}{2}\right)^x$, $f(x)=(0.8)^x$ funksiyalar ko‘rsatkichli funksiyalarga misol bo‘ladi.

Yodda saqlang! Darajali $y=x^2$ yoki $y=x^3$ funksiyalarda noma’lum son asosda qatnashadi, ko‘rsatkichli funksiyada esa $y=a^x$ asos o‘zgarmaydi, uning daraja ko‘rsatkichida noma’lum qatnashadi.

Ko‘rsatkichli funksiyalarning tatbiq qilinadigan sohalari judayam ko‘p. Keling, ko‘rsatkichli funksiyalar grafiklarini o‘rganishdan boshlaymiz.

1-misol. $f(x) = 2^x$ funksiya grafigini chizing.

Yechilishi: ► Dastlab, funksiya qiymatlarini hisoblab olamiz va jadval tuzamiz. 2^x ning qiymati har doim musbat sondan iborat.

$x = 0$ da $y = 2^0 = 1$;

$x = 0.5$ da $y = 2^{0.5} = \sqrt{2} \approx 1.4$;

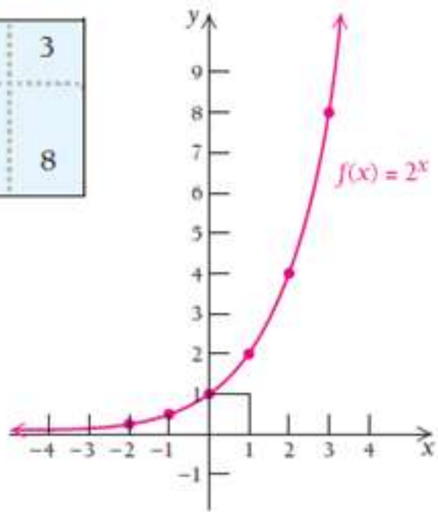
$x = 1$ da $y = 2^1 = 2$;

$x = 2$ da $y = 2^2 = 4$;

$x = 3$ da $y = 2^3 = 8$.

Topilgan nuqtalarni koordinata sistemasiga belgilab, ularni silliq chiziq bilan tutashtirib chiqamiz.

x	-2	-1	0	$\frac{1}{2}$	1	2	3
$y = f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	1.4	2	4	8



Grafik uzluksiz, o‘suvchi, biroz botiqroq. Shuningdek, x – o‘qi gorizontal asimptota ekanini ko‘rishimiz mumkin(3.3 bo‘lim), bundan

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} a^x = 0 \quad \text{va} \quad \lim_{x \to \infty} f(x) = \lim_{x \to \infty} a^x = \infty . ◄$$

2-misol. $g(x)=\left(\frac{1}{2}\right)^x$ funksiya grafigini chizing.

Yechilishi: ►Dastlab, funksiya ko‘rinishini ixchamlashtirib olamiz:

$$g(x)=\left(\frac{1}{2}\right)^x=(2^{-1})^x=2^{-x}.$$

So‘ngra funksiya qiymatlarini hisoblaymiz va jadval tuzamiz:

$x=-3$ da $y=2^{-(-3)}=2^3=8$;

$x=-2$ da $y=2^{-(-2)}=2^2=4$;

$x=-1$ da $y=2^{-(-1)}=2$;

$x=0$ da $y=2^0=1$;

$x=0.5$ da $y=2^{-0.5}=\sqrt{2^{-1}}=\frac{1}{\sqrt{2}}\approx\frac{1}{1.4}\approx 0.7$;

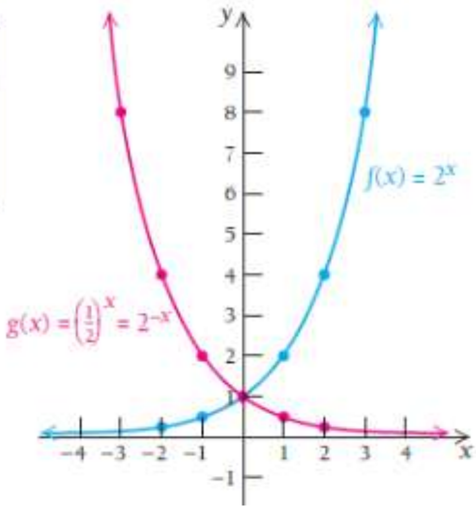
$x=1$ da $y=2^{-1}=\frac{1}{2}=0.5$;

$x=2$ da $y=2^{-2}=\frac{1}{4}=0.25$;

$x=3$ da $y=2^{-3}=\frac{1}{8}=0.125$.

Topilgan nuqtalarni koordinata sistemasiga belgilab, ularni silliq chiziq bilan tutashtirib chiqamiz.

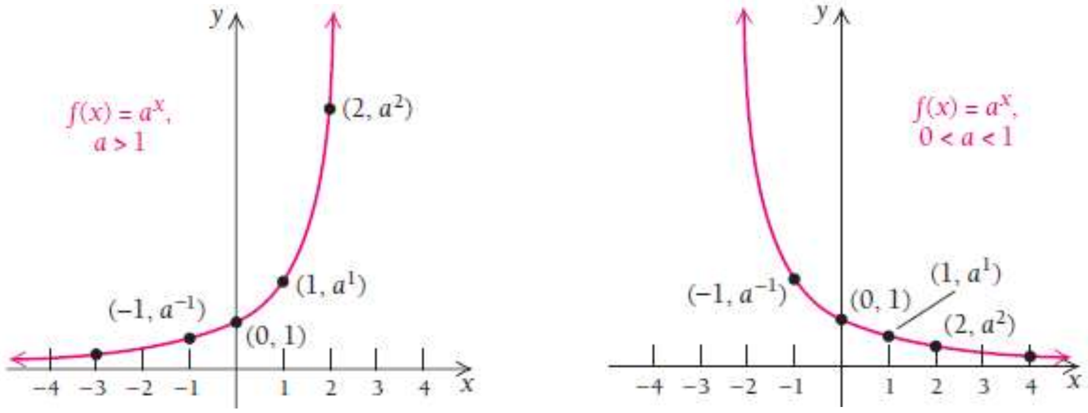
x	-3	-2	-1	0	$\frac{1}{2}$	1	2
$y=f(x)=\left(\frac{1}{2}\right)^x$	8	4	2	1	0.7	$\frac{1}{2}$	$\frac{1}{4}$



Grafik uzluksiz, kamayuvchi va biroz botiqroq. Shuningdek, $x-$ o‘qi gorizontal asimptota ekanini ko‘rishimiz mumkin (3.3 bo‘lim), bundan $\lim\limits_{x\to-\infty} g(x)=\infty$ va $\lim\limits_{x\to\infty} g(x)=0$ ga teng. 1-misoldagi $f(x)=2^x$ bilan solishtirib ko‘ramiz. Bu ikki grafik $y-$ o‘qiga nisbatan simmetrik. ◄

1- va 2- misollardan xulosa chiqaramiz.

$f(x)=a^x$ funksiya $a>1$ bo‘lganda o‘suvchi funksiya, $0<a<1$ bo‘lganda kamayuvchi funksiya bo‘ladi. Har doim (0, 1) nuqtadan o‘tadi.



$a=1$ bo‘lganda $f(x)=a^x=1^x=1$ o‘zgarmas funksiyaga aylanib qoladi, shuninh uchun ko‘rsatkichli funksiyaning asosi 1 ga teng bo‘lishi mumkin emas.

4.1.2. e soni va e^x ko‘rsatkichli funksiyaning hosilasi

Endi $f(x)=a^x$ ko‘rsatkichli funksiya hosilasini topishga harakat qilamiz. Hosila ta’rifiga ko‘ra,

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x}=\lim_{\Delta x\to 0}\frac{a^x\cdot a^{\Delta x}-a^x}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{a^x(a^{\Delta x}-1)}{\Delta x}=a^x\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}.$$

Misol uchun, $f(x)=2^x$ ning hosilasi $f'(x)=2^x\lim\limits_{\Delta x\to 0}\frac{2^{\Delta x}-1}{\Delta x}$ ekan. E'tibor bering, limit hosila olayotgan o'zgaruvchiga bog'liq emas, u qandaydir aniq chekli son. Hosila mavjud bo'lishi uchun $\lim\limits_{\Delta x\to 0}\frac{2^{\Delta x}-1}{\Delta x}$ limitni hisoblash kerak. 0 ning atrofida $\lim\limits_{\Delta x\to 0}\frac{2^{\Delta x}-1}{\Delta x}$ qabul qiladigan qiymatlarni jadvalga yozamiz.

h	$\frac{2^h-1}{h}$
0.5	0.8284
0.25	0.7568
0.175	0.7369
0.0625	0.7084
0.03125	0.7007
0.00111	0.6934
0.000001	0.6931

Jadvaldan ko'rinadiki, $f'(x)=2^x\lim\limits_{\Delta x\to 0}\frac{2^{\Delta x}-1}{\Delta x}\approx 2^x\cdot 0.7$, ya'ni $\left(2^x\right)'=(0.7)\cdot 2^x$;

Xuddi shuningdek, $g(x)=3^x$ ni hosilasini topish uchun $g'(x)=3^x\lim\limits_{\Delta x\to 0}\frac{3^{\Delta x}-1}{\Delta x}$ dan foydalanamiz, uning uchun ham limitni hisoblash jadvalini tuzamiz:

h	$\frac{3^h-1}{h}$
0.5	1.4641
0.25	1.2643
0.175	1.2113
0.0625	1.1372
0.03125	1.1177
0.00111	1.0993
0.000001	1.0986

Jadvaldan $g'(x)=3^x\lim\limits_{\Delta x\to 0}\frac{3^{\Delta x}-1}{\Delta x}\approx 3^x\cdot(1.1)$, ya'ni $\left(3^x\right)'=(1.1)\cdot 3^x$ ekanini ko'rish mumkin.

Shularga asosan $f(x)=a^x$ funksiya hosilasi uchun umumiy xulosa chiqaramiz.

Agar $f(x)=a^x$ bo‘lsa, uning hosilasi $f'(x)=a^x\lim\limits_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}$ ga teng bo‘ladi.

$\lim\limits_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}$ limit nimaga teng.

1) $a=2$ da $\lim\limits_{\Delta x\to 0}\frac{2^{\Delta x}-1}{\Delta x}\approx 0.6931$;

2) $a=3$ da $\lim\limits_{\Delta x\to 0}\frac{3^{\Delta x}-1}{\Delta x}\approx 1.0986$, ya’ni asosi 2 va 3 bo‘lgan ko‘rsatkichli funksiyalar limiti orasidagi son 1 ga yaqin. Shuning uchun

$\lim\limits_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}=1$ deb olamiz. Bundan

$$\frac{a^{\Delta x}-1}{\Delta x}=1$$

$$a^{\Delta x}=\Delta x+1$$

$$a=(\Delta x+1)^{\frac{1}{\Delta x}}.$$

U holda $\lim\limits_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}=\lim\limits_{\Delta x\to 0}1$ va $\lim\limits_{\Delta x\to 0}a=\lim\limits_{\Delta x\to 0}(\Delta x+1)^{\frac{1}{\Delta x}}$ ekvivalent bo‘ladi.

Δx	$(\Delta x+1)^{\frac{1}{\Delta x}}$
0.5	2.25
0.1	2.5937
0.01	2.7048
0.001	2.7169
-0.01	2.732
-0.001	2.7196
0.0001	2.7181

Shunda $\lim\limits_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}=1$ va $a=\lim\limits_{\Delta x\to 0}(\Delta x+1)^{\frac{1}{\Delta x}}$ lar ham ekvivalent.

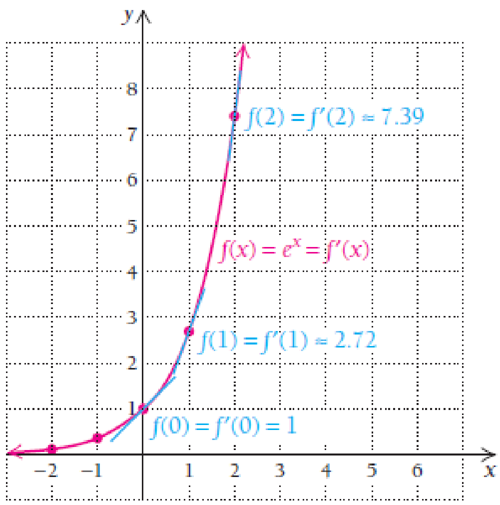
$a=\lim_{\Delta x\to 0}(\Delta x+1)^{\frac{1}{\Delta x}}$ tenglik biz izlayotgan maxsus songa teng, u "*e*" bilan belgilanadi. Bu belgilash shu sohada yangilik kiritgan Shvetsariyalik olim Leonard Eyler (1707-1783) sharafiga qo'yilgan.

Ta'rif. $$e=\lim_{\Delta x\to 0}(1+\Delta x)^{\frac{1}{\Delta x}}\approx 2.718281828459$$

tenglikni qanoatlantiruvchi *e* soniga **natural asos** deyiladi.

Bundan kelib chiqadiki, $f'(x)=e^x\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}=e^x\cdot 1=e^x$, ya'ni e^x ning hosilasi e^x ga teng.

1-teorema. $f(x)=e^x$ funksiyaning hosilasi shu funksiyaning o'ziga teng: $(e^x)'=e^x$ yoki $\frac{d}{dx}e^x=e^x$.



1-teoremadan $f(x)=e^x$ funksiya grafigiga

(0, 1) nuqtada o'tkazilgan urinmaning burchak koeffitsiyenti $m=1$ ga;

(1, *e*) nuqtada o'tkazilgan urinmaning burchak koeffitsiyenti $m=e$ ga;

(1, e^2) nuqtada o'tkazilgan urinmaning burchak koeffitsiyenti $m=e^2$ ga teng bo'lishi kelib chiqadi.

Shunday xossaga ega bo'lgan yagona ko'rsatkichli funksiya $f(x)=e^x$ dir.

4.1.3. Ikkinchi ajoyib limit

$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$ ifodaga **2-ajoyib limit** deyiladi.

2-ajoyib limitdan kelib chiqadigan natijalar:

1) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e\approx 2,71828$

2) $\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$

3) $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$

4) $\lim_{x\to 0}\frac{(1+x)^m-1}{x}=m$

3-misol. $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$ limitni hisoblang.

Yechilishi: ▶

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n=\left(\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right)^2=e^2.$$ ◀

4-misol. $\lim_{x\to\infty}\left(\frac{x^2+5x+4}{x^2-3x+7}\right)^x$ limitni hisoblang.

Yechilishi: ▶ $\lim_{x\to\infty}\left(\frac{x^2+5x+4}{x^2-3x+7}\right)^x$ berilgan limitni shakl almashtirishlar bajarib, 2-ajoyib limitga keltiramiz.

$$\lim_{x\to\infty}\left(\frac{x^2+5x+4}{x^2-3x+7}\right)^x=\lim_{x\to\infty}\left(1+\frac{8x-3}{x^2-3x+7}\right)^x=\lim_{x\to\infty}\left[\left(1+\frac{8x-3}{x^2-3x+7}\right)^{\frac{x^2-3x+7}{8x-3}}\right]^{\frac{x(8x-3)}{x^2-3x+7}}=$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{8x-3}{x^2-3x+7}\right)^{\frac{x^2-3x+7}{8x-3}}\right]^{\frac{8-3/x}{1-3/x+7/x^2}}=e^8$$

◄

4.1.4. e^x qatnashgan funksiyalar hosilasini topish

Biz ushbu mavzuda o‘tgan mavzulardagi teoremalardan hamda 1-teoremadan foydalanamiz.

3-misol. a) $y=5e^x$; b) $y=x^3e^x$; c) $y=\dfrac{e^x}{x^{-2}}$

funksiyalarning hosilalarini toping.

Yechilishi: ► a) $(5e^x)'=5(e^x)'=5e^x$ funksiya bilan o‘zgarmas son ko‘paytmasining hosilasi;

b) $(x^3e^x)'=3x^2e^x+x^3e^x=x^2e^x(3+x)$ ko‘paytmaning hosilasi;

c) $\left(\dfrac{e^x}{x^{-2}}\right)'=\dfrac{e^xx^{-2}-e^x\cdot(-2)x^{-3}}{x^{-4}}=\dfrac{x^{-2}e^x(1+2x^{-1})}{x^{-4}}=\dfrac{e^x)(1+2x^{-1})}{x^{-2}}$ bo‘linmaning hosilasi. ◄

4-misol. $y=e^{x^2+3x-1}$ funksiya hosilasini toping.

Yechilishi: ► Bu murakkab funksiya bo‘lib, $f(x)=g(h(x))=e^{h(x)}$, bunda $g(x)=e^{h(x)}$ va $h(x)=x^2+3x-1$. U holda 2.7-bo‘limdagi zanjir formulasiga ko‘ra $f'(x)=g'(h(x))\cdot h'(x)=e^{h(x)}\cdot h'(x)$ o‘rinli.

$$y'=\left(e^{x^2+3x-1}\right)'=e^{x^2+3x-1}\cdot(x^2+3x-1)'=e^{x^2+3x-1}(2x+3)$$ ◄

2-vazifa. a) $y = 3e^x$; b) $y = x^2 e^x$; c) $y = \dfrac{e^x}{x^3}$

funksiyalarning hosilasini toping.

2-teorema. Biror daraja ko'rsatkichli $e^{u(x)}$ funksiyaning hosilasi shu funksiya bilan daraja ko'rsatkichidan olingan hosilaning ko'paytmasiga teng:
$$\left(e^{u(x)}\right)' = e^{u(x)} \cdot u'(x)$$

5-misol. $y = e^{x^3 - 5x}$ funksiya hosilasini toping.

Yechilishi: ► $y' = \left(e^{x^3-5x}\right)' = e^{x^3-5x} \cdot (x^3 - 5x)' = e^{x^3-5x}(3x^2 - 5)$ ◄

6-misol. a) $y = e^{12x}$; b) $y = e^{-x^2+7x-3}$; c) $y = e^{\sqrt{x^2-8}}$

funksiyalarning hosilalarini toping.

Yechilishi: ► a) $y' = \left(e^{12x}\right)' = 12e^{12x}$; b) $y' = \left(e^{-x^2+7x-3}\right)' = e^{-x^2+7x-3}(7-2x)$;

c) $y' = \left(e^{\sqrt{x^2-8}}\right)' = \dfrac{xe^{\sqrt{x^2-8}}}{\sqrt{x^2-8}}$. ◄

3-vazifa. a) $y = e^{5x}$; b) $y = e^{x^3-2x+4}$; c) $y = e^{\sqrt{x^2+5}}$

funksiyalarning hosilasini toping.

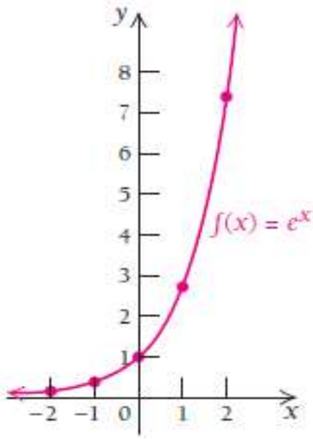
4.1.5. e^x, e^{-x} va $1 - e^{-kx}$ funksiyalarning grafiklari

Ushbu e^x, e^{-x} va $1 - e^{-kx}$ ko'rinishdagi funksiyalarning hosilalarini qanday topishni o'rganib oldik. Endi ularning grafiklarini 3.2 bo'limda tahlil qilganlarimiz asosida chizishga harakat qilamiz.

7-misol. $f(x) = e^x$ funksiyaning grafigini chizing.

Yechilishi: ► Kalkulyatordan foydalanib, ba'zi nuqtalarda funksiya qiymatlarini hisoblaymiz va jadvalga yozamiz. Shu nuqtalarni koordinata sistemasida belgilab, silliq chiziq bilan tutashtiramiz:

x	$f(x)$
-2	0.135
-1	0.368
0	1
1	2.718
2	7.389



Endi grafikni tahlil qilib chiqaylik:

1) **Hosilalari:** $f(x) = e^x$ funksiyaning 1- va 2-tartibli hosilalari $f'(x) = e^x$ va $f''(x) = e^x$.

2) **Kritik nuqtalari:** Barcha x haqiqiy sonlar uchun $f'(x) = e^x > 0$ va $f'(x) = e^x = 0$ bo'ladigan qiymatlar yo'q. Shu sababli kritik nuqtalar ham, maksimum yoki minimum nuqtalar ham yo'q.

3) **Funksiya o'suvchi:** $f'(x) = e^x > 0$ bo'lganligi uchun funksiya butun sonlar o'qida faqat o'sadi.

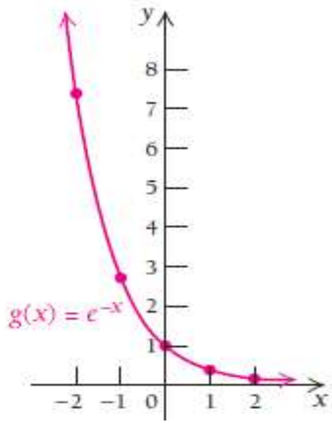
4) **Burilish nuqtalari:** Barcha x haqiqiy sonlar uchun $f''(x) = e^x > 0$ $f''(x) = e^x = 0$ bo'ladigan qiymatlar yo'qligi sababli funksiya burilish nuqtalariga ega emas.

5) **Funksiya botiq:** Barcha x haqiqiy sonlar uchun $f''(x) = e^x > 0$ bo'lganligi sababli butun son o'qida funksiya botiq shaklga ega. ◄

8-misol. $f(x) = e^{-x}$ funksiyaning grafigini chizing.

Yechilishi: ► Kalkulyatordan foydalanib, ba’zi nuqtalarda funksiya qiymatlarini hisoblaymiz va jadvalga yozamiz. Shu nuqtalarni koordinata sistemasida belgilab, silliq chiziq bilan tutashtiramiz:

x	$g(x)$
−2	7.389
−1	2.718
0	1
1	0.368
2	0.135



Endi grafikni tahlil qilib chiqaylik:

1) **Hosilalari:** $f(x)=e^{-x}$ funksiyaning 1- va 2-tartibli hosilalari $f'(x)=-e^{-x}$ va $f''(x)=e^{-x}$.

2) **Kritik nuqtalari:** Barcha x haqiqiy sonlar uchun

$f'(x)=-e^{-x}=-\dfrac{1}{e^x}<0$ va $f'(x)=-e^{-x}=0$ tenglamaning yechimi yo‘q. Shu sababli kritik nuqtalar ham, maksimum yoki minimum nuqtalar ham yo‘q.

3) **Funksiya o‘suvchi:** $f'(x)=-e^{-x}=-\dfrac{1}{e^x}<0$ bo‘lganligi uchun funksiya butun sonlar o‘qida faqat kamayadi.

4) **Burilish nuqtalari:** Barcha x haqiqiy sonlar uchun $f''(x)=e^{-x}>0$, $f''(x)=e^{-x}=0$ bo‘ladigan qiymatlar yo‘qligi sababli funksiya burilish nuqtalariga ega emas.

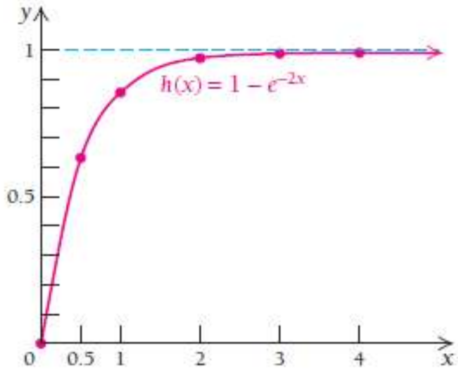
5) **Funksiya botiq:** Barcha x haqiqiy sonlar uchun $f''(x)=e^{-x}>0$ bo‘lganligi sababli butun son o‘qida funksiya botiq shaklga ega. ◄

$f(x)=1-e^{-kx}$, $x\geq 0$ funksiyadan amaliyotda juda ko‘p foydalaniladi.

9-misol. $f(x)=1-e^{-2x}$, $x\geq 0$ funksiya grafigini chizing va tahlil qiling.

Yechilishi: ► Kalkulyatordan foydalanib, ba’zi nuqtalarda funksiya qiymatlarini hisoblaymiz va jadvalga yozamiz. Shu nuqtalarni koordinata sistemasida belgilab, silliq chiziq bilan tutashtiramiz:

x	$h(x)$
0	0
0.5	0.63212
1	0.86466
2	0.98168
3	0.99752
4	0.99966
5	0.99995



Endi grafikni tahlil qilib chiqamiz:

1) **Hosilalari:** $f(x)=1-e^{-2x}$ funksiyaning 1- va 2-tartibli hosilalari $f'(x)=2e^{-2x}$ va $f''(x)=-4e^{-2x}$.

2) **Kritik nuqtalari:** $f'(x)=2e^{-2x}>0$ va $f'(x)=2e^{-2x}=0$ tenglamaning yechimi yo‘q. Shu sababli $(0,\infty)$ da kritik nuqtalar yo‘q.

3) **Funksiya o‘suvchi:** $f'(x)=2e^{-2x}>0$ bo‘lganligi uchun funksiya $(0,\infty)$ da faqat o‘sadi.

4) **Burilish nuqtalari:** $f''(x)=-4e^{-2x}<0$, $f''(x)=-4e^{-2x}=0$ bo‘ladigan qiymatlar yo‘qligi sababli funksiya burilish nuqtalariga ega emas.

5) **Funksiya qavariq:** $f''(x)=-4e^{-2x}<0$ bo‘lganligi uchun $(0,\infty)$ da funksiya qavariq shaklga ega. ◄

4-vazifa. a) $y = 3e^{-x}$; b) $y = 3e^{x}$; c) $f(x) = 1 - e^{-x}$ funksiyalarning grafiklarini chizing.

10-misol. Ishchining ish unumdorligi. Ishchi yangi narsni chiqarib boshlaganda sekin asta o‘rganib, samaradorligi ham oshib boradi, lekin ma’lum bir vaqtdan keyin unumdorlik bir xil bo‘lib qoladi. Firma 5G aqlli smartfonlar ishlab chiqaradi. t kundan keyin nechta telefon tayyorlanganiga qarab, ish unumdorligi aniqlanadi. Ishchilar unumdorligi $N(t) = 80 - 70e^{-0.13t}$ tenglik bilan modellashtirilsa, quyidagilarni aniqlang:

a) $N(0),\ N(1),\ N(5),\ N(10),\ N(20),\ N(30)$ qiymatlarni hisoblang;

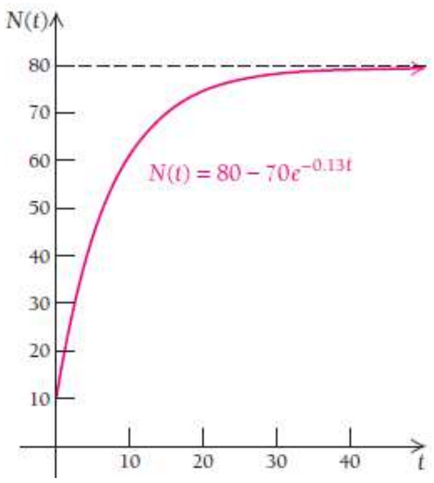
b) $N(t) = 80 - 70e^{-0.13t}$ ning grafigini chizing;

c) $N'(t)$ ni toping va interpretatsiya qiling;

d) Telefonlar sonidan qachon ishchining unumdorligi bir xil bo‘lib qolganligini bilish mumkinmi?

Yechilishi: ► a)-b) Kalkulyatordan foydalanib, berilgan nuqtalarda funksiya qiymatlarini hisoblaymiz va jadvalga yozamiz. Shu nuqtalarni koordinata sistemasida belgilab, silliq chiziq bilan tutashtiramiz:

t	0	1	5	10	20	30
$N(t)$	10	18.5	43.5	60.9	74.8	78.6



c) $N'(t) = \left(80 - 70e^{-0.13t}\right)' = -70e^{-0.13t} \cdot (-0.13) = 9.1e^{-0.13t}$

t kundan keyin kuniga tayyorlangan telefonlar soni $9.1e^{-0.13t}$ ga teng ekan.

d) diagrammani tahlil qilib, funksiya qiymatlarini o‘rganib, aytish mumkinki, ishchilar unumdorligi kuniga 80 ta telefon yasash bilan tenglashib ketadi. Undan ko‘p ishlab chiqarisha olmaydi.◄

5-vazifa. 10-misolni $N(t)=80-60e^{-0.12t}$ uchun bajaring.

MUSTAQIL YECHISH UCHUN MISOLLAR

1-10 misollarda funksiya grafigini chizing:

1. $y = 4^x$

2. $y = 5^x$

3. $y = (0.25)^x$

4. $y = (0.2)^x$

5. $f(x) = \left(\frac{3}{2}\right)^x$

6. $f(x) = \left(\frac{4}{3}\right)^x$

7. $g(x) = \left(\frac{2}{3}\right)^x$

8. $g(x) = \left(\frac{3}{4}\right)^x$

9. $f(x) = (2.5)^x$

10. $f(x) = (1.2)^x$

11-54 misollarda funksiya hosilasini toping:

11. $f(x) = e^{-x}$

12. $f(x) = e^x$

13. $g(x) = e^{3x}$

14. $g(x) = e^{2x}$

15. $f(x) = 6e^x$

16. $f(x) = 4e^x$

17. $F(x) = e^{-7x}$

18. $F(x) = e^{-4x}$

19. $G(x) = 2e^{4x}$

20. $g(x) = 3e^{5x}$

21. $f(x) = -3e^{-x}$

22. $G(x) = -7e^{-x}$

23. $g(x) = \frac{1}{2}e^{-5x}$

24. $f(x) = \frac{1}{3}e^{-4x}$

25. $F(x) = -\frac{2}{3}e^{x^2}$

26. $g(x) = -\frac{4}{3}e^{x^3}$

27. $G(x) = 7 + 3e^{5x}$

28. $F(x) = 4 - e^{2x}$

29. $f(x) = x^5 - 2e^{6x}$

30. $G(x) = x^3 - 5e^{2x}$

31. $g(x) = x^5e^{2x}$

32. $f(x) = x^7e^{4x}$

33. $F(x) = \frac{e^{2x}}{x^4}$

34. $g(x) = \frac{e^{3x}}{x^6}$

35. $f(x) = (x^2 + 3x - 9)e^x$

36. $f(x) = (x^2 - 2x + 2)e^x$

37. $f(x) = \frac{e^x}{x^4}$

38. $f(x) = \frac{e^x}{x^5}$

39. $f(x) = e^{-x^2+7x}$

40. $f(x) = e^{-x^2+8x}$

41. $f(x) = e^{-x^2/2}$

42. $f(x) = e^{x^2/2}$

43. $y = e^{\sqrt{x-7}}$

44. $y = e^{\sqrt{x-4}}$

45. $y = \sqrt{e^x - 1}$

46. $y = \sqrt{e^x + 1}$

47. $y = xe^{-2x} + e^{-x} + x^3$

48. $y = e^x + x^3 - xe^x$

49. $y = 1 - e^{-x}$

50. $y = 1 - e^{-3x}$

51. $y = 1 - e^{-kx}$

52. $y = 1 - e^{-mx}$

53. $g(x) = (4x^2 + 3x)e^{x^2-7x}$

54. $g(x) = (5x^2 - 8x)e^{x^2-4x}$

55-64 misollarda funksiya grafigini chizing va tahlil qiling:

55. $f(x) = e^{2x}$

56. $g(x) = e^{-2x}$

57. $g(x) = e^{(1/2)x}$

58. $f(x) = e^{(1/3)x}$

59. $f(x) = \frac{1}{2}e^{-x}$

60. $g(x) = \frac{1}{3}e^{-x}$

61. $F(x) = -e^{(1/3)x}$

62. $G(x) = -e^{(1/2)x}$

63. $g(x) = 2(1 - e^{-x}), \quad \text{for } x \geq 0$

64. $f(x) = 3 - e^{-x}, \quad \text{for } x \geq 0$

65. Eksport. O‘zbekistonning eksport hajmi eksponensial funksiya ko‘rinishida o‘sib bormoqda. 2009 yildan keyin t yil o‘tib, $V(t)=1.6e^{0.046t}$ formula bilan approksimatsiyalash mumkin. Bunda 2009 yil $t=0$ deb olinadi, V esa mlrd dollar hisobida.

a) 2009-2020 yillardagi eksport qiymatini baholang;

b) Qachon eksport qiymati 2 marta ko‘payadi?

66. Tabiiy mahsulotlar. Hozirgi paytda odamlar organik oziq-ovqatlarni sotib olishmoqda. 2005 yildan buyon t yilda tabiiy mahsulotlarga sarflangan $A(t)$ mlrd.lab so‘m $V(t)=2.43e^{0.18t}$ tenglik bilan approksimatsiyalanadi.

a) 2019 yilda tabiiy mahsulotlarga pul sarflagan kishilar sonini toping;

b) 2016 yildan buyon tabiiy mahsulotlarga talab o‘sganmi?

67. Ta’lim modeli. Faraz qilaylik, siz modulni 100% o‘zlashtirdingiz. Lekin inson miyasi t hafta o‘tib, o‘rganganlarini P foizini esda saqlab qoladi. Innovatsion ta’lim modeli ta’kidlashicha bizning saqlab qolish foizini $P(t)=q+(100-q)e^{-kt}$ tenglik bilan approksimatsiyalash mumkin. Bunda $q-$hech qachon esimizdan chiqmaydigan bilimlar foizi, $k=const$.

a) 0, 1, 2, 6, 10 haftadan keyin xotiramizda qoladigan bilimlar foizini hisoblang;

b) $\lim\limits_{t\to\infty} P(t)$ hisoblang;

c) $P(t)$ ning grafigini chizing;

d) $P'(t)$, ya’ni P ning vaqtga bog‘liq hosilasini toping.

68- 93 misollar limitlarni 2-ajoyib limit formulasidan foydalanib, hisoblashga doir:

68. $\lim\limits_{x\to\infty}\left(\frac{x+4}{x+8}\right)^{-3x}$; **69.** $\lim\limits_{x\to\infty}\left(\frac{x}{x+1}\right)^{2x-3}$;

70. $\lim\limits_{x\to\infty}\left(\frac{x-1}{x}\right)^{2-3x}$; **71.** $\lim\limits_{x\to\infty}\left(\frac{2x+5}{2x+1}\right)^{5x}$;

72. $\lim\limits_{x\to\infty}\left(\frac{x+3}{x}\right)^{-5x}$; **73.** $\lim\limits_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{1+2x}$;

74. $\lim\limits_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{x-4}$; **75.** $\lim\limits_{x\to\infty}\left(\frac{2x}{2x-3}\right)^{3x}$;

76. $\lim\limits_{x\to\infty}\left(\frac{x-7}{x}\right)^{2x+1}$; **77.** $\lim\limits_{x\to\infty}\left(\frac{x+4}{x+8}\right)^{-3x}$;

78. $\lim\limits_{x\to\infty}\left(\frac{2x+1}{2x-1}\right)^{x+2}$; **79.** $\lim\limits_{x\to\infty}\left(\frac{x-2}{x+1}\right)^{2x-3}$;

80. $\lim\limits_{x\to\infty}\left(\frac{x}{x-3}\right)^{x-5}$; **81.** $\lim\limits_{x\to\infty}\left(\frac{3x-4}{3x+2}\right)^{2x}$;

82. $\lim\limits_{x\to\infty}\left(\frac{2x-1}{2x+4}\right)^{3x-1}$; **83.** $\lim\limits_{x\to\infty}\left(\frac{2x}{1+2x}\right)^{-4x}$.

84. $\lim\limits_{x\to\infty}\left(\frac{x+3}{x-2}\right)^{x}$. **85** $\lim\limits_{x\to\infty}\left(\frac{2x-1}{2x+1}\right)^{x}$.

86. $\lim\limits_{x\to\infty}\left(\frac{4x+1}{4x}\right)^{2x}$. **87.** $\lim\limits_{x\to 0}(1+2x)^{1/x}$.

88. $\lim\limits_{x\to+\infty} x\left[\ln(x+1)-\ln x\right]$. **89.** $\lim\limits_{x\to+\infty}(1+2x)\left[\ln(x+3)-\ln x\right]$.

90. $\lim\limits_{x\to+\infty}(x-5)\left[\ln(x-3)-\ln x\right]$. **91.** $\lim\limits_{x\to 1}(7-6x)^{x/(3x-3)}$.

92. $\lim\limits_{x\to 2}(3x-5)^{2x/(x^2-4)}$. **93.** $\lim\limits_{x\to 3}(3x-8)^{2/(x-3)}$.

4.2. Logarifmik funksiya va uning hosilasi

4.2.1. Logarifmik funksiyaning grafigi va asosiy xossalari

$10^y = 1000$ tenglamani yechamiz. Buning uchun 1000 ga teng bo‘ladigan 10 ning darajasini izlaymiz. $10^3 = 1000$ tenglik 3 da o‘rinli bo‘ladi. Ana shu 3 soniga 1000 ning 10 **asosli logarifmi** deyiladi.

Ta’rif. Logarifm quyidagicha aniqlangan:

$$\log_a x = y \text{ bunda } a^y = x,\ a > 0,\ a \neq 1.$$

$\log_a x$ son y ga teng bo‘lib, u x ni hosil qilish uchun a ni ko‘tarish kerak bo‘lgan daraja ko‘rsatkichidir. a ni **logarifmning asosi** deyiladi. $\log_a x$ ni “logarifm a asosga ko‘ra iks” deb o‘qiymiz.

10 asosli logarifm $\log_{10} x$ uchun y son 10 ni ko‘tarish kerak bo‘lgan daraja ko‘rsatkichi $10^y = x$. Bundan ko‘rinadiki, logarifmik funksiyadan ko‘rsatkichli funksiyaga o‘tish mumkin va aksincha.

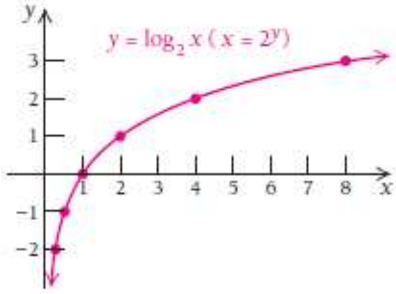
logarifmik funksiya	ko’rsatkichli funksiya
$\log_a M = N$	$a^N = M$
$\log_{10} 100 = 2$ $\log_5 \frac{1}{25} = -2$ $\log_{49} 7 = \frac{1}{2}$	$10^2 = 100$ $5^{-2} = \frac{1}{25}$ $49^{1/2} = 7$

Logarifmik funksiyaning grafigini chizish uchun unga ekvivalent bo‘lgan ko‘rsatkichli funksiya grafigidan foydalanishimiz mumkin.

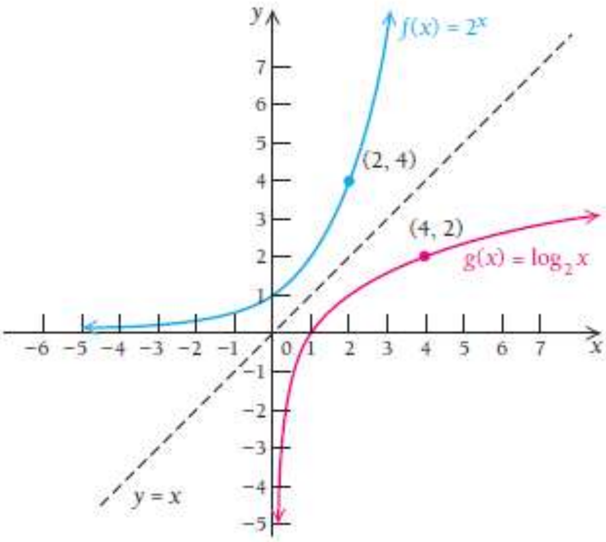
1-misol. $y = \log_2 x$ funksiyaning grafigini chizing.

Yechilishi: ► Dastlab unga ekvivalent bo‘lgan ko‘rsatkichli funksiyani yozib olamiz: $2^y = x$. So‘ngra y uchun qiymat tanlab, x ni hosil qilamiz. Esda tuting, x haliyam 1-koordinata, so‘ngra hosil bo‘lgan nuqtalarni silliq chiziq bilan tutashtirib chiqamiz.

x, 2^y	y
1	0
2	1
4	2
8	3
$\frac{1}{2}$	-1
$\frac{1}{4}$	-2



$f(x) = 2^x$ va $g(x) = \log_2 x$ funksiyalarning grafiklari bir-biriga $y = x$ chiziqqa nisbatan simmetrik.



Grafiklari mana shunday simmetrik bo‘lgan funksiyalar **bir-biriga inversiya** deyiladi.

Logarifmlarning asosiy xossalari

3-teorama. $M,\ N,\ a,\ b$ musbat sonlar va $a,\ b \neq 1,\ k$ biror haqiqiy son bo‘lsin. Quyidagi tengliklar o‘rinli:

1^0. $\log_a(MN) = \log_a M + \log_a N$

2^0. $\log_a(\frac{M}{N}) = \log_a M - \log_a N$

3^0. $\log_a(M^k) = k \cdot \log_a M$

4^0. $\log_a a = 1$

5^0. $\log_a a^k = k$

6^0. $\log_a 1 = 0$

7^0. $\log_b M = \dfrac{\log_a M}{\log_a b}$.

Ushbu xossalarning tatbiqlarini misollarda ko‘ramiz.

2-misol. $\log_a 2 = 0.301$ va $\log_a 3 = 0.477$ bo‘lsa, quyidagilarni hisoblang: a) $\log_a 6$ b) $\log_a \frac{2}{3}$ c) $\log_a 81$

d) $\log_a \frac{1}{3}$ e) $\log_a \sqrt{a}$ f) $\log_a (2a)$

Yechilishi: ► a) $\log_a 6 = \log_a(2\cdot 3) = \log_a 2 + \log_a 3 = 0.301 + 0.477 = 0.778$;

b) $\log_a \frac{2}{3} = \log_a 2 - \log_a 3 = 0.301 - 0.477 = -0.176$;

c) $\log_a 81 = \log_a 3^4 = 4\cdot\log_a 3 = 4\cdot 0.477 = 1.908$;

d) $\log_a \frac{1}{3} = \log_a 3^{-1} = -1\cdot\log_a 3 = -0.477$;

e) $\log_a \sqrt{a} = \log_a a^{\frac{1}{2}} = \frac{1}{2}\log_a a = \frac{1}{2}$;

f) $\log_a (2a) = \log_a 2 + \log_a a = 0.301 + 1 = 1.301$ ◄

O‘nli va natural logarifmlar

$\log_{10} x$ yoki $\lg x$ ga **o‘nli logarifm** deyiladi.

Ta’rif: Har qanday x musbat son uchun $\log_{10} x = \lg x$ tenhlik o‘rinli.

Quyidagi tengliklar o‘rinli:

$1000 = 10^3$	$\lg 1000 = 3$
$100 = 10^2$	$\lg 100 = 2$
$10 = 10^1$	$\lg 10 = 1$
$1 = 10^0$	$\lg 1 = 0$
$0.1 = 10^{-1}$	$\lg 0.1 = -1$
$0.01 = 10^{-2}$	$\lg 0.01 = -2$
$0.001 = 10^{-3}$	$\lg 0.001 = -3$

Agar $\lg 100 = 2$ va $\lg 1000 = 3$ ekanligi ma’lum bo‘lsa, u holda $2 < \lg 500 < 3$ bo‘lishi tabiiy. Logarifmik jadvallar dastlab mana shunday taqribiy hisoblashlar uchun ishlatilgan. Keyinchalik kalkulyatorlar paydo bo‘lgandan keyin bunday jadvallar kam ishlatiladigan bo‘ldi. Kalkulyator yordamida $\lg 500 \approx 2.6990$ ekanini topamiz.

O‘nli logarifmlarning maxsus belgilanishiga sabab, oldinlari 10 asosli logarifmlar hisoblashlarda ko‘p ishlatilgan. Shunday bo‘lishi ham kerak-da, chunki sanoq sistemamiz 10 lik sanoq sistemasiku.

Hozirgi kunda o‘nli logarifmik funksiyalar tarixda qoldi. Endilikda logarifmik funksiyalar ishlatiladigan masalalarda asosi $e \approx 2.718$ bo‘lgan $\log_e x$ funksiya muhim o‘rin egallaydi.

Asosi $e \approx 2.718$ bo‘lgan $\log_e x = \ln x$ funksiya juda ko‘p sohalarda o‘zining tatbig‘ini topgan.

Ta’rif. Ixtiyoriy musbat son x uchun $\log_e x = \ln x$ tenglik o‘rinli. $\ln x$ ga **natural logarifm** deyiladi.

Natural logarifmlar uchun quyidagi xossalar o‘rinli:

4-teorama. $M,\ N,\ a,\ b$ musbat sonlar va $a,\ b \neq 1,\ k \in R$ bo‘lsin.

1⁰. $\ln(MN) = \ln M + \ln N$ **2⁰.** $\ln\left(\frac{M}{N}\right) = \ln M - \ln N$

3⁰. $\ln(e^k) = k$ **4⁰.** $\ln e = 1$

5⁰. $\ln 1 = 0$ **6⁰.** $\ln a^k = k \ln a$ **7⁰.** $\ln M = \frac{\lg M}{\lg b}$.

Ushbu xossalarning tatbiqlarini misollarda ko‘ramiz.

3-misol. $\ln 2 = 0.6931$ va $\ln 3 = 1.0986$ bo‘lsa, quyidagilarni hisoblang: a) $\ln 6$ b) $\ln\frac{2}{3}$ c) $\ln 81$

d) $\ln\frac{1}{3}$ e) $\ln(2e^6)$

Yechilishi: ► a) $\ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3 = 0.6931 + 1.0986 = 1.7917$;

b) $\ln\frac{2}{3} = \ln 2 - \ln 3 = 0.6931 - 1.0986 = -0.4055$;

c) $\ln 81 = \ln 3^4 = 4\cdot\ln 3 = 4\cdot 1.0986 = 4.3944$;

d) $\ln\frac{1}{3} = \ln 3^{-1} = -1\cdot\ln 3 = -1.0986$;

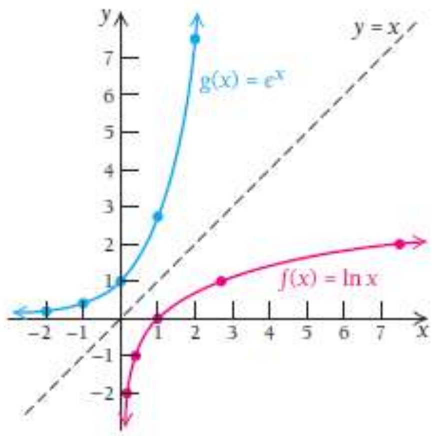
e) $\ln(2e^6) = \ln 2 + 6\ln e = 0.6931 + 6 = 6.6931$ ◄

4.2.2. Natural logarifmik funksiyaning grafigi

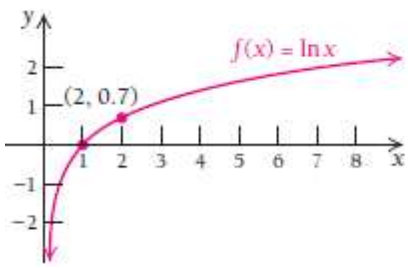
Natural logarifmik funksiya $y = \ln x$ grafigini chizishning 2 usuli mavjud:

1-usul. $y = \ln x$ ga ekvivalent bo‘lgan $x = e^y$ tenglama grafigini chizib, so‘ngra unga $y = x$ to‘g‘ri chiziqqa nisbatan simmetrik grafikni yasash

x, e^y	y
0.1	−2
0.4	−1
1.0	0
2.7	1
7.4	2
20.1	3



2-usul. $y = \ln x$ ning bir nechta qiymatini kalkulyatorda topib, sonlar juftliklarini hosil qilamiz. So‘ngra shu nuqtalardan o‘tuvchi silliq chiziqni yasaymiz. Misol uchun $\ln 1 = 0$ va $\ln 2 \approx 0.7$ deb olish mumkin, bulardan (1,0) va (2, 0.7) nuqtalar hosil bo‘ladi.



Funksiya grafigiga qarab, quyidagi xossa o‘rinli wkanligini ko‘rish mumkin.

5-teorama. $\ln x$ faqat $x > 0$ sonlar uchun o‘rinli.

$y = \ln x$ ning aniqlanish sohasi $(0;\infty)$.

$0 < x < 1$ da $\ln x < 0$;

$x = 1$ da $\ln x = 0$;

$x > 1$ da $\ln x < 0$.

$y = \ln x$ funksiya o‘zining aniqlanish sohasida faqat o‘suvchi.

$y = \ln x$ ning qiymatlar sohasi $(-\infty;\infty)$ dan iborat.

4.2.4. Natural logarifmik funksiyaning hosilasi

$f(x) = \ln x$ funksiyadan hosila olamiz. Buning uchun $e^{f(x)} = x$ dan foydalanamiz. Tenglikning har ikki tomonini differensiallaymiz:

$$\frac{d}{dx}e^{f(x)} = \frac{d}{dx}x$$

$$e^{f(x)} \cdot f'(x) = 1$$

$e^{f(x)} = x$ o‘rniga qo‘yamiz.

$$x \cdot f'(x) = 1$$

$$f'(x) = \frac{1}{x}.$$

6-teorama. $x > 0$ sonlar uchun $(\ln x)' = \frac{1}{x}$ munosabat o‘rinli.

4-misol. a) $y = 5\ln x$; b) $y = x^3 \ln x + 3x$; c) $y = \frac{\ln x}{x^2}$

funksiyalarning hosilalarini hisoblang.

Yechilishi: ► a) $y' = (5\ln x)' = \frac{5}{x}$;

b) $y' = (x^3 \ln x + 3x)' = 3x^2 \ln x + x^3 \cdot \frac{1}{x} + 3 = 3x^2 \ln x + x^2 + 3$,

c) $y'=\left(\dfrac{\ln x}{x^2}\right)'=\dfrac{\frac{1}{x}\cdot x^2-\ln x\cdot 2x}{x^4}=\dfrac{x-2x\ln x}{x^4}=\dfrac{1-2\ln x}{x^3}$. ◀

1-vazifa. a) $y=2\ln x$; b) $y=x^2\ln x-5x$; c) $y=\dfrac{1+\ln x}{x^3}$

funksiyalarning hosilalarini toping.

5-misol. $y=\ln(x^2-9x)$ funksiya hosilasini toping.

Yechilishi: ▶ Ushbu misolda murakkab funksiya berilgan. Agar $f(x)=g(h(x))=\ln(h(x))$, bunda $g(x)=\ln x$ va $h(x)=x^2-9x$. $g'(x)=\dfrac{1}{x}$

U holda 2.7-bo‘limdagi zanjir formulasiga ko‘ra

$$f'(x)=g'(h(x))\cdot h'(x)=\frac{1}{h(x)}\cdot h'(x) \text{ o‘rinli.}$$

Bizning misolda $h(x)=x^2-9x$ va $h'(x)=2x-9$. Shunda

$$y'=\left(\ln(x^2-9x)\right)'=\frac{1}{x^2-9x}\cdot(x^2-9x)'=\frac{2x-9}{x^2-9x}$$ ni hosil qilamiz. ◀

7-teorama. $y=\ln(h(x))$ murakkab funksiyaning hosilasi

$$y'=\frac{1}{h(x)}\cdot h'(x) \text{ yoki } y'=\frac{h'(x)}{h(x)} \text{ bo‘ladi.}$$

6-misol. a) $y=\ln(5x)$; b) $y=\ln(x^3+3x)$; c) $y=\ln\left(\dfrac{1+x}{x^2}\right)$

funksiyalarning hosilalarini hisoblang.

Yechilishi: ▶ a) $y'=(\ln(5x))'=\dfrac{5}{5x}=\dfrac{1}{x}$ yoki $y=\ln(5x)=\ln 5+\ln x$

deb olib, keyin hosila olish mumkin: $y'=(\ln 5+\ln x)'=0+\dfrac{1}{x}=\dfrac{1}{x}$;

b) $y'=\left(\ln(x^3+3x)\right)'=\dfrac{1}{x^3+3x}\cdot(3x^2+3)=\dfrac{3(x^2+1)}{x(x^2+3)}$;

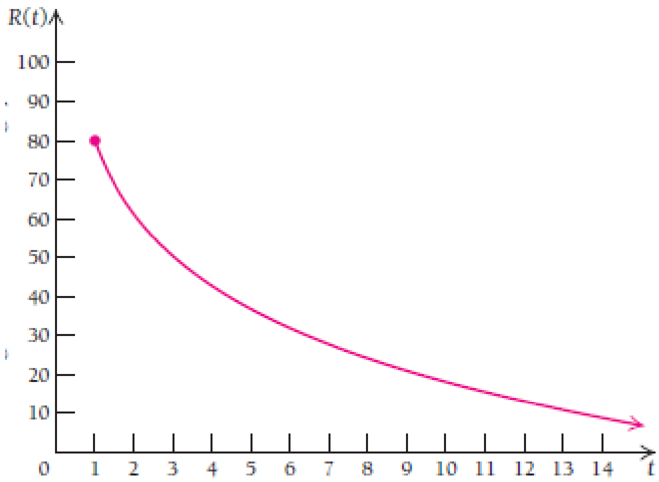
c) $y' = \left(\ln\left(\frac{1+x}{x^2}\right)\right)' = \frac{1}{\frac{1+x}{x^2}} \cdot \left(\frac{1+x}{x^2}\right)' = \frac{x^2}{1+x} \cdot \frac{x^2 - (1+x)\cdot 2x}{x^4} = \frac{-x(2+x)}{x^2(1+x)}$ ◀

2-vazifa. a) $y = \ln(3x)$; b) $y = \ln(2x^3 + 4)$; c) $y = \ln\left(\frac{x^4 - 3}{x}\right)$

d) $y = \ln(\ln(5x))$ funksiyalarning hosilalarini toping.

Natural logarifmik funksiyaning tatbiqlari

7-misol. **Psixologiya.** Talabalarga bir nechta ma'nosiz harf birikmalari aytilib, (masalan, rok, roz, med, …), shu birikmalarni har bir minutdan takrorlab turish so'raldi. t minutdan keyin xotirada saqlab qolgan talabalar soni foizining modeli $R(t) = 80 - 27\ln t$, $t \geq 1$ quyidagicha bo'ldi.



a) Necha foiz talaba 1 minutdan keyin berilgan birikmalarni xotirasida saqlab qolgan?

b) $R'(2)$ ni hisoblang va bu nimani bildirishini tushuntiring.

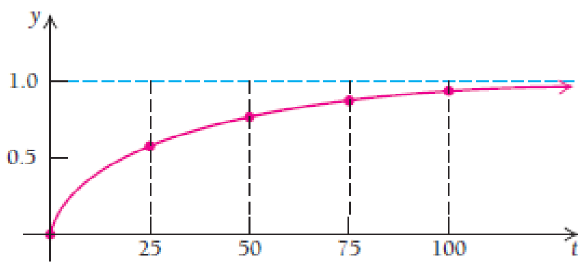
Yechilishi: ▶ a) $R(1) = 80 - 27\ln t = 80 - 27\ln 1 = 80\%$;

b) $R'(t) = (80 - 27\ln t)' = -\dfrac{27}{t}$ bundan $R'(2) = -\dfrac{27}{2} = -13.5$ chiqadi.

Demak, 2 minutdan keyin har minutda 13.5% talaba birikmalarni eslay olmay qoladi. ◄

7-misol. Reklama modeli. Qurilish kompaniyasi yangi uylarni sotish uchun radioda reklama berishni boshladi. Reklamaning davomiylik funksiyasi $f(t) = 1 - e^{-0.04t}$, bunda $t-$reklama berilgan kunlar soni. Radioeshituvchilar taxminan 1000000 kishi bo‘lsa. Kompaniya har bitta reklama e’loni uchun 0.5$ dan to‘lashga kelishdi va kuniga 1000$ berdi. Maksimal foyda olish uchun necha kun reklama berish kerak?

Yechilishi: ► Reklama qancha ko‘p berilsa, shuncha ko‘p odam eshitadi va savdo yaxshi bo‘ladi. $f(t) = 1 - e^{-0.04t}$ reklama asosida sotilgan uylar foizi modeliga ko‘ra, 0 dan 1 gacha o‘zgaradi.



Umumiy foyda formulasidan

Foyda=daromad-harajat= $F(t) = D(t) - T(x)$.

$D(t) =$(sotish foizi)·(reklama eshituvchilar)·(1 ta reklama narxi)=

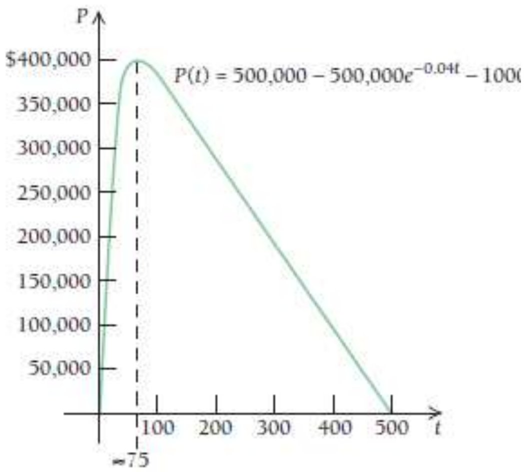
$= (1 - e^{-0.04t}) \cdot 1000000 \cdot 0.5 = 500000 - 500000e^{-0.04t}$

$T(x) =$(kuniga reklamaga sarflanadigan pul)·(kunlar soni)=1000t;

Shunda $F(t) = D(t) - T(x) = 500000 - 500000e^{-0.04t} - 1000t$.

$$F'(t) = -500000e^{-0.04t} \cdot (-0.04) - 1000 = 20000e^{-0.04t} - 1000,$$

1-tartibli hosilani nolga tenglaymiz: $20000\,e^{-0.04t} - 1000 = 0$



$$e^{-0.04t} = 0.05$$

$$\ln e^{-0.04t} = \ln 0.05$$

$$-0.04t = \ln 0.05$$

$$t \approx 75$$

Bu maksimum qiymat ekanligini tekshirish uchun 2-tartibli hosila olamiz: $F''(t) = (20000e^{-0.04t} - 1000)' = -800e^{-0.04t}$, $e^{-0.04t} > 0$ bo‘lgani sababli $F''(75) < 0$.

Demak, foydani maksimallashtirish uchun kompaniya 75 kun davomida reklama berib turishi kerak ekan.◄

4.2.4. Hosilani hisoblashda logarifmlash usuli

Funksiya hosilasini hisoblashda natural logarifmdan foydalaniladi. Bizga $y = f(x)$ funksiya berilgan bo‘lsin. Bu funksiya hosilasini hisoblash uchun

1) Tenglikning har ikki tomonini natural asosli logarifmlaymiz;

2) $\frac{y'}{y}$ dan y' ni topib olamiz.

Yoki quyidagi formuladan foydalanamiz: $\left(\ln f(x)\right)' = \frac{f'(x)}{f(x)}$

3-misol. $y = (\sin 2x)^x$ funksiyaning hosilasini toping.

Yechilishi: ► Funksiya hosilasini ikki usul bilan hisoblash mumkin:

1-usul. $y=(\sin 2x)^x$ tenglikni ikkala tomonini logarifmlaymiz va hosila olamiz: $\ln y = x\ln\sin 2x$

$$(\ln y)' = (x)'\ln\sin 2x + x(\ln\sin 2x)' = \ln\sin 2x + x\cdot\frac{1}{\sin 2x}\cdot 2\cos 2x$$

bu yerdan, $\frac{y'}{y} = \ln\sin 2x + 2x\cdot \text{ctg}2x$

Demak, $y' = y(\ln\sin 2x + 2x\cdot\text{ctg}2x) = (\sin 2x)^x(\ln\sin 2x + 2x\cdot\text{ctg}2x)$.

2-usul: $y=(\sin 2x)^x = e^{\ln(\sin 2x)^x} = e^{x\ln\sin 2x}$ ko`rinishga keltiramiz.

$$y' = \left(e^{x\ln\sin 2x}\right)' = e^{x\ln\sin 2x}(x\cdot\ln\sin 2x)' = (\sin 2x)^x(\ln\sin 2x + 2x\cdot\text{ctg}2x).$$ ◄

MUSTAQIL YECHISH UCHUN MISOLLAR

1-12 misollarda kalkulyatordan foydalanmasdan funksiya qiymatini hisoblang:

$\log_b 3 = 1.099$ va $\log_b 5 = 1.609$ bo'lsa, quyidagilarni hisoblang:

1. $\log_b \frac{5}{3}$ **2.** $\log_b \frac{1}{5}$ **3.** $\log_b 15$

4. $\log_b (5b)$ **5.** $\log_b 75$ **6.** $\log_b \sqrt[3]{5^4}$

$\ln 4 = 1.3863$ va $\ln 5 = 1.6094$ bo'lsa, quyidagilarni hisoblang:

7. $\ln 20$ **8.** $\ln 80$ **9.** $\ln(5e)$

10. $\ln\sqrt{e^5}$ **11.** $\ln\sqrt{16}$ **12.** $\ln(4e^3)$

13-37 misollarda funksiya hosilasini toping:

13. $y=-7\ln x$ **14.** $y = 9\ln x^2$ **15.** $y=\ln(6x)$

16. $y = x^4 \ln x - \frac{1}{2}x^2$ **17.** $y = x^6 \ln x - \frac{1}{4}x^4$

18. $y = x^2 \ln(6x)$ **19.** $y = x^5 \ln(5x)$

20. $y = \frac{\ln x}{x^4}$ **21.** $y = \frac{\ln x}{x^5}$

22. $y = \ln\left(\frac{x^4}{2}\right)$ **23.** $y = \ln\left(\frac{x^2}{4}\right)$

24. $y = \ln(3x^2 + 2x - 1)$ **25.** $y = \ln(7x^2 + 5x + 8)$

26. $y = \ln(x^2 + 1)^5$ **27.** $y = \ln(x^3 - x)^7$

28. $f(t) = [\ln(t + 6)]^5$ **29.** $f(t) = \ln[\ln(\ln(3t)]$

30. $y = \ln\frac{x^2 + 1}{x^2}$ **31.** $y = \ln\frac{x + 1}{x - 1}$

32. $y = \ln\sqrt{\frac{x - 1}{x^2}}$ **33.** $y = \ln\sqrt[3]{\frac{x + 1}{x - 1}}$

34. $g(t) = \ln[(t^3 + 6)(t^2 - 3)]$

35. $g(t) = \ln[(t + 5)^2(t - 4)^3]$

36. $y = \log_5 x$ **37.** $y = \log_7 x$.

Logarifmik funksiya tatbiqlariga doir masalalar:

38. Reklama. Xaridorlar talabi asosida $N(a) = 2000 + 500\ln a,\ a \geq 1$ reklama berilmoqda, bunda $N(a)$ sotilgan mahsulot soni, a – reklamaga ketgan mablag‘ (dollar hisobida).

1) Reklamaga 1000$ sarf qilgandan keyin qancha mahsulot sotilgan?

2) $N'(a)$ va $N'(10)$ toping;

3) Agar maksimum, minimum qiymatlar mavjud bo‘lsa, ularni toping;

4) $\lim_{a\to\infty} N'(a)$ ni aniqlang. Reklamaga ko‘p mablag‘ sarflash kerakmikan?

39. Tadbirkorlik. Xaridorlar talabiga ko‘ra mahsulot reklamasi $N(a)=1000+200\ln a,\ a\geq 1$ berilmoqda, bunda $N(a)$ sotilgan mahsulot soni, $a-$reklamaga ketgan mablag‘ (dollar hisobida).

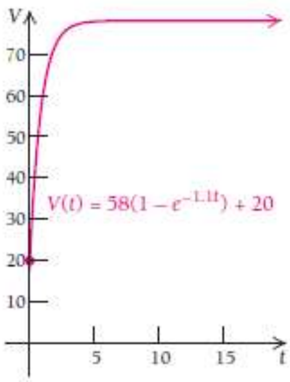
1) Reklamaga 1000$ sarf qilgandan keyin qancha mahsulot sotilgan?

2) $N'(a)$ va $N'(10)$ toping;

3) Agar maksimum, minimum qiymatlar mavjud bo‘lsa, ularni toping;

4) $\lim_{a\to\infty} N'(a)$ ni aniqlang. Reklamaga ko‘p mablag‘ sarflash kerakmikan?

40. Zahiradagi mahsulot narxining ortishi. Mahsulot sotib olingandan t oy o‘tib, uning qiymati $V(t)=58(1-e^{-1.1t})+20$ ga o‘sdi.



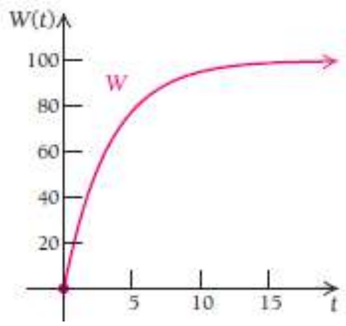
1) $V(1),\ V(12)$ ni hisoblang;

2) $V'(t)$ ni toping;

3) Nechchi oydan keyin mahsulot narxi 75$ bo‘ladi?

4) $\lim_{t\to\infty} V(t)$ ni aniqlang. Vaqt ko‘p o‘tsa, mahsulot narxi oshib boraveradimi? Nima uchun?

41. O‘rgatuvchi model. “Mashinistka” modeli kompyuterda minutiga W dona so‘z terishni o‘rgatadi: $W(t)=100(1-e^{-0.3t})$.



1) $W(1)$, $W(8)$ ni hisoblang;

2) $W'(t)$ ni toping;

3) Necha haftadah keyin minutiga 96 ta so‘z yazadigan bo‘lishadi?

4) $\lim\limits_{t\to\infty} W(t)$ ni aniqlang. Bu ifoda nimani anglatadi?

42- 60 misollarda funksiyalar hosilasini logarifm yordamida hisoblang:

42. $y=(\mathrm{cth}\,3x)^{\arcsin x}$;

43. $y=(\cos(x+2))^{\ln x}$;

44. $y=(\sin 3x)^{\arccos x}$;

45. $y=(\mathrm{th}\,5x)^{\arcsin(x+1)}$;

46. $y=(\mathrm{sh}(x+2))^{\arcsin 2x}$;

47. $y=(\cos 5x)^{\mathrm{arctg}\sqrt{x}}$;

48. $y=\left(\sqrt{3x+2}\right)^{\mathrm{arcctg}\,3x}$;

49. $y=(\ln(x+3))^{\sin\sqrt{x}}$;

50. $y=(\log_2(x+4))^{\mathrm{ctg}\,7x}$;

51. $y=(\mathrm{sh}\,3x)^{\mathrm{arctg}(x+2)}$;

52. $y=(\mathrm{ch}\,3x)^{\mathrm{ctg}\frac{1}{x}}$;

53. $y=(\arcsin 5x)^{\mathrm{tg}\sqrt{x}}$;

54. $y=(\arccos 5x)^{\ln x}$;

55. $y=(\mathrm{arctg}\,2x)^{\sin x}$;

56. $y=(\ln(x+7))^{\mathrm{ctg}\,2x}$;

57. $y=(\mathrm{ctg}(7x+4))^{\sqrt{x+3}}$;

58. $y=\left(\mathrm{th}\sqrt{x+1}\right)^{\mathrm{arctg}\,2x}$;

59. $y=\left(\mathrm{cth}\dfrac{1}{x}\right)^{\arcsin 7x}$;

60. $y=(\cos(x+5))^{\arcsin 3x}$.

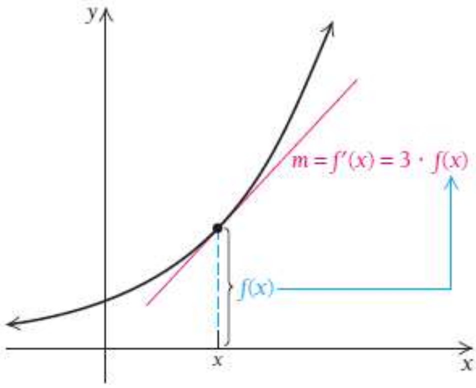
4.3. Erkin va chegaralangan o‘sish modellari

4.3.1. Eksponensial o‘sish. Erkin o‘sish modeli

$f(x)=5e^{3x}$ funksiyani qaraylik.

Funksiya differensiali $f'(x)=5e^{3x}\cdot 3=3f(x)$ ga teng. Bundan ko‘rinadiki, funksiya hosilasi yoki urinmasining burchak koeffitsiyenti funksiyaning o‘zidan 3 marta katta bo‘lar ekan.

$f(x)=ce^{kx}$ **ko‘rsatkichli funksiya hosilasi o‘zidan o‘zgarmas son marta katta bo‘ladigan yagona funksiyadir.**



8-teorema. Agar $f(x)=ce^{kx}$ bo‘lsa, u holda $f'(x)=k\cdot f(x)$ o‘rinli.

1-misol. $A'(t)=5A$ tenglikni qanoatlantiruvchi $A(t)$ funksiyaning umumiy ko‘rinishini yozing.

Yechilishi: ► Ushbu funksiya $A(t)=ce^{5t}$ ko‘rinishda bo‘ladi, chunki

$A'(t)=(ce^{5t})'=ce^{5t}\cdot 5=5\cdot A(t)$ ◄

1-vazifa. $f(x)=2e^{4x}$ funksiya hosilasini toping. $f'(x)$ ni $f(x)$ bilan bog‘liqligini ko‘rsating.

2-misol. $\dfrac{dP}{dt}=kP$ tenglikni qanoatlantiruvchi funksiyani toping.

Yechilishi: ► Ushbu funksiya $P(t) = ce^{kt}$ ko‘rinishda bo‘ladi, chunki $\frac{dP}{dt} = ce^{kt} \cdot k = kP$ o‘rinli. ◄

Biz bilamizki, algebraik tenglamalarning yechimi son chiqadi, lekin 1-va 2-misollarda sizga havola qilgan tenglamalarimizning yechimlari funksiyalardan iborat bo‘ldi. Misol uchun, $3x + 7 = 13$ tenglamaning yechimi 2 ga teng, $\frac{dP}{dt} = kP$ tenglamaning yechimi esa $P(t) = ce^{kt}$ funksiya.

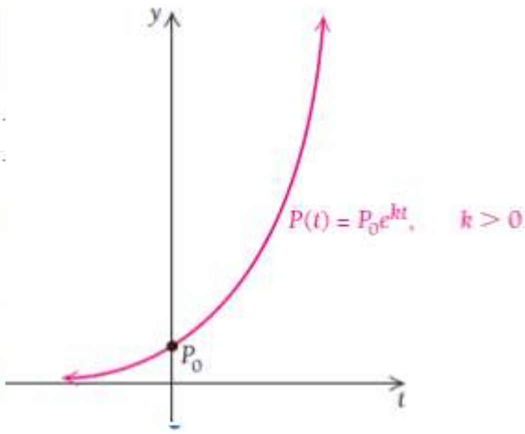
Hosila qatnashgan va yechimi funksiyadan iborat bo‘ladigan $\frac{dP}{dt} = kP$ ko‘rinishidagi tenglamaga **differensial tenglama** deyiladi.

2-vazifa. $\frac{dN}{dt} = kN$ tenglikni qanoatlantiruvchi funksiyani toping.

$\frac{dP}{dt} = kP$ yoki $P'(t) = kP(t),\ k > 0$ tenglamaga **erkin (chegaralanmagan) o‘sish modeli** deyiladi.

Bu tenglamani faqat $P(t) = ce^{kt}$ funksiya qanoatlantiradi, bu yerda $t -$vaqt, $k -$o‘sish koeffitsiyenti, $P(0) = ce^{k0} = c$. Ushbu funksiyaning elementlari odamlar, tabiatdagi bakteriyalar, kompaniyaga foyda keltiradigan davomli investitsiya bo‘lishi mumkin. Bu funksiyaning o‘sishi uchun hech qanday to‘sqinlik va qarshiliklar bo‘lmasligi kerak, bu funksiyaning o‘sishi uning o‘lchamiga bog‘liq bo‘ladi.

$P(t) = P_0 e^{kt},\ k > 0$ funksiya grafigi “demografik o‘sish” sur’atini ko‘rsatadi. Agar o‘limlar, tabiiy ofatlar bo‘lmaganida bu ideal formula.

$k = const$ songa eksponensial o‘sish sur’ati deyiladi. k soni $\frac{dP}{dt} = kP$ tenglikka bog‘liq ravishda o‘zgarmaydi, u $t-$vaqtning istalgan paytida o‘zgaradigan aholi sonining hosilasi emas.

k soni bank tomonidan to‘lanadigan har kungi foiz stavkasiga o‘xshaydi. Agar har kungi foiz stavkasi $\frac{0.07}{365}$ bo‘lsa, u holda ma’lum P balans dollar kursi bo‘yicha kuniga $\frac{0.07}{365} \cdot P$ ga o‘sib boradi. Chunki pulni 1 yilga 7% lik omonat turiga qo‘yganmiz. Agar foizlar uzluksiz qo‘shib borilsa, eksponensial o‘sish guvohi bo‘lasiz.

3-misol. Bankda omonatning o‘sishi. Faraz qiling, P_0 dollarni 1 yilga 7% lik omonat turiga qo‘ydingiz. U holda omonatingiz $\frac{dP}{dt} = 0.07P$ tenglik bo‘yicha o‘sib boradi.

a) Tenglamani qanoatlantiruvchi funksiyani toping, uni P_0 va 0.07 ga mos yozing.

b) Agar 100% qo‘yilsa, 1 yildan keyin qancha bo‘ladi?

c) Qancha vaqtdan keyin 100$ pulingiz 2 barobar ko‘payadi?

Yechilishi: ► a) $P(t) = P_0e^{0.07t}$, $P(0) = P_0$;

b) $P(1)=100e^{0.07\cdot 1}=100e^{0.07}\approx 100\cdot 1.072508\approx 107.25\$$;

c) $P(t)=200\$$ bo‘lishi uchun $200=100e^{0.07t}$

tenglikning har 2 tomonini 100 ga bo‘lamiz: $2=e^{0.07t}$

tenglikning har 2 tomonini natural logarifmlaymiz: $\ln 2=\ln e^{0.07t}$

$$\ln 2=0.07t$$

$$t=\frac{\ln 2}{0.07}\approx 9.9$$

Demak, 100$ pulimiz 9.9 yildan keyin 2 marta ko‘payar ekan.◀

3-vazifa. 3-misolni 4 % lik omonat turi uchun yeching.

Keling o‘sish sur’ati va omonatni 2 marta ko‘paytiradigan vaqt uchun umumiy formula chiqaramiz: $2P_0=P_0e^{kt}$

$$2=e^{kt}$$

$$\ln 2=\ln e^{kt}$$

$$\ln 2=kt.$$

E’tibor bering, ushbu tenglikda k va t orasidagi munosabat P_0 dastlabki qiymatga bog‘liq emas.

9-teorema. O‘sish koeffitsiyentini k va omonatni 2 marta ko‘paytiradigan t vaqtning bog‘liqlik tenglamasi $kt=\ln 2\approx 0.693147$ ga teng. Bundan $k=\frac{\ln 2}{t}\approx\frac{0.693147}{t}$ yoki $t=\frac{\ln 2}{k}\approx\frac{0.693147}{k}$ o‘rinli.

4-misol. Facebook abonentlari. Facebook ijtimoiy tarmog‘i odamlarni o‘zaro “do‘stlar” deb nomlangan aloqa bilan bog‘laydi. Facebook abonentlari har 6 oyda 2 marta ko‘payishi ma’lum bo‘lsa, uning eksponensial o‘sish koeffitsiyentini foizda toping.

Yechilishi: ► $k=\frac{0.693147}{t}$ dan foydalanamiz. $t=6$ oy shunga ko‘ra,

$k=\frac{0.693147}{t}=\frac{0.693147}{6}=0.116$ ni topamiz, ya’ni Facebook abonentlari har oyda 11.6% ga ko‘payar ekan. ◄

5-misol. Yer shari aholisining o‘sishi. 2000 yilda yer shari aholisi soni 6.04 mlrd ga yaqin bo‘lgan. Bu son o‘sish yiliga 1.6% ni tashkil qiladigan eksponensial formula $\frac{dP}{dt}=0.016P$ bo‘yicha hisoblangan.

a) Tenglamani qanoatlantiruvchi funksiyani toping va unga $P_0=6.04,\ k=0.016$ qiymatlarni qo‘ying.

b) 2020 yil boshida aholi soni qanchaga yetishini hisoblang, $t=20$.

c) 2000 yildan qancha vaqt o‘tib, aholi soni 2 marta ko‘p bo‘ladi?

Yechilishi: ► a) $P(t)=6.04e^{0.016t}$;

b) $P(20)=6.04e^{0.016\cdot 20}=6.04e^{0.32}\approx 8.3179$ mlrd;

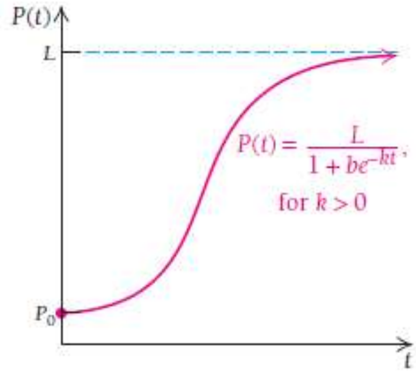
c) $t=\frac{\ln 2}{k}\approx\frac{0.693147}{0.016}=43.3$ yil.◄

4.3.3. Chegaralangan o‘sish modeli

Erkin o‘sish $P(t) = P_0 e^{kt}$, $k > 0$ modelining tatbiq qilinadigan sohalari juda ham ko‘p. Biz aholi sonini bashorat qilishda, omonat miqdorini aniqlashda qo‘llanilishini o‘tgan mavzuda ko‘rib chiqdik. Biroq o‘sishni ma’lum L – chegaradan oshirmaydigan sabablar mavjud, aytaylik, ochlik, yashash uchun joy tanqisligi yoki tabiat resurslarining yetishmasligi. Shunday modellardan biri

$$P(t) = \frac{L}{1 + be^{-kt}}, \quad k > 0$$

bo‘lib, unga **logistik funksiya** deyiladi.



6-misol. Sputnik aloqa abonentlari. Sputnik radiokompaniyasi abonentlariga 100 lab radiostansiyalar orqali tiniq signal, shuningdek, musiqa, so‘zlashuv, sport musobaqalarini taklif qiladi. MTC 2004 yilda ish boshladi. Ucell 2008 yilda ishga tushirildi. Nimadir ro‘y berdi va Ucell kompaniyasi 2014 yilda MTC ni qo‘shib olishiga to‘g‘ri keldi. Millionlab abonentga ega kompaniyaning t yildan keying o‘sish sur’ti

$$N(t) = \frac{19.362}{1 + 295.392e^{-1.11t}}$$ funksiya bilan modellashtiriladi.

a) (2014 yildan boshlab) 1 yildan, 3 yildan, 5 yilda, 8 yildan keyin Ucell kompaniyasi abonentlari soni qanchaga yetadi?

b) 8 yildan keyingi o‘sish koeffitsiyentini toping.

c) Funksiya grafigini chizing.

d) Nima sababdan bu model erkin o‘smaydi?

Yechilishi: ▶ a) Kalkulyatorda hisoblaymiz:

$$N(1)=\frac{19.362}{1+295.392e^{-1.11}}=0.197 \text{ mln}; \quad N(3)=\frac{19.362}{1+295.392e^{-1.11\cdot 3}}=1.673 \text{ mln};$$

$$N(5)=\frac{19.362}{1+295.392e^{-1.11\cdot 5}}=9.013 \text{ mln}; \quad N(8)=\frac{19.362}{1+295.392e^{-1.11\cdot 8}}=18.598 \text{ mln}.$$

Ya’ni 2015 yilda abonentlar soni 197 000 kishi;

2017 yilda abonentlar soni 1 673 000 kishi;

2019 yilda abonentlar soni 9 013 000 kishi;

2022 yilda abonentlar soni 18 598 000 kishi bo‘lar ekan.

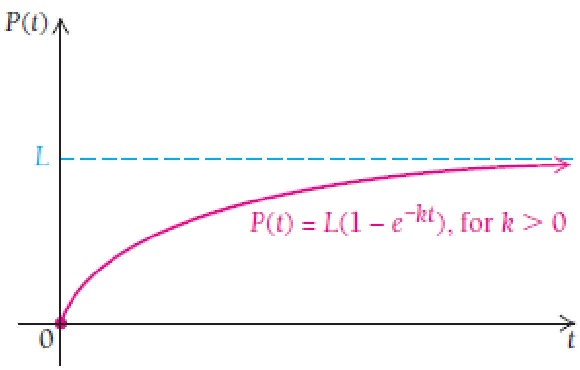
b) $N(t)=\dfrac{19.362}{1+295.392e^{-1.11t}}$

$$N'(t)=\frac{-19.362\cdot 295.392e^{-1.11t}(-1.11)}{(1+295.392e^{-1.11t})^2}=\frac{6348.533e^{-1.11t}}{(1+295.392e^{-1.11t})^2}$$

$N'(8)=0.815$ ga teng.

Demak, 8 yildan keyin abonentlar soni yiliga 0.815 tezlikda ko‘payadi, ya’ni yiliga 815 000 yangi abonentlar qo‘shiladi.

c) Funksiya grafigini chizamiz:



Bu model erkin o‘smaydi, chunki qancha odamlar ko‘p qo‘shilmasin, qo‘shilmaydigan odamlar ham bo‘ladi. Odamlar soni esa cheklangan. ◄

MUSTAQIL YECHISH UCHUN MISOLLAR

1. Agar $f'(x)=4f(x)$ bo‘lsa, $f(x)$ uchun umumiy formulani toping.

2. Agar $g'(x)=6g(x)$ bo‘lsa, $g(x)$ uchun umumiy formulani toping.

3. $\frac{dA}{dt}=-9A$ tenglamaning umumiy yechimini toping.

4. $\frac{dP}{dt}=-3P(t)$ tenglamaning umumiy yechimini toping.

5. $\frac{dP}{dt}=kP$ tenglamaning umumiy yechimini toping.

Iqtisodiyot. Tadbirkorlik.

6. Ilmiy tadqiqotlar. So‘nggi yillarda patent oluvchilar soni N keskin oshdi va yiliga o‘rtacha 4.6% ni tashkil qildi: $N'(t)=0.046N(t)$.

a) Ushbu tenglikni qanoatlantiruvchi funksiyani toping va patent uchun 112 000 ta ariza tushgan 2000 yildan $t=0$ deb boshlang.

b) 2025 yil uchun patentga arizalar sonini baholang.

c) $N(t)$ ikki marta ko‘payadigan vaqtni toping.

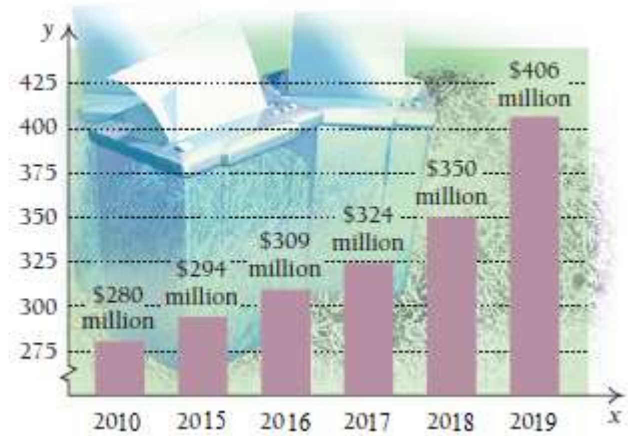
7. Butilkali suv savdosi. 2000 yildan boshlab butilkali suv savdosi hajmi H (mlrd gallon) yiliga o‘rtacha 9.3% ni tashkil qildi. Shundan kelib chiqib, t yildan so‘ng bu ko‘rsatkich $\frac{dH}{dt}=0.093H$ bo‘lishi kelib chiqadi.

a) Ushbu tenglikni qanoatlantiruvchi funksiyani toping. 2000 yilda 4.7 mlrd gallon suv sotilgan deb hisoblang.

b) 2025 yilda qancha gallon suv sotilishini baholang.

c) $H(t)$ qachon ikki marta ko‘payadi?

8. Qog‘oz maydalaydigan qurilma. Diagrammada oxirgi yillarda qog‘oz maydalaydigan qurilma savdosi keltirilgan.



a) Regressiya chizig‘idan foydalanib, $y = a \cdot b^x$ ko‘rsatkichli funksiyani toping. Agar y million dollar bo‘lsa, unda bu funksiyani asosi e bo‘lgan ko‘rsatkichli funksiyaga almashtiring va 2010 yildan keyingi o‘sish koeffitsiyentini toping.

b) 2010 -2019 yillarda jami nechta qog‘oz maydalagich sotilganini toping.

c) Qancha vaqtdan keyin qurilma savdosi 500 mln dollarga yetadi?

d) Qachon savdo 2 marta oshadi?

9. **Qog'oz maydalaydigan qurilma.** 8-misoldagi diagrammadan foydalanib, quyidagilarni toping:

a) Asosi e bo'lgan ko'rsatkichli funksiyani topishda ma'lumotlardan foydalaning. k ni (10; 280) va (19; 406) nuqtalardan foydalaning.

b) 2010 -2019 yillarda jami nechta qog'oz maydalagich sotilganini toping.

c) Qancha vaqtdan keyin qurilma savdosi 500 mln dollarga yetadi?

d) Qachon savdo 2 marta oshadi?

e) Javoblaringizni 8-misol javoblari bilan solishtiring va qaysi funksiya aniqroq ekanini tushuntiring.

10. **Umumiy daromad.** Kompyuter mikrosxemalari ishlab chiqaradigan Intel kompaniyasi 1986 yilda umumiy daromadini 1265 mln dollar deb, 2005 yilda esa 38.8 mlrd dollar deb e'lon qildi (AQSh qimmatli qog'ozlar va valyuta komission yig'imi bo'limi). Foyda funksiyasini eksponensial modelini tuzing va o'sish koeffitsiyentini aniqlang. Modeldan foydalanib, 2020 yildagi umumiy daromadni baholang.

4.4. a^x va $\log_a x$ funksiyalarning hosilalari

Ko‘rsatkichli funksiyaning hosilasi

Ixtiyoriy a asosli a^x ko‘rsatkichli funksiya hosilasini olishdan oldin uni e ning darajasi sifatida ifodalab olamiz.

Bilamizki, $b^{\log_b x} = x$. Shunga ko‘ra, $e^{\log_e A} = A$ yoki $e^{\ln A} = A$ o‘rinli. Agar A ni a^x bilan almashtirsak, $e^{\ln a^x} = a^x$ yoki $a^x = e^{\ln a^x}$ ga ega bo‘lamiz. Bu tenglikning har ikki tomonini differensiallaymiz:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{\ln a^x} = \frac{d}{dx}e^{x\ln a} = \frac{d}{dx}e^{(\ln a)x} = e^{(\ln a)x}\cdot\ln a = e^{\ln a^x}\ln a = a^x\ln a.$$

U holda quyidagi teorema o‘rinli bo‘ladi:

10-teorema: Ko‘rsatkichli funksiyaning hosilasi $\frac{d}{dx}a^x = a^x\ln a$ ga teng.

1-misol. Funksiyalar hosilasini toping:

a) $y = 2^x$ b) $y = (1.2)^x$ c) $y = 3^{5x}$

Yechilishi: ► a) $y' = (2^x)' = 2^x\ln 2$;

b) $y' = (1.2)^x\ln 1.2$;

c) $y' = 3^{5x}\ln 3\cdot(5x)' = 5\cdot 3^{5x}\ln 3$.

1-vazifa: a) $y = 5^x$; b) $y = 4^{3x}$; c) $y = (3.2)^x$ funksiyalar hosilasini toping.

$(a^x)' = a^x\ln a$ va $(e^x)' = e^x$ hosilalarni taqqoslab ko‘ramiz. $(e^x)' = e^x$ formula oddiy ko‘rinishda ekani uning asosida e bo‘lgani uchun. Tabiat hodisalarini e orqali ifodalaganimizda qo‘shimcha shartlar kiritish kerak bo‘ladi. Shuning uchun endi amaliy masalalarda a^x ni qo‘llasak, ana shu shartlarni kiritishga hojat qolmaydi.

Agar $f(x)=a^x$ bo‘lsa, $f'(x)=a^x\ln a$ ekanini isbotladik. Alternativ isbotni biz 4.1.2 mavzuda bergan edik, ya’ni $f(x)=a^x$ funksiya hosilasini $f'(x)=a^x\lim\limits_{\Delta x\to 0}\dfrac{a^{\Delta x}-1}{\Delta x}$ ga teng ekanini isbotlagan edik.

$a^x\ln a=a^x\lim\limits_{\Delta x\to 0}\dfrac{a^{\Delta x}-1}{\Delta x}$ tenglikni har ikki tomonini a^x ga bo‘lib yuboramiz. U holda quyidagi teorema kelib chiqadi:

11-teorema: $\ln a=\lim\limits_{\Delta x\to 0}\dfrac{a^{\Delta x}-1}{\Delta x}$ tenglik o‘rinli.

$\log_a x$ funksiyaning hosilasi

a^x funksiyaning hosilasini $\ln a$ ning ifodasi shaklida ifodaladik, bundan kelib chiqadiki, $\log_a x$ funksiya hosilasini ham $\ln a$ yordamida topish mumkin. Buning uchun dastlab $\log_a x$ ni $\ln a$ orqali ifodalab olamiz (4.2.2 mavzudagi 7-xossaga ko‘ra):

$$\log_a x=\frac{\log_e x}{\log_e a}=\frac{\ln x}{\ln a}.$$

Endi hosila olamiz:

$$(\log_a x)'=\left(\frac{\ln x}{\ln a}\right)'=\frac{1}{\ln a}\cdot(\ln x)'=\frac{1}{\ln a}\cdot\frac{1}{x}=\frac{1}{x\ln a}.$$

Bundan navbatdagi teorema kelib chiqadi:

12-teorema: $(\log_a x)'=\dfrac{1}{x\ln a}$ tenglik o‘rinli.

Eslatma: Biz e asosli logarifmdan foydalanishimizga sabab, hosila olishda ortiqcha $\frac{1}{\ln a}$ o‘zgarmas son hosil bo‘lishidan qochar ekanmiz.

2-misol. Funksiyalar hosilasini toping:

a) $y = \log_3 x$ b) $y = \lg x$

c) $y = \log_8(x^2 + 3)$ d) $y = x^2 \log_5 x$

Yechilishi: ▶ a) $y' = (\log_3 x)' = \frac{1}{x \ln 3}$;

b) $y' = (\lg x)' = \frac{1}{x \ln 10}$;

c) $y' = (\log_8(x^2 + 3))' = \frac{1}{(x^2 + 3)\ln 8} \cdot (x^2 + 3)' = \frac{2x}{(x^2 + 3)\ln 8}$;

d) $y' = (x^2)' \cdot \log_5 x + x^2 \cdot (\log_5 x)' = 2x \log_5 x + \frac{x^2}{x \ln 5} = x\left(2\log_5 x + \frac{1}{\ln 5}\right)$.

2-vazifa: a) $y = \log_2 x$; b) $y = -7\lg x$;

c) $y = x^5 \lg x$; d) $y = \log_5(x^3 - 1)$

funksiyalar hosilasini toping.

MUSTAQIL YECHISH UCHUN MISOLLAR

1-30 misollar funksiya hosilasini topishga doir:

1. $y=7^x$ **2.** $y=\left(\frac{1}{3}\right)^x$ **3.** $y=\left(\frac{1}{6}\right)^x$

4. $y=x^3 7^x$ **5.** $y=x^5(1.3)^x$ **6.** $y=15^x$

7. $y=7^{x^4+2}$ **8.** $y=4^{x^2+5}$ **9.** $y=e^{x^2}$

10. $y=3^{x^4-1}$ **11.** $y=12^{5x+4}$ **12.** $y=e^{6x}$

13. $y=\log_4 x$ **14.** $y=\log_5(x+1)$

15. $y=\log_6(5x-3)$ **16.** $y=\log_8(5x^2+1)$

17. $y=\log_2(x^3+x+6)$ **18.** $y=\log_2(x^4-x+10)$

19. $y=\log_2(\sqrt{x^3}-2)$ **20.** $y=-\log_6(\sqrt[3]{x}-1)$

21. $y=6^x\log_5 x$ **22.** $y=5^x\log_2 x$

23. $y=\log_3^2 x$ **24.** $y=\log_9^7 x$

25. $y=\dfrac{7^x}{3x+1}$ **26.** $y=\dfrac{6^x}{5x-4}$

27. $y=6^{2x^3+1}\log_5(6x+7)$ **28.** $y=4^{2x^2-3}\lg(5x-10)$

29. $y=(x+1)^3\log_5 x+12$ **30.** $y=(2x-7)^2\log_3 2x-15$

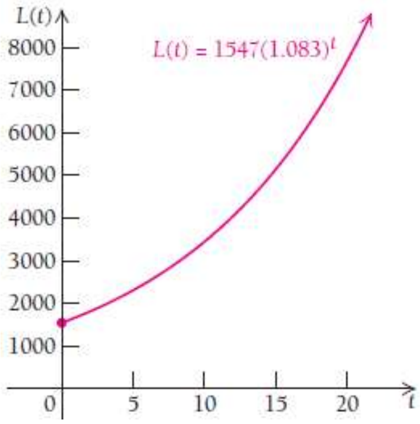
Iqtisodiyot. Tadbirkorlik.

31. Alyumin qutilarini qayta ishlash. Har yili 45% ichimlik alyumin qutilar qayta ishlanadi. Kompaniya 250 000 funt alyumin banka ishlatadi. t yilda qayta ishlangan qutilar soni $N(t)=250000(0.45)^t$ bo‘lsin.

a) $N'(t)$ ni toping;

b) $N'(t)$ nimani anglatishishi tushuntiring.

32. Oila byudjeti. Oilaning moliyaviy byudjeti 1994 yildan hisoblaganda t yilda $L(t)=1547(1.083)^t$ ga yetadi.

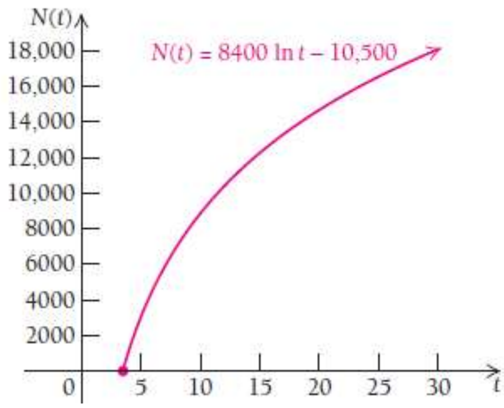


a) 2020 yilda oila byudjeti qanchani tashkil qiladi?

b) $L'(25)$ ni toping;

c) $L'(25)$ nimani anglatishishi tushuntiring.

33. Kichik biznes. Xo‘jalik mollari ishlab chiqaradigan tadbirkorning daromadi 1998 yildan hisoblaganda t yilda $N(t)=8400\ln t-10500$ ga yetadi.



a) 2020 yilda u jami qancha daromad qilgan bo‘ladi?

b) $N'(25)$ ni toping;

c) $N'(25)$ nimani anglatishishi tushuntiring.

Zilzila kuchi. Rixter shkalasi bo‘yicha o‘lchanganda uning intensivligi I quyidagi formula yordamida hisoblanadi: $R=\lg\frac{I}{I_0}$.

Bunda I_0 minimal intensivlik. Agar bir zilzila ikkinchisidan 10 marta intensivroq bo‘lsa, uning magnitudasi Rixter shkalasi bo‘yicha 1 birlik yuqori bo‘ladi. Agar 100 marta intensiv bo‘lsa, 2 birlik, … deb baholanadi. U holda Rixter shkalasi bo‘yicha 6 ballik yer silkinishi 5 ballikdan 10 marta intensive bo‘lar ekan.

34. Zilzila ofatlari. 2020 yil 30 oktabrda Turkiyadagi zilzila ko‘plab uylarning qulab tushishiga sabab bo‘ldi. Uning intensivligi $I_0 10^7$ bo‘ldi. Rixter shkalasi bo‘yicha zilzila necha ball bo‘lgan?

35. Zilzila ofatlari. 2010 yil 27 fevralda Chilidagi zilzila intensivligi $I_0 10^{8.8}$ bo‘lgan. Rixter shkalasi bo‘yicha zilzila necha ball bo‘lgan?

V BOB. INTEGRALLASH

5.1. Boshlang‘ich funksiya (antidifferensiallash)

5.2. Integrallash usullari

5.3. Kasr -ratsional va irratsional funksiyalarni integrallash

5.4. Trigonometrik funksiyalarni integrallash

5.5. Integral yordamida yuzalarni hisoblash

5.6. Aniq integralning tatbiqlari

Ushbu bobda nimalarni o‘rganamiz?

Agar transport vositasining tezlik funksiyasi haqida ma’lumotga ega bo‘lsak, uning bosib o‘tgan yo‘lini aniqlashimiz mumkinmi?

Agar kompaniyaning chegaraviy foyda funksiyasini bilsak, uning sof foydasi qancha ekanini aniqlay olamizmi?

Hisob fanining ikkinchi bo‘limi bo‘lgan integrallash jarayonidan foydalanib, yuqorida keltirilgan masalalarni hal qilish mumkin. Bilamizki, hisob fanining birinchi bo‘limi – bu differensiallash edi.

Integrallash yordamida fanda, biznesda, statistikada juda ko‘p tatbiq qilinadigan egri chiziq bilan chegaralangan soha yuzasini hisoblash mumkin.

5.1. Boshlang‘ich funksiya (antidifferensiallash)

Faraz qiling, biz differensiallashni teskarisini bajaramiz, ya’ni hosilasi berilgan funksiyaga teng bo‘lgan funksiyani izlaymiz. Bunga **antidifferensiallash (boshlang‘ich funksiyani topish)** deyiladi. Integrallash mavzusi ushbu bobning asosiy mavzusi bo‘lib, uni hisob fanining ikkinchi bo‘g‘ini deyish mumkin, chunki birinchi bo‘g‘in differensiallash jarayoni hisoblanadi.

Integrallashning juda ko‘p muhim tatbiqlari mavjud, xususan yuqoridan yoki pastddan egri chiziq bilan chegaralangan yopiq sohaning yuzasini hisoblash masalasi shular jumlasidandir.

Antidifferensiallash – differensiallash jarayoniga teskari jarayon. Bizga $f(x)$ funksiya berilgan bo‘lsin. Boshqa $F(x)$ funksiyani shunday aniqlaymiz: $F(x)$ funksiyaning hosilasi $f(x)$ ga teng, ya’ni $F'(x) = f(x)$.

Misol, agar $f(x) = 2x$ bo‘lsa, u holda uning boshlang‘ich funksiyasi $F(x) = x^2$ bo‘ladi, chunki $$F'(x) = (x^2)' = 2x.$$
Biroq boshqa bir funksiyaning hosilasi ham $2x$ bo‘lishi mumkin, aytaylik, $y = x^2 + 5$ yo $y = x^2 - 1$ yoki $y = x^2 + 560$.
Sababi x^2 ning hosilasi $2x$, o‘zgarmas sonlarning hosilasi esa nolga teng. Shuning uchun ham $f(x) = 2x$ funksiyaning boshlang‘ich funksiyasini umumiy ko‘rinishda $F(x) = x^2 + C$ deb yoziladi, bunda $C = \text{const}$.

1-teorema. $[F(x)+C]' = f(x)$ o‘rinli bo‘lsa, $F(x)+C$ funksiyalar to‘plami $f(x)$ ning boshlang‘ich funksiyasi bo‘ladi. $C-$o‘zgarmas songa **integrallash o‘zgarmasi** deyiladi.

1-teoremadan shunday xulosa qilish mumkin: agar $F(x)$ va $G(x)$ funksiyalarning hosilalari bir xil bo‘lsa, u holda ular o‘zgarmas songa farq qiladi: $F(x)=G(x)+C$.

Agar $F(x)$ funksiya $f(x)$ ning boshlang‘ich funksiyasi bo‘lsa, uni

$$\int f(x)dx = F(x)+C$$

ko‘rinishda yozamiz. Bu tenglikni x o‘zgaruvchiga nisbatan $f(x)$ ning aniqmas integrali $F(x)+C$ ga teng deb o‘qiladi. $\int$ belgi integral belgisi va integral olish uchun buyruq, $f(x)$ integral ostidagi funksiya va dx esa integrallash x o‘zgaruvchi bo‘yicha bajarilishini bildiradi. Chap tomondagi ifodaga **aniqmas integral** deyiladi.

1-misol. Quyidagi aniqmas integrallarni hisoblang va to‘g‘ri hisoblanganligini hosila yordamida tekshiring:

a) $\int 5dx$; b) $\int 5x^2dx$; c) $\int e^x dx$; d) $\int \frac{1}{x}dx$; e) $\int \sin x dx$.

Yechilishi: ►

a) $\int 5dx = 5x + C$. Tekshirish: $(5x+C)' = 5$

b) $\int 5x^2dx = \frac{5}{3}x^3 + C$. Tekshirish: $\left(\frac{5}{3}x^3 + C\right)' = 5x^2$.

c) $\int e^x dx = e^x + C$. Tekshirish: $(e^x + C)' = e^x$.

d) $\int \frac{1}{x}dx = \ln x + C$. Tekshirish: $(\ln x + C)' = \frac{1}{x}$.

e) $\int \sin x dx = -\cos x + C$. Tekshirish: $(-\cos x + C)' = \sin x$. ◀

Har doim olingan integral to‘g‘ri ekanligini hosila yordamida tekshirib ko‘ring.

2-teorema. Aniqmas integralning ba’zi qoidalari

1⁰. O‘zgarmas sonning integrali: $\int k\,dx = kx + C$;

2⁰. Darajali funksiyaning integrali: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$;

3⁰. Natural logarifm qoidasi: $\int \frac{1}{x}dx = \ln x + C$, $x > 0$;

4⁰. Eksponensial funksiya qoidasi: $\int e^{ax}dx = \frac{1}{a}e^{ax} + C$, $a \neq 0$.

2-misol. $f(x) = x^6$ funksiyaning boshlang‘ich funksiyasini toping. So‘ngra $\int x^6 dx$ ni hisoblang.

Yechilishi: ▶ $f(x) = x^6$ funksiyaning boshlang‘ich funksiyasini topish uchun 2-qoidaga ko‘ra, daraja ko‘rsatkichiga 1 ni qo‘shamiz va shu hosil bo‘lgan darajadagi songa bo‘lamiz: $F(x) = \frac{x^7}{7} + C$. Buni to‘g‘riligini hosila olib tekshirib ko‘rish mumkin: $\left(\frac{x^7}{7} + C\right)' = \frac{1}{7} \cdot 7x^6 = x^6$. ◀

Darajali funksiyaning integralini topish 2 qadamdan iborat:

1-qadam. Daraja ko‘rsatkichiga 1 qo‘shiladi;

2-qadam. Hosil bo‘lgan daraja ko‘rsatkichidagi songa bo‘linadi.

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$

3-misol. Quyidagi aniqmas integrallarni hisoblang:

a) $\int x^9 dx$; b) $\int x^{55} dx$; c) $\int \frac{1}{x^7} dx$; d) $\int \sqrt{x}\,dx$.

Yechilishi: ► a) $\int x^9 dx = \frac{x^{9+1}}{9+1} + C = \frac{1}{10}x^{10} + C;$

b) $\int x^{55} dx = \frac{x^{55+1}}{55+1} + C = \frac{1}{56}x^{56} + C;$

c) $\int \frac{1}{x^7} dx = \frac{x^{-7+1}}{-7+1} + C = -\frac{1}{6}x^{-6} + C = -\frac{1}{6x^6} + C;$

d) $\int \sqrt{x}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}\sqrt{x^3} + C.$ ◄

1-vazifa. Quyidagi aniqmas integrallarni hisoblang:

a) $\int x^{15} dx$; b) $\int x^{205} dx$; c) $\int \frac{1}{x^6} dx$; d) $\int \sqrt[9]{x}\,dx$.

Darajali funksiya x^n ning integralini hisoblaganimizda daraja ko‘rsatkichiga 1 ni qo‘shib, hosil bo‘lgan darajaga bo‘lamiz, faqat $n = -1$ da integral bu qonuniyat asosida hisoblanmaydi: $\int \frac{1}{x} dx = \ln x + C, \quad x > 0.$

4-misol. Quyidagi aniqmas integrallarni hisoblang:

a) $\int e^x dx$; b) $\int e^{5x} dx$; c) $\int e^{-7x} dx$.

Yechilishi: ► a) $\int e^x dx = e^x + C$; b) $\int e^{5x} dx = \frac{1}{5}e^{5x} + C$;

c) $\int e^{-7x}dx = -\frac{1}{7}e^{-7x} + C$. ◀

2-vazifa. Quyidagi aniqmas integrallarni hisoblang:

a) $\int e^{3x}dx$; b) $\int e^{\frac{x}{3}}dx$; c) $\int e^{-x\sqrt{5}}dx$.

3-teorema. Aniqmas integralning xossalari

1^0. O‘zgarmas ko‘paytuvchini integral belgisidan tashqariga chiqarish mumkin: $\int k\cdot f(x)\,dx = k\cdot\int f(x)dx$;

2^0. Funksiyalar algebraik yig‘indisining integrali shu funksiyalar integrallarining algebraik yig‘indisiga teng:

$$\int[f(x)\pm\varphi(x)]\,dx = \int f(x)dx \pm \int \varphi(x)dx.$$

5-misol. Quyidagi aniqmas integrallarni hisoblang:

a) $\int(2x-\sqrt[5]{x^3}-4)dx$; b) $\int(x^7-\frac{2}{\sqrt[3]{x}}+5^x)dx$

Yechilishi: ▶ a) $\int(2x+\sqrt[5]{x^3}-4)dx = x^2+\frac{x^{\frac{3}{5}+1}}{\frac{3}{5}+1}-4x+C = x^2+\frac{5}{8}\sqrt[5]{x^8}-4x+C$;

b) $\int(x^7-\frac{2}{\sqrt[3]{x}}+5^x)dx = \frac{x^8}{8}-2\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+\frac{5^x}{\ln 5}+C = \frac{x^8}{8}-3\sqrt[3]{x^2}+\frac{5^x}{\ln 5}+C$. ◀

3-vazifa. Quyidagi aniqmas integrallarni hisoblang:

$\int(3x^2+\sqrt[3]{x^8}-\frac{6}{x})dx$; b) $\int(x^3-\frac{1}{\sqrt[4]{x^3}}+6^x-1)dx$; c) $\int\frac{x^2+\sqrt[3]{x^5}-5x}{x^3}dx$.

5.1.1. Boshlang‘ich shartlar

Integrallashdagi C o‘zgarmas son ba’zi amaliy masalalarda ahamiyatli hisoblanadi. Shuning uchun bunday masalalarda funksiyaning aniqmas integralini C ga nisbatan yechib, nuqtani aniqlab olishimiz mumkin. Bu nuqtaga **boshlang‘ich shart** deyiladi.

6-misol. $f'(x) = 2x + 3$ funksiyaning $f(1) = -2$ nuqtadagi boshlang‘ich funksiyasini toping.

Yechilishi: ► Aniqmas integralni hisoblaymiz va quyidagi tenglikni

$$\int (2x+3)dx = x^2 + 3x + C$$

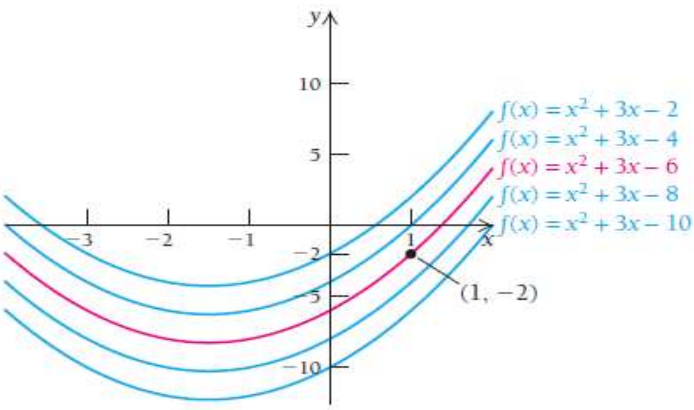
hosil qilamiz. Endi funksiyaning $f(1) = -2$ nuqtadagi qiymatini topiamiz.

$$f(x) = x^2 + 3x + C$$

$$-2 = 1^2 + 3 \cdot 1 + C$$

$$C = -6.$$

Bundan aytish mumkinki, $f'(x) = 2x + 3$ funksiyaning $f(1) = -2$ nuqtadagi boshlang‘ich funksiyasi $f(x) = x^2 + 3x - 6$ dan iborat ekan.



◄

4-vazifa. $f'(x) = e^{3x}$ funksiyaning $(0;\ 3)$ nuqtadagi boshlang‘ich funksiyasini toping.

Juda ko‘p sohalarda aniqmas integralning tatbiqlarini ko‘rish mumkin. 2.8 bo‘limda yo‘ldan hosila olib, tezlikni topgan edik. Shuning uchun ham tezlikning boshlang‘ich funksiyasi o‘tilgan yo‘lga teng bo‘ladi. Agar biror t vaqt oralig‘ida bosib o‘tilgan yo‘l qiymati ma’lum bo‘lsa, bu ma’lumot boshlang‘ich shart hisoblanadi.

7-misol. **Fizika. Tepaga otilgan ob’yektning tushish balangligi.** 10 fut balandlikda turib, tosh 50 fut/sekund tezlik bilan tepaga uloqtirildi. Toshning tezligi $v(t) = -30t + 50$ funksiya bilan modellashtirildi, bunda t sekund hisobida bo‘lib, $t = 0$ tosh otilgan moment.

a) h masofani (balandlikni) t ning funksiyasi sifatida aniqlang. Bunda 10 fut balandlikni e’tiborga oling.

b) 3 sekund o‘tgandagi toshning tezligi va balandligini hisoblang.

Yechilishi: ► a) $h(t) = \int(-30t + 50)dt = -15t^2 + 50t + C$.

Integrallashdagi o‘zgarmas C sonni topish uchun boshlang‘ich tezlik 0 va boshlang‘ich balandlik 10 fut bo‘lganligini inobatga olib, (0; 10) juftlikni $h(t)$ ga olib borib qo‘yamiz: $10 = -15 \cdot 0^2 + 50 \cdot 0 + C$.

$$C = 10.$$

Shunday qilib, balandlik funksiyasini aniqlaymiz: $h(t) = -15t^2 + 50t + 10.$

b) 3 sekund o‘tgandagi toshning tezligi va balandligini hisoblaymiz:

$v(t) = -30t + 50 = -30 \cdot 3 + 50 = -40$ fut/sek;

$h(3) = -15 \cdot 3^2 + 50 \cdot 3 + 10 = 25$ fut.

Tosh 3 sekundda yerdan 25 fut balandlikda bo‘lgan, tezlikdagi manfiy ishora esa toshning tezligi kamayayotganini bildiradi.◄

5-vazifa. Sergeli tumanida 2000 yildan buyon aholi soning o'sishi $A'(t)=11.7e^{0.026t}$ tenglik bilan modellashtirildi. Bunda $A'(t)-$ bir yilda aholining ko'payish soni (ming kishi hisobida).

a) Agar 2010 yilda aholi soni 25000 kishi bo'lsa, shu tuman uchun o'sish sur'atini modellashtiring;

b) 2025 yilda Sergeli tumanida aholi soni qanchaga yetadi?

5.2. Integrallash usullari

5.2.1. Bevosita integrallash usuli

Agar funksiyaning integralini hisoblashda integrallash jadvalidan foydalanib, to'g'ridan-to'g'ri hisoblash mumkin bo'lsa, unga **bevosita integrallash** deyiladi. 5.1-bo'limda keltirilgan misollarning barchasi bevosita integrallash usulida yechildi.

Quyida integrallash jadvalini keltiramiz:

1. $\int du = u + C$

2. $\int u^n du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \dfrac{du}{u^2} = -\dfrac{1}{u} + C$

4. $\int \dfrac{du}{\sqrt{u}} = 2\sqrt{u} + C$

5. $\int \dfrac{du}{u} = \ln|u| + C$

6. $\int a^u du = \dfrac{a^u}{\ln a} + C$

7. $\int e^u du = e^u + C$

8. $\int \sin u du = -\cos u + C$

9. $\int \cos u du = \sin u + C$

10. $\int \dfrac{du}{\cos^2 u} = tgu + C$

11. $\int \frac{du}{\sin^2 u} = -ctgu + C$

12. $\int \frac{du}{\sin u} = \ln\left|tg\frac{u}{2}\right| + C$

13. $\int \frac{du}{\cos u} = \ln\left|tg\left(\frac{u}{2} + \frac{\pi}{2}\right)\right| + C$

14. $\int tgudu = -\ln|\cos u| + C$

15. $\int ctgudu = \ln|\sin u| + C$

16. $\int \frac{du}{u^2 + a^2} = \frac{1}{a} arctg\frac{u}{a} + C$

17. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$

19. $\int \frac{du}{\sqrt{u^2 + a^2}} = \ln\left|u + \sqrt{u^2 + a^2}\right| + C$.

20. $\int \frac{1}{x(a+bx)}dx = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right| + C$

21. $\int \frac{1}{x(a+bx)^2}dx = \frac{1}{a(a+bx)} + \frac{1}{a^2}\ln\left|\frac{x}{a+bx}\right| + C$

22. $\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right|\right] + C$

23. $\int x\sqrt{a+bx}\,dx = \frac{2}{15b^2}(3bx - 2a)(a+bx)^{3/2} + C$

24. $\int x^2\sqrt{a+bx}\,dx = \frac{2}{105b^3}(15b^2x^2 - 12abx + 8a^2)(a+bx)^{3/2} + C$

25. $\int \frac{x\,dx}{\sqrt{a+bx}} = \frac{2}{3b^2}(bx - 2a)\sqrt{a+bx} + C$

26. $\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2}{15b^3}(3b^2x^2 - 4abx + 8a^2)\sqrt{a+bx} + C$

5.2.2. Integrallashda differensial belgisi ostiga kiritish usuli

Differensial belgisi ostiga kiritish usulida integral ostidagi ifodani quyidagicha almashtirish mumkin:

$$\int dx = \int d(x-a) = \int d(x+a),$$

$$\int dx = \frac{1}{k}\int d(kx) = \frac{1}{k}\int d(kx+a) = \frac{1}{k}\int d(kx-a),$$

$$\int xdx = \frac{1}{2}\int d(x^2) = \frac{1}{2}\int d(x^2-a) = \frac{1}{2}\int d(x^2+a),$$

$$\int \cos xdx = \int d(\sin x), \qquad \int \frac{dx}{x} = \int d(\ln x), \quad \dots$$

8-misol. Integralni hisoblang: $\int \frac{dx}{x\ln x}$.

Yechilishi: ▶ $\int \frac{dx}{x\ln x} = \int \frac{d(\ln x)}{\ln x} = \ln(\ln x) + C$, bunda $t = \ln x$. ◀

9-misol. Integralni hisoblang: $\int e^{-x^2}\, x\, dx$.

Yechilishi: ▶ $\int e^{-x^2}\, x\, dx = -\frac{1}{2}\int e^{-x^2} d(-x^2) = -\frac{1}{2}e^{-x^2} + C$. ◀

6-vazifa. Quyidagi integrallarni hisoblang.

a) $\int \frac{arctg\, x}{1+x^2}dx$; b) $\int (\ln x)^3 \frac{dx}{x}$

5.2.3. Integrallashda o‘zgaruvchini almashtirish usuli

Jadvalda berilmagan $\int f(x)dx$ integralni hisoblash kerak bo‘lsin. x ni t erkli o‘zgaruvchining biror differensiallanuvchi funksiyasi sifatida ifodalab, integrallashda yangi t o‘zgaruvchini kiritamiz.

$x=\varphi(t)$, u holda tenglikning har ikki tomonini differensiallasak $dx=\varphi'(t)dt$ hosil bo‘ladi. Topilganlarni integraldagi x va dx larning o‘rniga qo‘yamiz:

$$\int f(x)dx=\int f[\varphi(t)]\cdot\varphi'(t)dt.$$

Shunday qilib, integral yangi t o‘zgaruvchi bo‘yicha hisoblanadi. Tenglikning o‘ng qismida integrallashdan so‘ng eski x o‘zgaruvchiga qaytiladi.

10-misol. Integralni hisoblang: $\int \frac{\cos x}{\sqrt[3]{\sin x}}dx$.

Yechilishi: ►

$$\int \frac{\cos x}{\sqrt[3]{\sin x}}dx=\int \frac{d(\sin x)}{\sqrt[3]{\sin x}}=\int \frac{dt}{t^{\frac{1}{3}}}=\frac{t^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+C=\frac{3}{2}t^{\frac{2}{3}}+C=\frac{3}{2}\sqrt[3]{\sin^2 x}+C$$ ◄

7-vazifa. Quyidagi integrallarni hisoblang.

a) $\int \sqrt{\cos x}\sin x\,dx$; b) $\int \frac{x^3}{2+x^4}dx$;

c) $\int \frac{x}{\sqrt{1-2x^2}}dx$; d) $\int \frac{x^3}{2x^4+5}dx$

5.2.4. Bo‘laklab integrallash

$u = u(x)$ va $v = v(x)$ funksiyalar berilgan bo‘lsin. Bu ikki funksiya ko‘paytmasini differensiallaymiz:

$$(u \cdot v)' = u \cdot v' + v \cdot u'$$

yoki $$d(u \cdot v) = u \cdot dv + v \cdot du$$

bundan $$u \cdot dv = d(u \cdot v) - v \cdot du$$ kelib chiqadi.

Oxirgi tenglikning ikkala tomonini integrallaymiz va quyidagini topamiz:

$$\int u \cdot dv = \int d(u \cdot v) - \int v \cdot du,$$

bunda $\int d(u \cdot v) = u \cdot v$ o‘rinli. Natijada

$$\int udv = uv - \int v \cdot du$$

bo‘laklab integrallash formulasi hosil bo‘ladi.

Odatda, ushbu formula integral ostidagi funksiya turli sinfdagi funksiyalar ko‘paytmasidan, masalan, darajali va ko‘rsatkichli, darajali va trigonometrik, trigonometrik va ko‘rsatkichli va hakozo funksiyalarning ko‘paytmasidan iborat bo‘lganda qo‘llaniladi. Bunday integrallarning ikki turini ajratib ko‘rsatish mumkin, ularda qaysi funksiyani u deb va nimani dv deb qabul qilishni aniqlab olamiz.

Birinchi tur bo‘laklab integrallashda integral ostidagi ifoda $P_n(x)$ ko‘phad bilan ko‘rsatkichli yoki ko‘phad bilan trigonometrik funksiyaning ko‘paytmasidan iborat bo‘lsa, u holda u orqali $P_n(x)$ ko‘phad belgilanadi, qolgan hamma ifoda dv deb olinadi:

11-misol. Integralni hisoblang: $\int x\cos 2x\,dx$.

Yechilishi: ►

$$\int x\cos 2x\,dx = \left| \begin{matrix} x = u & \cos 2x dx = dv \\ dx = du & \dfrac{\sin 2x}{2} = v \end{matrix} \right| = \frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2}dx =$$

$$= \frac{x\sin 2x}{2} - \left(-\frac{\cos 2x}{4}\right) + C = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C. \quad ◄$$

12-misol. Integralni hisoblang: $\int xe^{-x}\,dx$.

Yechilishi: ► $\int xe^{-x}\,dx = \left| \begin{matrix} x = u & e^{-x}dx = dv \\ dx = du & -e^{-x} = v \end{matrix} \right| = -xe^{-x} - \int -e^{-x}dx =$

$$= -xe^{-x} + \int e^{-x}dx = C - xe^{-x} - e^{-x} = C - e^{-x}(x+1). \quad ◄$$

Ikkinchi tur bo'laklab integrallashda integral ostidagi ifoda $P_n(x)$ ko'phad bilan logarifmik funksiya yoki ko'phad bilan teskari trigonometrik funksiyaning ko'paytmasidan iborat bo'lsa, unda dv bilan $P_n(x)dx$ ifoda belgilanadi, qolgan hamma ifoda u deb olinadi:

13-misol. Integralni hisoblang: $\int x^3 \ln x\,dx$.

Yechilishi: ► $\int x^3 \ln x\,dx = \left| \begin{matrix} \ln x = u & x^3dx = dv \\ \dfrac{1}{x}dx = du & \dfrac{x^4}{4} = v \end{matrix} \right| = \dfrac{x^4 \ln x}{4} - \int \dfrac{x^4}{4}\cdot\dfrac{1}{x}dx =$

$$= \frac{x^4\ln x}{4} - \frac{1}{4}\int x^3dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C = \frac{1}{4}x^4(\ln x - 0.25) + C. \quad ◄$$

Agar bo'laklab integrallashni bir necha marta bajarishga to'g'ri kelsa, u holda jadval ko'rinishida integrallash maqsadga muvofiq bo'ladi:

14-misol. Integralni hisoblang: $\int x^3 e^{2x}\,dx$.

Yechilishi: ► Bu integralni hisoblash uchun 3 marta bo‘laklab integrallash formulasini qo‘llaymiz:

$$1)\quad \int x^3 e^{2x}\,dx = \left| \begin{matrix} x^3 = u & e^{2x}dx = dv \\ 3x^2dx = du & \frac{1}{2}e^{2x} = v \end{matrix} \right| = \frac{1}{2}x^3e^{2x} - \frac{3}{2}\int x^2e^{2x}dx$$

$$2)\quad \int x^2 e^{2x}\,dx = \left| \begin{matrix} x^2 = u & e^{2x}dx = dv \\ 2xdx = du & \frac{1}{2}e^{2x} = v \end{matrix} \right| = \frac{1}{2}x^2e^{2x} - \frac{2}{2}\int xe^{2x}dx$$

$$3)\quad \int x e^{2x}\,dx = \left| \begin{matrix} x = u & e^{2x}dx = dv \\ dx = du & \frac{1}{2}e^{2x} = v \end{matrix} \right| = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx$$

$$4)\quad \int e^{2x}\,dx = \frac{1}{2}e^{2x} + C\,.$$

Agar bu integrallarni quyidagi jadval yordamida hisoblaydigan bo‘lsak, juda oson chiqadi:

$f(x)$ va takroriy differensial	Ko’paytma ishorasi	$g(x)$ va takroriy integrallash
x^3	$(+)$	e^{2x}
$3x^2$	$(-)$	$\frac{1}{2}e^{2x}$
$6x$	$(+)$	$\frac{1}{4}e^{2x}$
6	$(-)$	$\frac{1}{8}e^{2x}$
0		$\frac{1}{16}e^{2x}$

$$\int x^3 e^{2x}\,dx = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{6}{8}xe^{2x} - \frac{6}{16}e^{2x} + C = e^{2x}\left(\frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8}\right) + C.$$ ◄

8-vazifa. Quyidagi integrallarni hisoblang.

a) $\int (x-1)\,e^{2x}\,dx$;   b) $\int (2x+1)\sin 3x\,dx$.

MUSTAQIL YECHISH UCHUN MISOLLAR

1-37 misollar aniqmas integrallarni hisoblash usullariga doir:

1. $\int (x^3+2x)dx$ 2. $\int (x^{\frac{1}{3}}-4x^2)dx$ 3. $\int \left(\sqrt{x}-\frac{x}{2}\right)dx$

4. $\int (5t^2+4t+6)dt$ 5. $\int (2t-\frac{3}{\sqrt{t}})dt$ 6. $\int (t^4+\frac{4}{t}-1)dt$

7. $\int (y^2-4y+b)dy$ 8. $\int (y-\frac{1}{\sqrt{y}})dy$ 9. $\int (y+\frac{a}{y})^2 dy$

10. $\int \sqrt{\ln x}\,\frac{dx}{x}$; 11. $\int (8x-2)\sin 5x\,dx$;

12. $\int \frac{x}{\sqrt{1-2x^2}}dx$; 13. $\int (x-3)e^{-2x}dx$;

14. $\int \frac{x^3}{2x^4+5}dx$; 15. $\int \sqrt{x}\ln 3x\,dx$;

16. $\int arctg\,2x\,dx$; 17. $\int (2x+8)\,e^{-7x}dx$;

18. $\int \frac{\sin x}{\cos^2 x}dx$; 19. $\int x^3\ln x\,dx$;

20. $\int \frac{x^2}{2x^3+3}dx$; 21. $\int (3x+7)\cos 5x\,dx$;

22. $\int \sqrt{5x^4+3}\,x^3\,dx$; 23. $\int (12x+2)\sin 3x\,dx$;

24. $\int x^2 e^{x^3+1}dx$; 25. $\int \sqrt[3]{x}\ln 2x\,dx$;

26. $\int \frac{x^3}{\sqrt{8x^4-1}}dx$; 27. $\int x\sin 8x\,dx$;

28. $\int \frac{x}{2x^2+3}\,dx$;

29. $\int \arccos 5x\,dx$;

30. $\int \arcsin^2 x \frac{dx}{\sqrt{1-x^2}}$;

31. $\int \arcsin 2x\,dx$;

32. $\int \frac{\sqrt{arctg\,x}}{1+x^2}\,dx$;

33. $\int (2x-1)\cos 3x\,dx$;

34. $\int \frac{\ln x+3}{x}\,dx$;

35. $\int (8x-10)\sin 7x\,dx$;

36. $\int \sqrt{1+x^2}\,x\,dx$;

37. $\int \ln 8x\,dx$;

5.3. Kasr-ratsional va irratsional funksiyalarni integrallash

Quyidaqi to‘g‘ri kasrlarga **oddiy ratsional funksiyalar** deyiladi:

I. $\frac{A}{x-a}$,

II. $\frac{A}{(x-a)^k}$, $(k \geq 2$ va butun son$)$

III. $\frac{Ax+B}{x^2+px+q}$, (maxrajning diskreminanti $D = p^2-4q<0$).

IV. $\frac{Ax+B}{(x^2+px+q)^k}$, ($k \geq 2$ va butun, $D<0$).

Bu yerda A, B, a, p, q - haqiqiy sonlar.

Ushbu ratsional funksiyalarning integrallarini hisoblaymiz:

1. $\int \frac{A}{x-a}dx = A\ln|x-2|+C$ (jadvaldan topiladi.)

2. $\int \frac{A}{(x-a)^k}dx = A\int (x-a)^{-k}dx = -\frac{A}{(k-1)(x-a)^{k-1}}+C$

(darajali funksiyaning integrali formulasidan topiladi.)

3. $\int \frac{Ax+B}{x^2+px+q}dx$ integralda $A \neq 0$ bo‘lsa, suratida maxrajining hosilasini hosil qilib olamiz:

$$\blacktriangleright \int \frac{Ax+B}{x^2+px+q}dx = \frac{A}{2}\int \frac{(2x+p)+\left(\frac{2B}{A}-p\right)}{x^2+px+q}dx == \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx +$$

$$+\left(B-\frac{Ap}{2}\right)\int \frac{dx}{x^2+px+q} = \frac{A}{2}\ln\left(x^2+px+q\right)+\left(B-\frac{Ap}{2}\right)\int \frac{dx}{\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)}$$

Oxirgi integralda $q-\frac{p^2}{4}=\frac{4q-p^2}{4}>0$ $(D<0)$ bo‘lgani uchun, jadvaldagi $\int \frac{du}{u^2+a^2}$ integralga keladi. Demak,

$$\int \frac{Ax+B}{x^2+px+q}dx = \frac{A}{2}\ln\left(x^2+px+q\right)+\frac{2B-Ap}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}}+C\text{ . }\blacktriangleleft$$

1-misol. $\int \frac{3x-2}{2x^2+8x+26}dx$ integralni hisoblang.

Yechilishi: ► Avval maxrajidan 2 ko‘paytuvchi qavsdan chiqaramiz, suratida maxrajining hosilasini hosil qilib olamiz.

$$\frac{1}{2}\int \frac{3x-2}{x^2+4x+13}dx = \frac{3}{4}\int \frac{2x+4-4-\frac{4}{3}}{x^2+4x+13}dx = \frac{3}{4}\int \frac{2x+4}{x^2+4x+13}dx - 4\int \frac{dx}{(x+2)^2+3^2} =$$

$$= \frac{3}{4}\ln\left(x^2+4x+13\right)-\frac{4}{3}arctg\frac{x+2}{3}+C\text{ . }\blacktriangleleft$$

4. $\dfrac{Ax+B}{(x^2+px+q)^k}$, $(k \geq 2$ va butun, $D<0)$ kasrni intagrallaymiz.

$$\blacktriangleright \int \frac{Ax+B}{\left(x^2+px+q\right)^k}dx = \frac{A}{2}\int \frac{2x+p}{\left(x^2+px+q\right)^k}dx + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{\left(\left(x+\frac{p}{2}\right)^2+\frac{4q-p^2}{4}\right)^k}.$$

Bunda $\displaystyle \frac{A}{2}\int \frac{2x+p}{\left(x^2+px+q\right)^k}dx = -\frac{A}{2}\cdot \frac{1}{(k-1)\left(x^2+px+q\right)^{k-1}},$

oxirgi integralda esa $u = x+\dfrac{p}{2}$, $a = \dfrac{\sqrt{4q-p^2}}{2}$ almashtirish bajaramiz.

$$\int \frac{du}{\left(u^2+a^2\right)^k} = \frac{1}{a^2}\int \frac{\left(u^2+a^2\right)-u^2}{\left(u^2+a^2\right)^k}du = \frac{1}{a^2}\int \frac{du}{\left(u^2+a^2\right)^{k-1}} - \frac{1}{a^2}\int \frac{u^2}{\left(u^2+a^2\right)^k}du.$$

Birinchi integral berilgan integralning tartibi bittaga kamaygan holi, ikkinchi integralni bo‘laklab integrallash mumkin.

Natijada, quyidagi rekkurent formulani hosil qilamiz:

$$\int \frac{du}{\left(u^2+a^2\right)^k} = -\frac{u}{2a^2(k-1)\left(u^2+a^2\right)^{k-1}} + \frac{2k-3}{2a^2(k-1)}\int \frac{du}{\left(u^2+a^2\right)^{k-1}}. \blacktriangleleft$$

Eslatma. Agar maxrajda ax^2+bx+c ko‘phad bo‘lsa, avval a qavsdan chiqariladi: $ax^2+bx+c = a\left(x^2+\dfrac{b}{a}x+\dfrac{c}{a}\right)$

2-misol. $\displaystyle\int \frac{7x+3}{\left(x^2+2x+5\right)^2}dx$ integralni hisoblang.

Yechilishi: ▶

$$\int \frac{7x+3}{\left(x^2+2x+5\right)^2}dx = \frac{7}{2}\int \frac{2x+2-2+\frac{6}{7}}{\left(x^2+2x+5\right)^2}dx = \frac{7}{2}\int \frac{2x+2}{\left(x^2+2x+5\right)^2}dx - 4\int \frac{dx}{\left((x+1)^2+2^2\right)^2}.$$

Birinchi qo‘shiluvchi

$$\frac{7}{2}\int\frac{2x+2}{\left(x^2+2x+5\right)^2}dx=-\frac{7}{2}\cdot\frac{1}{x^2+2x+5}\ \text{ga teng.}$$

Ikkinchi integral uchun rekkurent formulani qo‘llaymiz:

$$\int\frac{dx}{\left((x+1)^2+2^2\right)^2}=-\frac{x+1}{2\cdot 2^2(2-1)\left((x+1)^2+2^2\right)^2}+\frac{2\cdot 2-3}{2\cdot 2^2(2-1)}\int\frac{d(x+1)}{(x+1)^2+2^2}=$$

$$=-\frac{x+1}{8\left((x^2+2x+5\right)^2}+\frac{1}{8}\cdot\frac{1}{2}arctg\frac{x+1}{2}.$$

Demak,

$$\int\frac{7x+3}{\left(x^2+2x+5\right)^2}dx=-\frac{7}{2\left(x^2+2x+5\right)}-\frac{x+1}{8(x^2+2x+5)^2}+\frac{1}{16}arctg\frac{x+1}{2}+C.\blacktriangleleft$$

5.3.1. Integrallashda noma’lum koeffitsiyentlar usuli

Ma’lumki, har qanday haqiqiy koeffitsiyentli ko‘phad quyidagi ko‘paytma shaklida ifodalanadi:

$$P_n(x)=a_0(x-\alpha_1)^{k_1}\cdot\ldots\cdot(x-\alpha_\beta)^{k_\beta}(x^2+p_1x+q_1)^{t_1}\cdot\ldots\cdot(x^2+p_sx+q_s)^{t_s},\ \text{(a)}$$

bu yerda $\alpha_1,\ldots,\alpha_\beta$ lar ko‘phadning $k_1,\ldots,k_\beta$ karrali haqiqiy ildizlari,

$$p_i^2-4q_i<0,\ (i=\overline{1,s})\ \text{va}\ k_1+\ldots+k_\beta+2t_1+\ldots+2t_s=n.$$

Teorema (to‘g‘ri kasrni oddiy kasrlar yig‘ndisiga ajratish haqida).

Maxraji (a) shakldа tasvirlangan har qanday to‘g‘ri ratsional kasrni I-IV turdagi oddiy kasrlar yig‘indisiga yoyish mumkin. Bu yoyilmada $P_n(x)$ ko‘phadning har bir k_r karrali α_r haqiqiy ildiziga ($(x-\alpha_r)^{k_r}$ ko‘paytuvchisiga)

$$\frac{A_1}{x-\alpha_r}+\frac{A_2}{(x-\alpha_r)^2}+\frac{A_3}{(x-\alpha_r)^3}...+\frac{A_{k_r}}{(x-\alpha_r)^{k_r}} \qquad \text{(b)}$$

ko‘rinishdagi k_r ta oddiy kasrlar yig‘indisi mos keladi. $P_n(x)$ ko‘phadning har bir juft qo‘shma - kompleks ildiziga ($(x^2+p_\gamma x+q_\gamma)^{t_\gamma}$ ko‘paytuvchisiga)

$$\frac{M_1x+N_1}{x^2+p_\gamma x+q_\gamma}+\frac{M_2x+N_2}{(x^2+p_\gamma x+q_\gamma)^2}+\frac{M_3x+N_3}{(x^2+p_\gamma x+q_\gamma)^3}+...+\frac{M_{t_\gamma}x+N_{t_\gamma}}{(x^2+p_\gamma x+q_\gamma)^{t_\gamma}} \qquad \text{(c)}$$

ko‘rinishdagi t_γ ta oddiy kasrlar yig‘indisi mos keladi.

Demak, integral ostidagi $R(x)$ to‘g‘ri ratsional kasrni (b) va (c) formulalarni e’tiborga olib noma’lum koeffitsiyentli oddiy kasrlarlarga yoyiladi. So‘ng bu kasrlarga umumiy maxraj beriladi. Yoyilmadagi A, M, N koeffitsiyentlarning qiymatlari esa

1) noma’lum koeffitsiyentlar usulini;

2) o‘rniga qo‘yish usulidan birini yoki ikkalasini qo‘llab aniqlanadi.

3-misol. $\int \frac{x^2-3x+2}{x(x+1)^2}dx$ integralni hisoblang.

Yechilishi: ► Maxrajdagi ko‘phadning $x=0$ bir karrali haqiqiy va $x=-1$ ikki karrali ildizlari bor bo‘lgani uchun

$$\frac{x^2-3x+2}{x(x+1)^2}=\frac{A}{x}+\frac{B}{(x+1)^2}+\frac{C}{x+1}.$$

umumiy maxraj berib, suratdagi ko‘phadlarni tenglaymiz:

$$x^2-3x+2\equiv Ax^2+2xA+A+Bx+Cx^2+Cx$$

yoki $$x^2-3x+2\equiv x^2(A+C)+x(2A+B+C)+A.$$

Noma’lum koeffitsiyentlar usulidan foydalanamiz, x ning darajalari oldidagi koeffitsiyentlarni tenglaymiz:

$$\begin{aligned} x^2: &\quad A+C=1; \\ x: &\quad 2A+B+C=-3; \\ x^0: &\quad A=2. \end{aligned}$$

Bundan , $A=2,\ B=-6,\ C=-1$.

Demak, $$\int \frac{x^2-3x+2}{x(x+1)^2}dx = \int \frac{2}{x}dx - \int \frac{6}{(x+1)^2}dx - \int \frac{1}{x+1}dx =$$

$$= 2\ln|x| + \frac{6}{x+1} - \ln|x+1| + C = \ln\frac{x^2}{|x+1|} + \frac{6}{x+1} + C\ . ◄$$

4-misol. $\int \frac{(x^2+3)dx}{x(x-1)(x+2)}$ integralni hisoblang.

Yechilishi: ► Integral ostida ifoda to‘g‘ri ratsional kasr bo‘lib, u I turdagi sodda kasrlar yig‘indisiga ajraladi:

$$\frac{x^2+3}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{D}{x+2},$$

bundan $x^2+3 = A(x-1)(x+2) + Bx(x+2) + Dx(x-1).$

A, B, D koeffitsiyentlarni topish uchun o‘rniga qo‘yish usulidan foydalanamiz:

$x=0$ bo‘lganda $3=-2A$, bundan $A=-\frac{3}{2}$;

$x=1$ bo‘lganda $4=3B$, bundan $B=\frac{4}{3}$;

$x=-2$ bo‘lganda, $7=6D$, bundan $D=\frac{7}{6}$.

Shunday qilib , quyidagini hosil qilamiz :

$$\int \frac{(x^2+3)dx}{x(x-1)(x+2)} = -\frac{3}{2}\int \frac{dx}{x} + \frac{4}{3}\int \frac{d(x-1)}{x-1} + \frac{7}{6}\int \frac{d(x+2)}{x+2} =$$

$$= -\frac{3}{2}\ln|x| + \frac{4}{3}\ln|x-1| + \frac{7}{6}\ln|x+2| + C = \ln\sqrt[6]{\frac{(x-1)^8|x+2|^7}{|x|^9}} + C\,. ◀$$

5-misol. $\int \frac{dx}{x^3+8}$ integralni hisoblang.

Yechilishi: ► Integral ostida to‘g‘ri ratsional kasrning maxrajidagi ko‘phadni ko‘paytuvchilarga ajratamiz va sodda kasrlar yig‘indisi shaklida ifodalaymiz:

$$\frac{1}{x^3+8} = \frac{1}{(x+2)(x^2-2x+4)} = \frac{A}{x+2} + \frac{Mx+N}{x^2-2x+4}.$$

Umumiy maxraj berib, suratlarini tenglaymiz:

$$A(x^2-2x+4) + Bx(x+2) + C(x+2) \equiv 1$$

A, M, N koeffitsiyentlarni topish uchun yuqoridagi usullarni birga qo‘llaymiz:

$$x=-2: \quad 12A = 1$$
$$x^2: \quad A + B = 0;$$
$$x^0: \quad 4A + 2C = 1.$$

Bundan , $A = 1/12$, $B = -1/12$, $C = 1/3$ va $\frac{1}{x^3+8} = \frac{1}{12(x+2)} - \frac{x-4}{12(x^2-2x+4)}$.

Endi integralni hisoblaymiz:

$$\int \frac{dx}{x^3+8} = \frac{1}{12}\int \frac{dx}{x+2} - \frac{1}{12}\int \frac{x-4}{x^2-2x+4}dx = \frac{1}{12}\ln|x+2| - \frac{1}{12\cdot 2}\int \frac{(2x-2)-6}{x^2-2x+4}dx =$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\int \frac{(2x-2)dx}{x^2-2x+4} + \frac{1}{4}\int \frac{d(x-1)}{(x-1)^2+\left(\sqrt{3}\right)^2} =$$

$$=\frac{1}{12}\ln|x+2|-\frac{1}{24}\ln\left|x^2-2x+4\right|+\frac{1}{4\sqrt{3}}arctg\frac{x-1}{\sqrt{3}}+C=$$

$$=\ln\sqrt[24]{\frac{(x+2)^2}{x^2-2x+4}}+\frac{1}{4\sqrt{3}}arctg\frac{x-1}{\sqrt{3}}+C\ .◀$$

Eslatma. Ratsional kasr ifodani integrallash 4 qadamdan iborat:

1) uning to‘g‘ri yoki noto‘g‘ri kasr ekanligi tekshiriladi, agar noto‘g‘ri kasr bo‘lsa, u holda butun qismi ajratiladi, ko‘phad va to‘g‘ri ratsional kasr hosil qilinadi;

2) to‘g‘ri ratsional kasrni oddiy kasrlar yig‘indisiga ajratiladi;

3) yoyilmaning koeffitsiyentlari topiladi;

4) ifoda integrallanadi.

5.3.2. Ba’zi irratsional funksiyalarni integrallash

Har qanday irratsional funksiyani elementar funksiya ko‘rinishidagi boshlang‘ich funksiyasini aniqlab bo‘lmaydi. Quyida ayrim almashtirishlar yordamida ratsional funksiyalar integrallariga olib kelinadigan irratsional funksiyalarning integrallarini ko‘rib chiqamiz.

I. Ushbu

$$\int R\left(x,\left(\frac{ax+b}{cx+d}\right)^{\frac{r_1}{s_1}},\left(\frac{ax+b}{cx+d}\right)^{\frac{r_2}{s_2}},\ldots\left(\frac{ax+b}{cx+d}\right)^{\frac{r_n}{s_n}}\right)dx\ ,$$

(bu yerda R-ratsional funksiya, a, b, c , d - o‘zgarmas sonlar, r_i , s_i musbat butun sonlar) integral $\frac{ax+b}{cx+d}=t^m$ almashtirish yordamida ratsionallashtiriladi. Bu yerda m - $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots \frac{r_n}{s_n}$ kasrlarning umumiy maxraji, ya'ni $m = EKUB(s_1, s_2, \ldots s_n)$.

Xususan, $\int R\left(x, x^{\frac{r_1}{s_1}}, x^{\frac{r_2}{s_2}}, \ldots x^{\frac{r_n}{s_n}}\right)dx$ integral $x = t^m$ almashtirish yordamida ratsionallashtiriladi.

6-misol. $\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+4}dx$ integralni hisoblang.

Yechilishi: ► $EKUB(2, 4) = 4$ bo'lgani uchun,

$$\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+4}dx = \begin{vmatrix} x = t^4 \\ dx = 4t^3dt \end{vmatrix} = 4\int \frac{t^5}{t^3+4}dt = 4\int\left(t^2 - \frac{4t^2}{t^3+4}\right)dt = \frac{4}{3}t^3 - \frac{16}{3}\ln\left|t^3+4\right| + C =$$

$$= \left|t = \sqrt[4]{x}\right| = \frac{4}{3}\sqrt[4]{x^3} - \frac{16}{3}\ln\left|\sqrt[4]{x^3}+4\right| + C. ◄$$

II. $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$ ko'rinishidagi integralni hisoblaymiz.

Dastlab, kasrning suratida ildiz ostidagi kvadrat uchhadning differensialini hosil qilamiz ($A \neq 0$), kvadrat uchhaddan to'la kvadrat ajratamiz:

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx = \frac{A}{2a}\int \frac{(2ax+b)dx}{\sqrt{ax^2+bx+c}} + \left(B - \frac{Ab}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}} =$$

$$= \frac{A}{a}\sqrt{ax^2+bx+c} + \left(B - \frac{Ab}{2a}\right)\int \frac{dx}{\sqrt{a\left(x+\frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)}}.$$

Agar $c \neq \frac{b^2}{4a}$, $a > 0$ bo'lsa, oxirgi integralni

$$\int \frac{du}{\sqrt{u^2+k}} = \ln\left|u+\sqrt{u^2+k}\right| + C$$

integralga keltirib hisoblash mumkin.

Agar $c > \frac{b^2}{4a}$, $a < 0$ bo‘lsa, $\int \frac{du}{\sqrt{k^2-u^2}} = \arcsin\frac{u}{k} + C$

integralga keltirib hisoblash mumkin.

Eslatma. Qulaylik uchun, kvadrat uchhadni to‘la kvadrat ajratishdan avval a ning modulini ildizdan chiqarish kerak.

7-misol. $\int \frac{5x+3}{\sqrt{x^2-4x+8}}dx$ integralni hisoblang.

Yechilishi: ►

$$\int \frac{5x+3}{\sqrt{x^2-4x+8}}dx = \frac{5}{2}\int \frac{2x-4}{\sqrt{x^2-4x+8}}dx + \left(3-\frac{5\cdot 4}{2}\right)\int \frac{dx}{\sqrt{x^2-4x+8}} =$$

$$= 5\sqrt{x^2-4x+8} - 7\int \frac{dx}{\sqrt{(x-2)^2+4}} = 5\sqrt{x^2-4x+8} - 7\ln\left|x-2+\sqrt{(x-2)^2+4}\right| + C$$

◄

8-misol. $\int \frac{3x-2}{\sqrt{10-8x-2x^2}}dx$ integralni hisoblang.

Yechilishi: ► Qulaylik uchun, avval 2 ni ildizdan chiqarib olamiz.

$$\int \frac{3x-21}{\sqrt{5-8x-2x^2}}dx = \frac{1}{\sqrt{2}}\int \frac{3x-2}{\sqrt{5-4x-x^2}}dx = \frac{\sqrt{2}}{2}I_1.$$

Hosil bo‘lgan integralni hisoblaymiz.

$$I_1 = \int \frac{3x-2}{\sqrt{5-4x-x^2}}dx = \int \frac{-\frac{3}{2}(-4-2x)+8}{\sqrt{5-4x-x^2}}dx = -\frac{3}{2}\int \frac{(-4-2x)dx}{\sqrt{5-4x-x^2}} +$$

$$+8\int \frac{dx}{\sqrt{1-(x+2)^2}} = -3\sqrt{5-4x-x^2} + 8\arcsin(x+2) + C_1.$$

Demak, $\int \frac{3x-2}{\sqrt{10-8x-2x^2}}dx = -\frac{3\sqrt{2}}{2}\sqrt{5-4x-x^2} + 4\sqrt{2}\arcsin(x+2) + C$.◀

III. Agar integral $\int \frac{Ax+B}{(x-\alpha)\sqrt{ax^2+bx+c}}dx$ ko‘rinishda bo‘lsa, u holda integral $x-\alpha = \frac{1}{t}$ almashtirish yordamida hisoblanadi.

9-misol. $\int \frac{dx}{(x+1)\sqrt{x^2+2x+10}}$ integralni hisoblang.

Yechilishi:▶ $\int \frac{dx}{(x+1)\sqrt{x^2+2x+10}} = \left| \begin{matrix} x+1 = \frac{1}{t} \\ dx = -\frac{1}{t^2}dt \end{matrix} \right| = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}+9}} = -\int \frac{dt}{\sqrt{9t^2+1}} =$

$= -\frac{1}{3}\ln\left|3t + \sqrt{9t^2+1}\right| + C = -\frac{1}{3}\ln\left|\frac{3}{x+1} + \sqrt{\frac{9}{(x+1)^2}+1}\right| + C$.◀

10-misol. Ushbu integralni hisoblang. $\int \frac{dx}{x\sqrt{5x^2-2x+1}}$.

Yechilishi:▶ O‘rniga qo‘yishni bajaramiz: $z = \frac{1}{x}$, $x = \frac{1}{z}$, $dx = -\frac{dz}{z^2}$.

So‘ngra ushbuni hosil qilamiz:

$$\int \frac{dx}{x\sqrt{5x^2-2x+1}} = -\int \frac{dz}{z^2 \cdot \frac{1}{z}\sqrt{\frac{5-2z+z^2}{z^2}}} =$$

$$= -\int \frac{dz}{\sqrt{5-2z+z^2}} - \int \frac{dz}{\sqrt{(z-1)^2+4}} =$$

$$= C - ln\left|z - 1 + \sqrt{5-2z+z^2}\right| = C - ln\left|\frac{1}{x} - 1 + \sqrt{5 - \frac{2}{x} + \frac{1}{x^2}}\right| \blacktriangleleft$$

IV. Agar integral $\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$ ko‘rinishda bo‘lsa, kvadrat uchhadni to‘la kvadratga ajratib, quyidagi

1) $\int R\left(u, \sqrt{k^2-u^2}\right)du$, $u = k\sin t (u = k\cos t)$ almashtirish;

2) $\int R\left(u, \sqrt{k^2+u^2}\right)du$, $u = ktgt (u = kctgt)$ almashtirish;

3) $\int R\left(u, \sqrt{u^2-k^2}\right)du$, $u = \dfrac{k}{\sin t}\left(u = \dfrac{k}{\cos t}\right)$ almashtirish

yordamida hisoblanadigan integrallardan biriga keltirish mumkin.

11-misol. $\int \sqrt{3+2x-x^2}\,dx$ integralni hisoblang.

Yechilishi: ► $\int \sqrt{3+2x-x^2}\,dx = \int \sqrt{4-(x-1)^2}\,dx = \begin{vmatrix} x-1 = 2\sin t \\ dx = 2\cos t dt \end{vmatrix} =$

$$= \int \sqrt{4-4\sin^2 t}\cdot 2\cos t dt = 4\int \cos^2 t dt = 2\int (1+\cos 2t)dt = 2t + \sin 2t + C =$$

$$= 2t + 2\sin t\sqrt{1-\sin^2 t} + C = 2\arcsin\frac{x-1}{2} + \frac{(x-1)\sqrt{3+2x-x^2}}{2} + C. ◄$$

12-misol. $\int \dfrac{dx}{\sqrt{\left(x^2+4x+5\right)^3}}$ integralni hisoblang.

Yechilishi: ►

$$\int \frac{dx}{\sqrt{\left(x^2+4x+5\right)^3}} = \int \frac{dx}{\sqrt{\left((x+2)^2+1\right)^3}} = \begin{vmatrix} x+2 = tgt \\ dx = \dfrac{dt}{\cos^2 t} \end{vmatrix} = \int \frac{dt}{\cos^2 t\sqrt{\left(tg^2 t+1\right)^3}} =$$

$$= \int \frac{dt}{\cos^2 t\sqrt{\left(tg^2 t+1\right)^3}} = \int \cos t dt = \sin t + C = \frac{tgt}{\sqrt{tg^2 t+1}} + C = \frac{x+2}{\sqrt{x^2+4x+5}} + C. ◄$$

MUSTAQIL YECHISH UCHUN MISOLLAR

1-24 misollar ratsional va irratsional ifodalarni integrallashga doir:

1. $\int \frac{x^3 dx}{x-2}$;

2. $\int \frac{x^4 dx}{x^2+2}$;

3. $\int \frac{3x+5}{x^2-4x+5} dx$;

4. $\int \frac{5x+2}{x^2+2x+10} dx$;

5. $\int \frac{(x+1)^3 dx}{x^2-x}$;

6. $\int \frac{2x^2-5x+1}{x^3-2x^2+x} dx$;

7. $\int \frac{7x-15}{x^3-2x^2+5} dx$;

8. $\int \frac{3x^2+2x+1}{(x+1)^2(x^2+1)} dx$;

9. $\int \frac{2x+1}{(x^2+2x+5)^2} dx$;

10. $\int \frac{x+1}{x^4+4x^2+4} dx$;

11. $\int \frac{x+20}{x^3-8} dx$;

12. $\int \frac{3x+1}{x^3+x} dx$;

13. $\int \frac{3x-1}{x^3+3x} dx$;

14. $\int \frac{x^3}{2+x^4} dx$

15. $\int \frac{8x+5}{x^3+x^2+2x+2} dx$;

16. $\int \frac{3x+10}{x^2-8x+10} dx$

17. $\int x^3 \left(1+2x^2\right)^{-3/2} dx$;

18. $\int \frac{dx}{\sqrt[4]{1+x^4}}$;

19. $\int \frac{7x-2}{x^3-3x^2+x-3} dx$;

20. $\int \frac{5x-11}{x^3+4x} dx$

21. $\int \frac{dx}{x^4 \sqrt{1+x^2}}$;

22. $\int \frac{dx}{x \sqrt[3]{1+x^5}}$;

23. $\int \frac{dx}{x^2 \left(2+x^3\right)^{5/3}}$;

24. $\int \frac{dx}{\sqrt{x^3} \sqrt[3]{1+\sqrt[4]{x^3}}}$;

5.4. Trigonometrik funksiyalarni integrallash

Bizga faqat trigonometrik funksiyalar qatnashgan ratsional ifoda berilgan bo‘lsin. Uni har doim trigonometrik formulalardan foydalanib, $sinx$ va $cosx$ orqali ifodalash mumkin. Bu ifodani $R(sinx,\ cosx)$ deb belgilaymiz. Ushbu $\int R(\sin x, \cos x)dx$ turdagi integralni $tg\frac{x}{2} = z$ o‘rniga qo‘yish bilan z o‘zgaruvchili rasional funksiyaning integraliga almashtirish mumkin. Integralni bunday almashtirishga **rasionallashtirish** deyiladi.

Agar $tg\frac{x}{2} = z$ belgilash kiritsak, undan x ni topib olamiz:

$$x = 2arctgz$$

Tenglikni har ikki tomonini differensiallaymiz va dx ni aniqlaymiz:

$$dx = \frac{2dz}{1+z^2}.$$

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2} = \frac{2z}{1+z^2}; \qquad \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2} = \frac{1-z^2}{1+z^2};$$

$sinx$ va $cosx$ larni ham yarim burchak tangensiga almashtiramiz va berilgan integralga qo‘yamiz. Natijada quyidagini hosil qilamiz:

$$\int R(\sin x, \cos x)dx = \int R\left(\frac{2z}{1+z^2}, \frac{1-z^2}{1+z^2}\right)\cdot\frac{2dz}{1+z^2}.$$

1-misol. Ushbu $\int\frac{dx}{\sin x + 3\cos x + 1}$ integralni hisoblang.

Yechilishi: ▶ Universal $tg\frac{x}{2}=z$ o‘rniga qo‘yishdan foydalanamiz:

$$\int\frac{dx}{\sin x+3\cos x+4}=\int\frac{\frac{2dz}{1+z^2}}{\frac{2z}{1+z^2}+3\cdot\frac{1-z^2}{1+z^2}+4}=\int\frac{2dz}{z^2+2z+7}=\int\frac{dz}{(z+1)^2+\left(\sqrt{6}\right)^2}=$$

$$=\frac{1}{\sqrt{6}}arctg\frac{z+1}{\sqrt{6}}+C=\frac{1}{\sqrt{6}}arctg\frac{tg\frac{x}{2}}{\sqrt{6}}+C.$$ ◀

$tg\frac{x}{2}=z$ o‘rniga qo‘yish $R(sinx,\ cosx)$ ko‘rinishdagi har qanday funksiyani integrallashga imkon beradi, shuning uchun uni **universal trigonometrik almashtirish** deyiladi. Biroq amaliyotda bu almashtirish ancha murakkab ratsional funksiyaga olib keladi. Shuning uchun ko‘pincha undan foydalanmasdan soddaroq almashtirishlardan foydalaniladi.

5.4.1. $z=\cos x,\ z=\sin x,\ z=tgx$ almashtirishlar

a) Agar $R(sinx,\ cosx)$ funksiya ***sinx* ga nisbatan toq** bo‘lsa, ya’ni

$$R(-sinx,\ cosx)=-R(sinx,\ cosx)$$

bo‘lsa, u holda $z=cosx,\ \ dz=-sinxdx$

o‘rniga qo‘yishdan foydalaniladi.

2-misol. $I=\int\frac{\sin^3 x}{2+\cos x}dx$ integralni hisoblang.

Yechilishi: ▶ Integral ostidagi funksiya $sinx$ ga nisbatan toq funksiya. Shuning uchun $z=\cos x,\ dz=-\sin xdx$ almashtirishni bajaramiz:

$$I=\int\frac{\sin^2 x\cdot\sin xdx}{2+\cos x}=\int\frac{(1-\cos^2 x)\sin xdx}{2+\cos x}=-\int\frac{(1-z^2)dz}{2+z}=\int\frac{z^2-1}{2+z}dz=$$

$$=\int\left(z-2+\frac{3}{z+2}\right)dz=\frac{z^2}{2}-2z+3\ln|z+2|+C=\frac{\cos^2 x}{2}-2\cos x+3\ln|\cos x+2|+C \blacktriangleleft$$

b) Agar $R(sinx,\ cosx)$ funksiya $\boldsymbol{cosx}$ **ga nisbatan toq** bo‘lsa, ya’ni $R(sinx,-cosx)=-R(sinx,cosx)$ bo‘lsa, u holda

$$z=sinx,\ \ dz=cosxdx$$

o‘rniga qo‘yishdan foydalaniladi.

c) Agar $R(sinx,\ cosx)$ funksiya $\boldsymbol{sinx}$ **va** $\boldsymbol{cosx}$ **ga nisbatan juft** bo‘lsa, ya’ni $R(-sinx,-cosx)=R(sinx,cosx)$ bo‘lsa, u holda

$$z=tgx,\ \ dz=\frac{dz}{1+z^2}$$

almashtirishdan foydalaniladi. Bu holda

$$sin^2x=\frac{tg^2x}{1+tg^2x}=\frac{z^2}{1+z^2},\quad cos^2x=\frac{1}{1+tg^2x}=\frac{1}{1+z^2}.$$

3-misol. $I=\int\frac{dx}{1+\sin^2 x}$ integralni hisoblang.

Yechilishi: ►Integral belgisi ostidagi funksiya juft funksiya, shuning uchun $z=tgx$ almashtirishni bajaramiz. U holda,

$$x=arctgz,\ \ dx=\frac{dz}{1+z^2};\qquad \sin^2 x=\frac{z^2}{1+z^2}.$$

Natijada ,

$$I=\int\frac{dx}{1+\sin^2 x}=\int\frac{\frac{dt}{1+z^2}}{1+\frac{z^2}{1+z^2}}=\int\frac{\frac{dz}{1+z^2}}{\frac{1+z^2+z^2}{1+z^2}}=\int\frac{dz}{1+2z^2}=\frac{1}{2}\int\frac{dz}{\frac{1}{2}+z^2}=\frac{1}{2}\int\frac{dz}{\left(\sqrt{\frac{1}{2}}\right)^2+z^2}=$$

$$=\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{1}{2}}}arctg\frac{z}{\sqrt{\frac{1}{2}}}+C=\frac{\sqrt{2}}{2}arctg\sqrt{2}Z+C=\frac{\sqrt{2}}{2}arctg\sqrt{2}tgx+C. ◄$$

d) Agar $R(\sin x,\cos x)$ funksiya *sinx* va *cosx* darajalarining ko‘paytmasi bo‘lsa, ya’ni $\int\sin^n x\cdot\cos^m xdx$ ko‘rinishdagi integralni hisoblashda, *m* va *n* ga bog‘liq holda turli almashtirishlar bajariladi:

1) agar $n>0$ va toq bo‘lsa, u holda $\cos x=z, \sin xdx=-dz$ almashtirish bajariladi;

2) agar $m>0$ va toq bo‘lsa, u holda $\sin x=z, \cos xdx=dz$ almashtirish bajariladi.

4-misol. $I=\int\frac{\sin^3 x}{\cos^4 x}dx$ integralni hisoblang.

Yechilishi: ► $\cos x=z, \sin xdx=-dz$ almashtirishni bajaramiz:

$$I=\int\frac{\sin^2 x\sin x}{\cos^4 x}dx=\int\frac{(1-\cos^2 x)\sin xdx}{\cos^4 x}=-\int\frac{(1-z)^2dz}{z^4}=-\int\frac{dz}{z^4}+\int\frac{z^2}{z^4}dz=$$

$$=\frac{1}{3z^3}-\frac{1}{z}+C=\frac{1}{3\cos^3 x}-\frac{1}{\cos x}+C. ◄$$

3) agar darajalardan biri nolga teng, ikkinchisi manfiy toq son bo‘lsa, u holda $tg\frac{x}{2}=z$ almashtirish bajariladi.

5-misol. $I=\int\frac{dx}{\sin^3 x}$ integralni hisoblang.

Yechilishi: ► Quyidagicha almashtirish bajaramiz:

$$tg\frac{x}{2}=z;\ dx=\frac{2dz}{1+z^2};\ \sin x=\frac{2z}{1+z^2}$$

Natijada, $I=\int\frac{dx}{\sin^3 x}=\int\frac{\frac{2dz}{1+z^2}}{\left(\frac{2z}{1+z^2}\right)^3}=\frac{1}{4}\int\frac{\left(1+z^2\right)^2}{z^3}dz=$

$$=\frac{1}{4}\int\frac{1+2z^2+z^4}{z^3}dz=\frac{1}{4}\int\left(\frac{1}{z^3}+\frac{2}{z}+z\right)dz=-\frac{1}{8z^2}+\frac{1}{2}\ln|z|+\frac{1}{4}\cdot\frac{z^2}{2}+C=$$

$$=-\frac{1}{8}ctg^2\frac{x}{2}+\frac{1}{2}\ln\left|tg\frac{x}{2}\right|+\frac{1}{8}tg^2\frac{x}{2}+C.$$ ◄

4) agar $m+n=-2k\leq 0$ (juft, nomusbat) bo‘lsa, u holda $tgx=z$ yoki $ctgx=z$ almashtirish integralni darajali funksiyalarning integrallari yig‘indisiga olib keladi. Xususan, $n<0$, $m<0$ va $m+n=-2k\leq 0$ bo‘lsa, kasrning suratini $1=(\sin^2 x+\cos^2 x)^s$ ifodaga almashtirish mumkin, bu yerda $s=\frac{|m+n|}{2}-1$.

6-misol . $I=\int\frac{\sin^2 x}{\cos^6 x}dx$ integralni hisoblang.

Yechilishi: ► Bu yerda $n=2, m=-6,\ m+n=-4<0$, $tgx=z,\ x=arctgz,\ dx=\frac{dz}{1+z^2}$ almashtirishni bajaramiz.

$$\frac{\sin^2 x}{\cos^6 x}=\frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\cos^4 x}=tg^2x\left(\frac{1}{\cos^4 x}\right)=tg^2x(1+tg^2x)^2=z^2(1+z^2)^2,$$

Natijada ,

$$I=\int\frac{\sin^2 x}{\cos^6 x}dx=\int z^2(1+z^2)^2\frac{dz}{1+z^2}=\int(z^2+z^4)dz=$$

$$=\frac{z^3}{3}+\frac{z^5}{5}+C=\frac{tg^3x}{3}+\frac{tg^5x}{5}+C. \blacktriangleleft$$

7-misol. $I=\int\frac{dx}{\sin^3 x\cdot\cos x}$ integralni hisoblang.

Yechilishi: ► Bu yerda $n=-3,\ m=-1,\ m+n=-4<0$

$$I=\int\frac{dx}{\sin^3 x\cdot\cos x}=\int\frac{\sin^2 x+\cos^2 x}{\sin^3 x\cdot\cos x}dx=\int\frac{1}{\sin x\cdot\cos x}dx+\int\frac{\cos x}{\sin^3 x}dx=$$

$$=2\int\frac{dx}{\sin 2x}+\int\frac{d(\sin x)}{\sin^3 x}=\ln|tgx|-\frac{1}{2\sin^2 x}+C. \blacktriangleleft$$

5.4.2. Darajani pasaytirish va yig‘indi formulalaridan foydalanib, integrallash

Agar ikkala *n* va *m* ko‘rsatkichlar juft va nomanfiy bo‘lsa, u holda trigonometriyadan ma`lum bo‘lgan

$$\sin^2 x=\frac{1-\cos 2x}{2};\quad \cos^2 x=\frac{1+\cos 2x}{2}$$

darajani pasaytirish formulalaridan foydalanamiz.

8-misol. $I=\int\sin^4 xdx$ integralni hisoblang.

Yechilishi: ► Darajani pasaytirish formulasidan foydalanamiz:

$$I=\int\sin^4 xdxd=\int(\sin^2 x)^2dx=\int\left(\frac{1-\cos 2x}{2}\right)^2dx=\frac{1}{4}\int\left(1-2\cos 2x+\cos^2 2x\right)dx=$$

$$=\frac{1}{4}\int\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)dx=\frac{1}{4}x-\frac{1}{2}\cdot\frac{\sin 2x}{2}+\frac{1}{8}x+\frac{1}{8}\cdot\frac{\sin 4x}{4}+C=$$

$$=\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C. \blacktriangleleft$$

Quyidagi ko‘rinishdagi integrallarni qarab chiqamiz:

$$\int \cos nx \cdot \cos mx dx,$$
$$\int \sin nx \cdot \cos mx dx,$$
$$\int \sin nx \cdot \sin mx dx.$$

Bunday integrallarni hisoblash uchun trigonometrik funksiyalarning ko‘paytmasini yig‘indiga almashtiruvchi formulalar qo‘llanadi:

$$\sin nx \cos mx = \frac{1}{2}\left[\sin(n+m)x + \sin(n-m)x\right]$$

$$\cos nx \cos mx = \frac{1}{2}\left[\cos(n+m)x + \cos(n-m)x\right]$$

$$\sin nx \sin mx = \frac{1}{2}\left[\cos(n-m)x - \cos(n+m)x\right]$$

9-misol. $I = \int \sin 3x \cdot \cos 2x dx$ integralni hisoblang.

►Integral ostidagi ko‘paytmani yig‘indiga almashtirib integrallaymiz.

$$I = \int \sin 3x \cdot \cos 2x dx = \frac{1}{2}\int (\sin 5x + \sin x) dx = -\frac{1}{2}\cdot\frac{\cos 5x}{5} - \frac{1}{2}\cdot\cos x + C =$$

$$= -\frac{1}{10}\cdot\frac{\cos 5x}{1} - \frac{1}{2}\cdot\cos x + C. ◄$$

5.4.3. Trigonometrik almashtirishlardan foydalanib, irratsional ifodalarni integrallash

Agar integral belgisi ostida $\sqrt{ax^2+bx+c}$ irratsional ifoda bo‘lsa-chi, uni qanday integrallaymiz?

Bizga $\int R\left(x,\ \sqrt{ax^2+bx+c}\right)dx$ ko‘rinishdagi integral berilgan bo‘lsin. Kvadrat uchhaddan to‘liq kvadrat ajratamiz:

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a},$$

va $x+\frac{b}{2a}=z,\ \ dx=dz$ belgilash kiritamiz, dastlabki integralni a va (b^2-4ac) ning ishoralariga bog‘liq holda quyidagi ko‘rinishdagi integrallardan birini topishga keltiramiz:

a) agar $a>0$ va $b^2-4ac<0$ bo‘lsa, u holda

$$\int R_1\left(z,\sqrt{m^2+n^2z^2}\right)dz,$$

bu yerda $n^2=a,\ m^2=-\frac{b^2-4ac}{4a}>0;$

b) agar $a>0$ va $b^2-4ac>0$ bo‘lsa, u holda

$$\int R_2\left(z,\sqrt{n^2z^2-m^2}\right)dz$$

bo‘ladi, bu yerda $n^2=a,\ m^2=\frac{b^2-4ac}{4a}>0;$

v) agar $a<0$ va $b^2-4ac>0$ bo‘lsa, u holda

$$\int R_3\left(z,\sqrt{m^2-n^2z^2}\right)dz$$

bo‘ladi, bu yerda $n^2 = -a,\ m^2 = -\frac{b^2-4ac}{4a} > 0$;

Bu integrallarni $\int R(sint; cost)dt$ ko‘rinishdagi integrallarga quyidagi trigonometrik almashtirishlar yordamida keltirish mumkin:

a) $z = \frac{m}{n}tgt,\ dz = \frac{m}{n} \cdot \frac{dt}{cos^2 t}$,

b) $z = \frac{m}{n}sect,\ dz = \frac{m}{n} \cdot sect \cdot tgtdt,$

v) $z = \frac{m}{n}sint,\ dz = \frac{m}{n} \cdot costdt.$

10-misol. Integralni hisoblang: $\int \frac{dx}{\sqrt{(x^2+2x+5)^3}}$

Yechilishi: ► Kvadrat uchhaddan to‘liq kvadrat ajratamiz:

$$5 + 2x + x^2 = (x+1)^2 + 4.$$

faraz qilaylik, $x + 1 = z,\ \ dx = dz$

bo‘lsin, u holda $\int \frac{dx}{\sqrt{(x^2+2x+5)^3}} = \int \frac{dz}{\sqrt{(4+z^2)^3}}$ tenglikni hosil qilamiz.

a) ko‘rinishdagi integralni hosil qilamiz. O‘rniga qo‘yishni bajaramiz:

$$z = 2tgt,\ \ dz = \frac{2dt}{cos^2 t}, \qquad 4 + z^2 = 4 + 4tg^2 t = \frac{4}{cos^2 t}.$$

Shunday qilib,

$$\int \frac{2dt}{cos^2 t\sqrt{\frac{4^3}{cos^6 t}}} = \frac{1}{4}\int costdt = \frac{1}{4}sint + C = \frac{1}{4} \cdot \frac{tgt}{\sqrt{1+tg^2 t}} + C =$$

$$= \frac{1}{4}\frac{\frac{x+1}{2}}{\sqrt{1+\frac{(x+1)^2}{4}}} = \frac{x+1}{4\sqrt{(x+1)^2+4}} + C = \frac{x+1}{4\sqrt{x^2+2x+5}} + C.$$

◄

MUSTAQIL YECHISH UCHUN MISOLLAR

1. $\int \sqrt{\cos x}\sin x\,dx$;

2. $\int \frac{dx}{3+5\cos x}$;

3. $\int (2x+1)\sin 3x\,dx$;

4. $\int \frac{dx}{\sin x+\cos x}$;

5. $\int \frac{arctg\,x}{1+x^2}dx$;

6. $\int \frac{\cos x}{1+\cos x}dx$.

7. $\int \frac{\cos x}{\sqrt[3]{\sin x}}dx$;

8. $\int x\cos 2x\,dx$;

9. $\int \frac{\sin x}{1-\sin x}dx$.

10. $\int arctg\,2x\,dx$;

11. $\int \frac{dx}{8-4\sin x+7\cos x}$;

12. $\int \frac{dx}{2\sin x+\cos x+3}$.

13. $\int (8x-2)\sin 5x\,dx$;

14. $\int \frac{1+tg\,x}{1-tg\,x}dx$;

15. $\int \frac{\sin x}{(1-\cos x)^3}dx$.

16. $\int \frac{\sin 2x}{1+\sin^2 x}dx$.

17. $\int \frac{\cos 2x}{\cos^4 x+\sin^4 x}dx$.

18. $\int \frac{\sin x}{\cos^2 x}dx$;

19. $\int \frac{\cos x}{\sin^2 x-6\sin x+5}dx$.

20. $\int (3x+7)\cos 5x\,dx$;

21. $\int \frac{dx}{1+\sin x+\cos x}$.

22. $\int (12x+2)\sin 3x\,dx$;

23. $\int x\sin 8x\,dx$;

24. $\int \arccos 5x\,dx$;

25. $\int \arcsin^2 x\frac{dx}{\sqrt{1-x^2}}$;

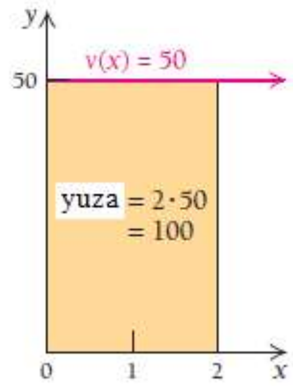
26. $\int \arcsin 2x\,dx$;

5.5. Integral yordamida yuzani hisoblash

Integral hisobdan foydalangan holda biror oraliqda funksiya grafigi va Ox o‘qi bilan chegaralangan soha yuzasi hisoblanadi. Soha turli xil shakllarda berilishi mumkin. Bu bo‘limda biz funksiyalarni faqat nomanfiy bo‘lgan holini qaraymiz.

1-misol. **Fizika. O‘tilgan yo‘lni hisoblash.** Transport vositasi 50 mill/soat tezlik bilan 2 soat harakatlandi. Transport vositasi qancha masofani bosib o‘tdi?

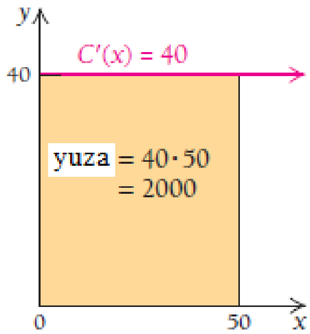
Yechilishi: ► Javob 100 mill. Transport vositasining tezligini funksiya deb qaraymiz: $v(x)=50$. Ushbu funksiyaning grafigini chizamiz, $x=2$ vertikal chiziqni o‘tkazamiz. Natijada gorizontal o‘lchami 2 birlik, vertikal o‘lchami 50 birlikka teng to‘g‘ri to‘rtburchak hosil bo‘ladi. Transport vositasining bosib o‘tgan yo‘lining qiymati ana shu shaklning yuzasiga teng bo‘ladi: $2\,\text{soat}\cdot\dfrac{50\,\text{mill}}{1\,\text{soat}}=100\,\text{mill}$.



◄

2-misol. **Tadbirkorlik. Umumiy tannarx.** Skeytbord ishlab chiqarishni boshlagan tadbirkor dastlabki 50 donasini yasadi. Uning bir donasi 40$ ga tushdi. Jami qancha harajat sarflangan?

Yechilishi: ► Tannarxning chegaraviy funksiyasi $C'(x)=40, \quad 0 \le x \le 50$ bo‘lib, uning grafigi gorizontal chiziqdan iborat. Agar Ox o‘qidan 50 birlikni ajratsak, u holda to‘g‘ri to‘rtburchak hosil qilamiz:



Bu to‘g‘ri to‘rtburchakning yuzasi $40 \cdot 50 = 2000$. Shunga ko‘ra 50 ta skeytbord ishlab chiqarish uchun jami 2000$ sarflangani kelib chiqadi:

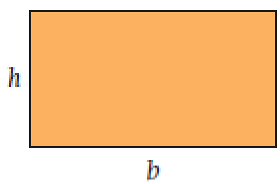
$$50\,\text{skeytbord} \cdot 40\,\frac{\text{dollar}}{\text{skeytbord}} = 2000\,\text{dollar}. \qquad ◄$$

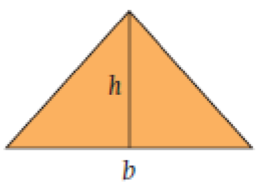
5.5.1. Integralning geometrik tatbiqlari

Agar chiziqli funksiyalar yordamida hosil qilingan yuzalarni hisoblamoqchi bo‘lsak, quyidagi formulalar bizga asqotadi.

Asosi b, balandligi h bo‘lgan to‘g‘ri to‘rtburchak yuzasi $S_{to'rt} = bh$;

Asosi b, balandligi h bo‘lgan teng yonli uchburchak yuzasi $S_{uchburch} = \frac{1}{2}bh$





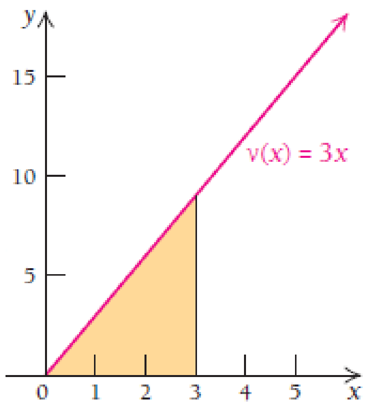
1-misoldagi bosib o‘tilgan yo‘lni va 2-misoldagi umumiy tannarxni topishning boshqa usullarini o‘rganamiz.

3-misol. Fizika. O‘tilgan yo‘lni hisoblash. Harakatlanayotgan ob’yektning tezligi $v(x)=3x$ funksiya bilan berilgan, bunda $x-$ soat hisobida, $v-$esa mill/soat hisobida o‘lchangan. Geometrik formulalar asosida quyidagilarni hisoblang:

a) Dastlabki 3 soatda ob’yekt qancha masofani o‘tgan $0\le x\le 3$?

b) 3- va 5-soatlar orasida qancha masofani bosib o‘tgan $3\le x\le 5$?

Yechilishi: ► Tezlik grafigini chizamiz:

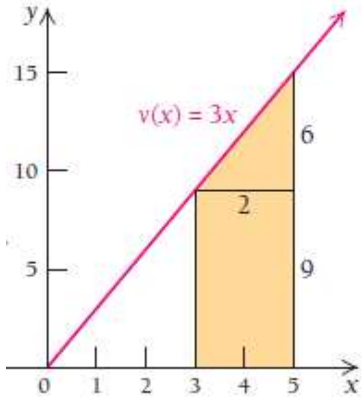


a) Chizmadan ko‘rinadiki, dastlabki 3 soatda, ob’yektning $0\le x\le 3$ vaqt oralig‘ida bosib o‘tgan yo‘li asosi 3 ga va balandligi $v(3)=9$ ga teng bo‘lgan uchburchak yuziga teng: $S=\frac{1}{2}\cdot 3\cdot 9=13.5$

Demak, ob’yekt dastlabki 3 soatda 13.5 mill masofani o‘tgan.

b) $3\le x\le 5$ soat oralig‘ida bosib o‘tgan yo‘li esa trapetsiya yuziga teng. Chizmaga ko‘ra, trapetsiya yuzi to‘g‘ri to‘rtburchak va uchburchak yuzalari yig‘indisiga teng. To‘rtburchakning asosi 2, balandligi 9, uchburchakning asosi 2 va balandligi 6 ga teng. Shunday qilib,

$$S = S_1 + S_2 = 2\cdot 9 + \frac{1}{2}\cdot 2\cdot 6 = 18 + 6 = 24$$.



Demak, ob'yekt 3- va 5-soatlar oralig'ida 24 mill masofani o'tgan.◀

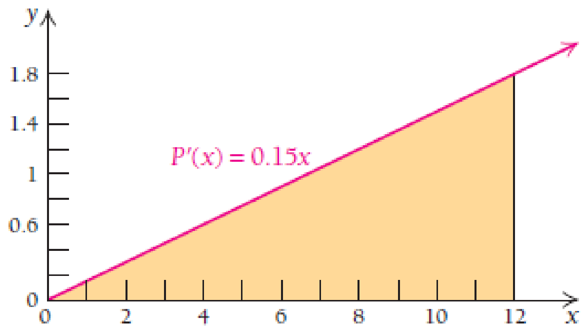
1-vazifa. Ob'yekt $v(t) = \frac{1}{3}t$ tezlik bilan harakatlanmoqda, bunda t – minut hisobida, $v(t)$ – metr hisobida.

a) Dastlabki 30 minutda ob'yekt qancha masofani bosib o'tadi?

b) Dastlabki 1- va 2-soatlar oralig'ida ob'yekt bosib o'tgan yo'lni toping.

4-misol. Tadbirkorlik. Umumiy foyda. "Anhor" ko'ngilochar bog'ining chegaraviy foydasi oyiga $P'(x) = 0.15x$ ming dollarga teng. Bog'ning 1 yillik foydasini hisoblang ($0 \le x \le 12$).

Yechilishi: ▶ $P'(x) = 0.15x$ ning grafigini chizamiz:



12 oylik davrdagi foyda to‘g‘ri burchakli uchburchak yuziga teng:

$$S = \frac{1}{2} \cdot 12\,\text{oy} \cdot 0.15 \cdot 12 \frac{\text{ming dollar}}{\text{oy}} = 10.8\,\text{ming dollar}.$$

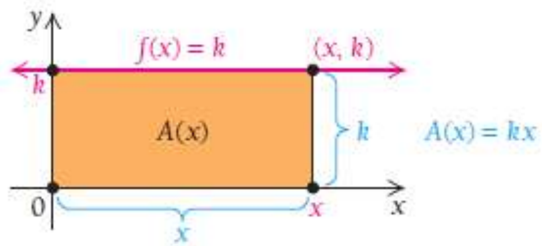
Demak, “Anhor” ko‘ngilochar bog‘i yiliga 10 800 $ foyda ko‘rar ekan◄

2-vazifa. 4-misol sharti bo‘yicha 5- va 12-oylar oralig‘idagi foydani toping.

5.5.2. Riman yig‘indilari[4]

Funksiya grafigi ostidagi soha yuzasini qanday hisoblash mumkin?

a) Agar $f(x) = k$ funksiya grafigi (bunda k – o‘zgarmas son) balandligi k ga, asosi [0, x] teng bo‘lgan to‘g‘ri to‘rtburchakni hosil qilsa, uning yuzasi $A(x) = kx$ ga teng bo‘ladi:



b) Agar $f(x) = mx$ funksiya grafigi og‘ma bo‘lib, u asosi [0, x] teng bo‘lgan to‘g‘ri burchakli uchburchakni hosil qilsa, bu funksiya ostidagi soha yuzasi $A(x) = \frac{1}{2}mx^2$ ga teng bo‘ladi:

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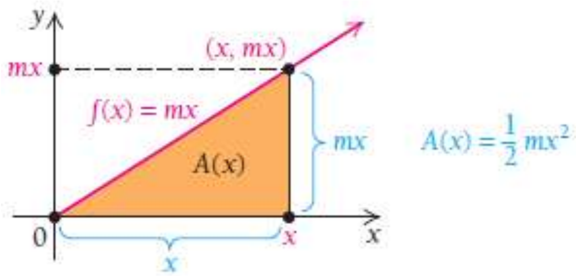
[4] **Nemis matematigi G.F.Bernxard Riman (1826-1866) tomonidan fanga kiritilgan.**



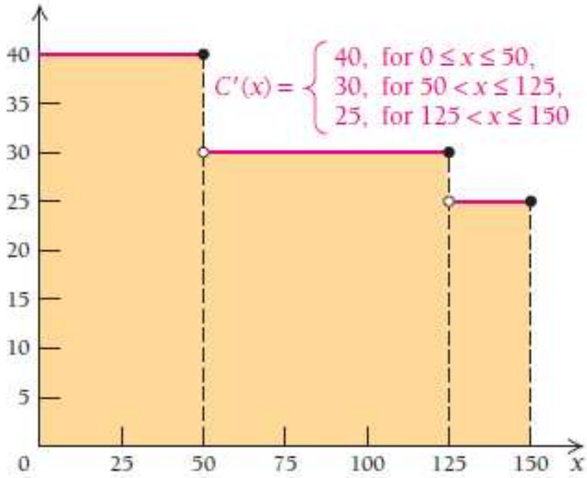
Bu ikki holatda ham $A(x)$ yuza funksiyasi – berilgan $f(x)$ funksiyaning integraliga teng. Bu har doim to‘g‘rimi? Ixtiyoriy funksiya grafigi ostidagi soha yuzasi shu funksiyadan olingan integralga tengmi? Agar funksiya egri chiziqli bo‘lsa-chi, unda soha yuzasini hisoblay olamizmi? Ushbu savollarga geometrik nuqtai nazardan javob izlaymiz. Bunda “Riman yig‘indilari” deb nomlanadigan formulalardan foydalanamiz.

5-misol. Tadbirkorlik. Umumiy tannarx. Skeytbord ishlab chiqarishda dastlabki 50 donasida uning har bir donasi 40$ bo‘ladi. Biroq 51 dan 125 donagacha ishlab chiqarilganda donasi 30$ dan tushadi. 125 donadan ko‘p chiqarilsa, tannarx 25 $ dan iborat bo‘ladi. Agar x skeytbordlar soni bo‘lsa, u holda chegaraviy tannarx funksiyasi

$$C'(x)=\begin{cases}40, & \text{agar } 0\le x\le 50\\ 30, & \text{agar } 50<x\le 125\\ 25, & \text{agar } 125<x\le 150\end{cases}$$

150 skeytbord ishlab chiqarishdagi umumiy tannarxni toping?

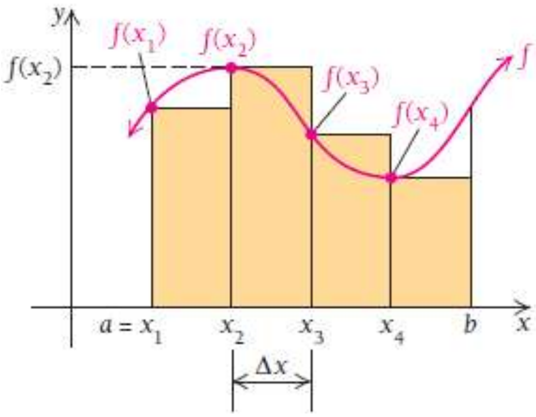
Yechilishi: ► 2-misolga o‘xshash tannarxning chegaraviy funksiyasi grafigini chizib, to‘g‘ri to‘rtburchaklarni hosil qilamiz. Shu to‘g‘ri to‘rtburchaklar yuzalarini qo‘shib chiqsak, umumiy tannarxni topgan bo‘lamiz.



Umumiy tannarx=40·50+30·75+25·25=4875 $. ◄

5-misol Riman yig‘indilarining sodda ko‘rinishini ifodalaydi.

Bizga biror $[a,b]$ kesmada $y = f(x)$ egri chiziqli funksiya berilgan bo‘lsin. 5-misolga o‘xshab, bu yerda ham $[a,b]$ kesmani 4 ta teng oraliqqa ajratib, asoslari $\Delta x = \frac{b-a}{4}$ bo‘lgan to‘g‘ri to‘rtburchaklarni hosil qilamiz.



Chizmadan ko‘rinadiki, to‘g‘ri to‘rtburchaklarning balandliklari:

$$f(x_1), \quad f(x_2), \quad f(x_3), \quad f(x_4) \qquad \text{ga teng.}$$

Egri chiziq ostidagi sohaning yuzasi taqriban shu 4 ta to‘g‘ri to‘rtburchak yuzalari yig‘indisiga teng:

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x.$$

Ushbu yig‘indini umumiy belgilashlar sistemasiga kiritilgan grek alfavitining katta “sigma” harfi bilan belgilaymiz: $\sum_{i=1}^{4} f(x_i)\Delta x$.

Bu belgilash “ $f(x_i)\Delta x$ ko‘paytmaning yig‘indisi $i=1$ dan $i=4$ gacha” deb o‘qiladi. Uni yoyib yozish uchun i ning o‘rniga 1 dan 4 gacha sonlarni va natijalar orasiga “+” belgisini qo‘yib chiqamiz.

Keling bu belgilashlarni misollarda ko‘rib chiqamiz.

6-misol. 2+4+6+8+10 yig‘indining umumiy hadi uchun formula yozing.

Yechilishi: ► Yig‘indining har bir hadi 2 ga ko‘payib bormoqda, shuning uchun $2+4+6+8+10=\sum_{n=1}^{5} 2n$ formula o‘rinli. ◄

7-misol. $g(x_1)\Delta x + g(x_2)\Delta x + ... + g(x_{22})\Delta x$ yig‘indining umumiy hadi uchun formula yozing.

Yechilishi: ► $g(x_1)\Delta x + g(x_2)\Delta x + ... + g(x_{22})\Delta x = \sum_{i=1}^{22} g(x_i)\Delta x$ ◄

8-misol. $\sum_{n=1}^{5} 2^n$ yig‘indini hisoblang.

Yechilishi: ► $\sum_{n=1}^{5} 2^n = 2^1+2^2+2^3+2^4+2^5 = 2+4+8+16+32=62$ ◄

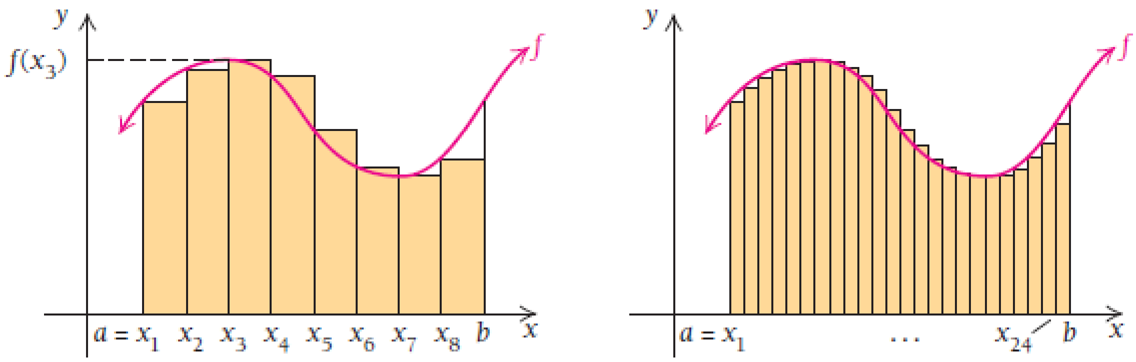
3-vazifa.

a) $5+10+15+20+25$ yig‘indi uchun umumiy formula yozing;

b) $33+44+55+66+77$ yig‘indi uchun umumiy formula yozing;

c) $\sum_{n=1}^{7} (n^2+n)$ yig‘indini hisoblang.

Agar oraliqlarni juda kichik o‘lchamda olsak va imkoni boricha ko‘proq to‘g‘ri to‘rtburchaklar hosil qilsak, bu to‘g‘ri to‘rtburchaklar yuzalarining yig‘indisi haqiqiy yuza qiymatiga yaqin bo‘ladi. Masalan, quyidagi chizmadan ko‘rinadiki, 8 ta oraliqqa ajratishdan 24 oraliqqa bo‘lib, yuzalarni qo‘shish haqiqatga ancha yaqin bo‘ladi.



9-misol. [0; 600] kesmada $f(x) = 600x - x^2$ funksiya grafigi bilan chegaralangan sohaning yuzasini taqribiy toping. Bunda

a) Kesmani 6 ta qismiy oraliqlarga ajrating;

b) Kesmani 12 ta qismiy oraliqlarga bo‘lib, hisoblang;

Yechilishi: ► a) [0; 600] kesmani 6 ta qismiy oraliqqa ajratsak,

$$\Delta x = \frac{600-0}{6} = 100, \text{ bunda } x_1 = 0 \text{ dan } x_6 = 500 \text{ bo‘ladi.}$$

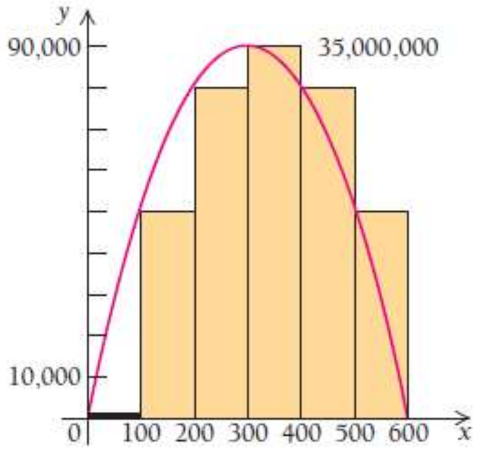
U holda sohaning yuzasi

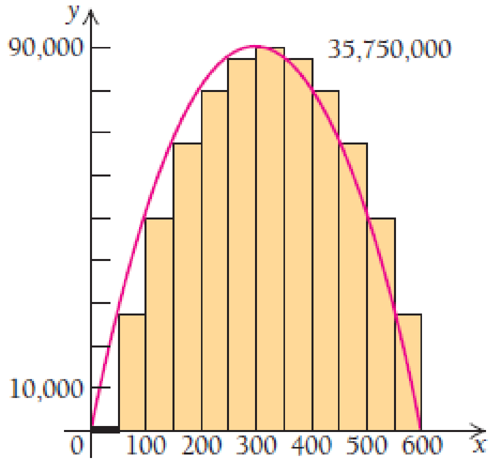
$$\sum_{i=1}^{6} f(x_i)\Delta x = f(0)\cdot 100 + f(100)\cdot 100 + f(200)\cdot 100 + f(300)\cdot 100 +$$

$$+ f(400)\cdot 100 + f(500)\cdot 100 = 0\cdot 100 + 50\,000\cdot 100 + 80\,000\cdot 100 + 90\,000\cdot 100 +$$

$$+ 80\,000\cdot 100 + 50\,000\cdot 100 = 35\,000\,000$$

kv.birlikka teng bo‘ladi.





b) [0; 600] kesmani 12 ta qismiy oraliqqa ajratsak,

$$\Delta x = \frac{600-0}{12} = 50, \text{ bunda } x_1 = 0 \text{ dan } x_{12} = 550 \text{ bo‘ladi.}$$

U holda sohaning yuzasi

$$\sum_{i=1}^{12} f(x_i)\Delta x = f(0)\cdot 50 + f(50)\cdot 50 + f(100)\cdot 50 + f(150)\cdot 50 + f(200)\cdot 50 +$$
$$+ f(250)\cdot 50 + ... + f(500)\cdot 50 + f(550)\cdot 50 = 0\cdot 50 + 27\,500\cdot 50 + 50\,000\cdot 50 + 67\,000\cdot 50 +$$
$$+ 80\,000\cdot 50 + 87\,500\cdot 50 + ... + 50\,000\cdot 50 + 27\,500\cdot 50 = 35\,750\,000. \quad ◀$$

9-misolda soha yuzasi kesmani $n = 6$ bo‘lakka ajratib hisoblagandan ko‘ra $n = 12$ bo‘lakka bo‘lib hisoblagan haqiqatga yaqinroq chiqadi.

10-misol. [1; 16] kesmada $f(x) = 0.1x^3 - 2.3x^2 + 12x + 25$ funksiya grafigi bilan chegaralangan sohaning yuzasini kesmani 5 ta bo‘lakka bo‘lib, taqribiy hisoblang.

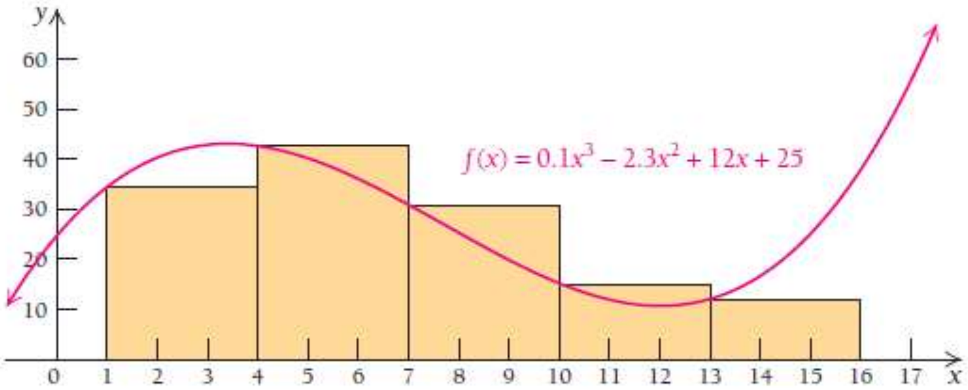
Yechilishi: ► [1; 16] kesmani 5 ta qismiy oraliqqa ajratsak,

$$\Delta x = \frac{16-1}{5} = 3, \text{ bunda } x_1 = 1 \text{ dan } x_5 = 13 \text{ bo‘ladi.}$$

U holda sohaning yuzasi

$$\sum_{i=1}^{5} f(x_i)\Delta x = f(1)\cdot 3 + f(4)\cdot 3 + f(7)\cdot 3 + f(10)\cdot 3 + f(13)\cdot 3 =$$

$$= 34.8 \cdot 3 + 42.6 \cdot 3 + 30.6 \cdot 3 + 15 \cdot 3 + 12 \cdot 3 = 405.$$



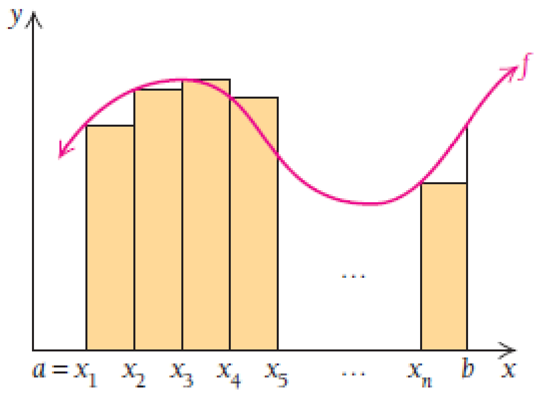
◀

4-vazifa. [0; 12] kesmada $f(x) = 0.1x^3 - 2.3x^2 + 12x + 25$ funksiya grafigi bilan chegaralangan sohaning yuzasini kesmani 6 ta bo‘lakka bo‘lib hisoblang.

Riman yig‘indilari jarayonining qadamlari:

Biror $[a,b]$ kesmada $y = f(x)$ egri chiziqli funksiya berilgan bo‘lsin.

1. $y = f(x)$ funksiya grafigini chizamiz.



2. $[a,b]$ kesmani n ta teng qismga ajratamiz: $\Delta x = \dfrac{b-a}{n}$.

3. Har bir bo‘lakda 1 ta to‘g‘ri to‘rtburchak hosil qilamiz. Shunda to‘rtburchakning tepadagi chap uchi grafik bilan umumiy nuqtaga ega bo‘ladi.

4. Har bir to‘g‘ri to‘rtburchakning yuzasini topamiz.

5. Topilgan yuzalar qiymatlarini qo‘shib chiqamiz. Natijada egri chiziqli soha yuzasining taqribiy qiymatiga ega bo‘lamiz.

5.5.3. Aniq integral

$[a,b]$ kesmada $y = f(x)$ uzluksiz funksiya berilgan bo‘lsin. Quyidagi qadamlarni bajaramiz:

1) $[a,b]$ kesmani quyidagi nuqtalar bilan n ta ixtiyoriy qismga bo‘lamiz, ularni **qismiy oraliqlar** deb ataymiz:

$$a = x_0 < x_1 < x_2 < ... < x_{i-1} < x_i < ... < x_n = b.$$

2) Qismiy oraliqlarning uzunliklarini bunday belgilaymiz:

$$\Delta x_1 = x_1 - a, \quad \Delta x_2 = x_2 - x_1, \ \ldots, \ \Delta x_i = x_i - x_{i-1}, \ \ldots,$$
$$\Delta x_n = b - x_{n-1}.$$

3) Har bir qismiy oraliqning ichidan bittadan ixtiyoriy nuqta tanlab olamiz: $\xi_1, \xi_2, \ldots, \xi_i, \ldots, \xi_n$.

4) Tanlangan nuqtalarda berilgan funksiyaning qiymatini hisoblaymiz: $f(\xi_1), f(\xi_2), \ldots, f(\xi_i), \ldots, f(\xi_n)$.

5) Funksiyaning hisoblangan qiymatlarini mos qismiy oraliq uzunligiga ko‘paytiramiz:

$$f(\xi_1)\Delta x_1, f(\xi_2)\Delta x_2, \ldots, f(\xi_i)\Delta x_i, \ldots, f(\xi_n)\Delta x_n.$$

6) Hosil bo‘lgan ko‘paytmalarni qo‘shamiz va yig‘indini σ bilan belgilaymiz:

$$\sigma = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \ldots + f(\xi_i)\Delta x_i + \ldots + f(\xi_n)\Delta x_n.$$

σ yig‘indiga $f(x)$ funksiya uchun $[a,\ b]$ kesmada tuzilgan **integral yig‘indi** yoki **Riman yig‘indilari** deyiladi va quyidagicha belgilanadi:

$$\sigma = \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$$

Integral yig‘indining geometrik ma’nosi ravshan: agar $f(x) \geq 0$ bo‘lsa, u holda σ-asoslari $\Delta x_1, \Delta x_2, \ldots, \Delta x_i, \ldots, \Delta x_n$ va balandliklari mos ravishda $f(\xi_1), f(\xi_2), \ldots, f(\xi_i), \ldots, f(\xi_n)$ bo‘lgan to‘g‘ri to‘rtburchak yuzalarining yig‘indisidan iborat.

Endi bo‘lishlar soni n ni orttira boramiz $(n \to \infty)$ va bunda eng katta oraliqning uzunligi nolga intiladi, ya’ni $max\ \Delta x_i \to 0$ deb faraz qilamiz:

$$\lim_{max\ \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$$

Bundan quyidagi teorema kelib chiqadi.

Integral hisobning asosiy teoremasi

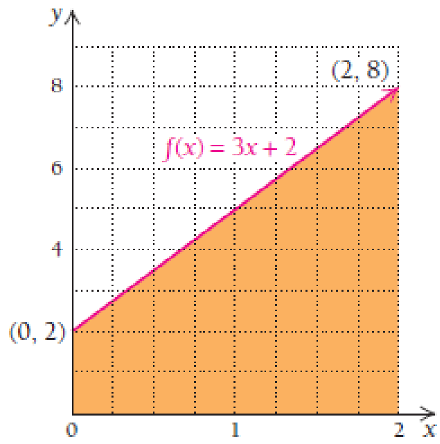
$f(x)$ funksiya $[a;\ b]$ kesmada uzluksiz funksiya bo‘lsin. U holda

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$ o‘rinli.

(“$f(x)$ dan x bo‘yicha a va b gacha olingan aniq integral” deb o‘qiladi). Bu yerda $f(x)$-integral ostidagi funksiya, $[a,\ b]$ kesma-integrallash oralig‘i, a va b sonlar integrallashning **quyi va yuqori chegarasi** deyiladi.

11-misol. $\int_0^2 (3x+2)dx$ hisoblang.

Yechilishi: ► $\int_0^2 (3x+2)dx = \left(\frac{3}{2}x^2 + 2x\right)\Big|_0^2 = \left(\frac{3}{2}\cdot 2^2 + 2\cdot 2\right) - \left(\frac{3}{2}\cdot 0 + 2\cdot 0\right) = 10$



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5-vazifa. Integrallarni hisoblang:

a) $\int_0^4 (x+3)dx$; b) $\int_2^5 (x^2-3x)dx$

MUSTAQIL YECHISH UCHUN MISOLLAR

1-12 misollar integralning tatbiqlariga doir:

Iqtisodiyot va tadbirkorlik. Chegaraviy tannarxdan umumiy tannarxni topish.

1. "Coffee Pho" qahva ishlab chiqaruvchi korxonaning uni qadoqlash uchun chegaraviy tannarxi bir funtiga $C'(x) = -0.012x + 6.5, \quad x \le 300$ dollar deb belgilangan. 200 funt qahvani qadoqlash umumiy tannarxini toping.

2. “Tillo domor” MChJ bir kg pishloq tayyorlash uchun $C'(x) = -0.003x + 4.25, \quad x \le 500$ dollar sarflaydi. 400 kg pishloqning umumiy tannarxini toping.

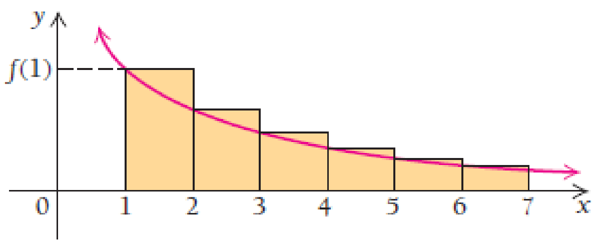
3. Tabiat manzarasi tushirilgan suratlarni bir donasini tayyorlash uchun $C'(x) = -0.04x + 85, \quad x \le 1000$ dollar sarflaydi. 650 dona suratning umumiy tannarxini toping.

4. “PRD+” korxonasi maxsus to‘qilgan gazlamaning bir yardiga $C'(x) = -0.007x + 12, \quad x \le 350$ dollar sarflaydi. 200 yard gazlamaning umumiy tannarxini toping.

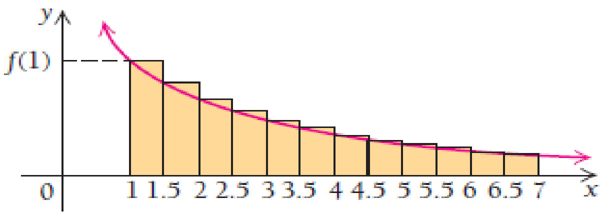
5. Maxsus kiyimga yopishtiriladigan bezak donasiga $C'(x) = -0.00002x^2 + 0.04x + 45, \quad x \le 800$ sent sarflanadi. [0; 800] kesmani 5 ta bo‘lakka bo‘lib, 800 ta bezakning umumiy tannarxini hisoblang.

6. “Rose” atirining unsiyasi (10 gr.)ga $C'(x) = -0.0005x^2 - 0.1x + 30, \quad x \le 125$ dollar sarflanadi. [0; 100] kesmani 5 ta bo‘lakka bo‘lib, 100 unsiyasining umumiy tannarxini hisoblang.

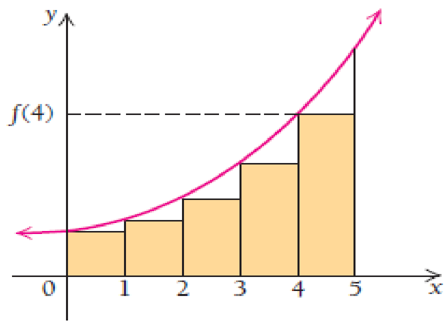
7. [1; 7] kesmada $f(x) = \frac{1}{x^2}$ funksiya grafigi ostidagi sohani 6 ta bo‘lakka ajratib, yuzasini toping.



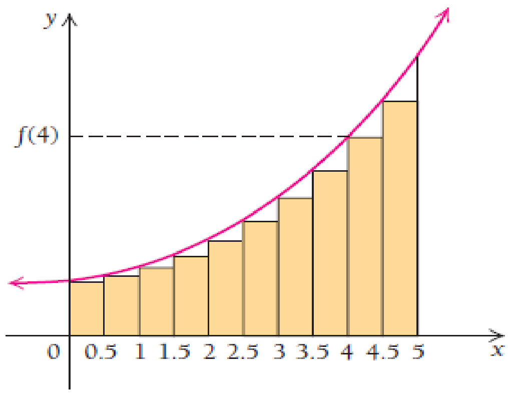
8. [1; 7] kesmada $f(x) = \frac{1}{x^2}$ funksiya grafigi ostidagi sohani 12 ta bo‘lakka ajratib, yuzasini toping va 7-misoldagi natija bilan taqqoslang.



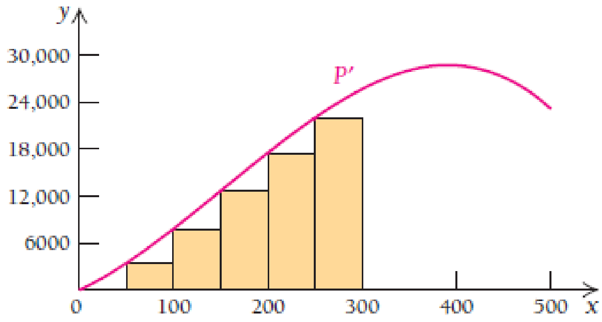
9. [0; 5] kesmada $f(x) = x^2 + 1$ funksiya grafigi ostidagi sohani 5 ta bo‘lakka ajratib, yuzasini toping.



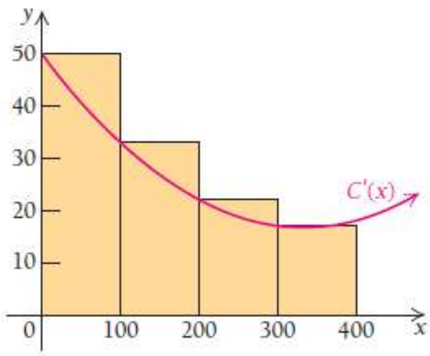
10. [0; 5] kesmada $f(x) = x^2 + 1$ funksiya grafigi ostidagi sohani 10 ta bo‘lakka ajratib, yuzasini toping va 9-misol natijasi bilan taqqoslang..



11. Chegaraviy foydadan umumiy foydani topish. Sport klubi har bir qatnashchidan $P'(x) = -0.0006x^3 + 0.28x^2 + 55.6x,\ \ x \le 500$ foyda ko‘radi. Agar 300 ta klub qatnashchisi bo‘lsa, quyidagi yig‘indidan foydalanib, umumiy foydani toping: $\sum_{i=1}^{6} P'(x_i)\Delta x,$ bunda $\Delta x = 50$.



12. Chegaraviy tannarxdan umumiy tannarxni topish. Korxona 1 litr pista yog‘i uchun $C'(x)=0.0003x^2-0.2x+50$ dollar sarflaydi. 400 litr yog‘ning umumiy tannarxini toping.



5.6. Aniq integralning tatbiqlari

5.5 bo‘limda biz egri chiziq ostidagi soha yuzasini hisoblashda berilgan sohani to‘g‘ri to‘rtburchaklarga ajaratib, ularning yuzalarini hisobladik, so‘ngra shu to‘rtburchaklar yuzalari yig‘indisini oldik. Bunda soha yuzasi taqribiy aniqlanadi.

Egri chiziq ostidagi soha yuzasini aniq hisoblay olamizmi?

Albatta, egri chiziq ostidagi soha yuzasini aniq hisoblash jarayoniga integrallash deyiladi.

Bizga $[a;b]$ oraliqda aniqlangan, uzluksiz, nomahfiy $y = f(x)$ funksiya berilgan bo‘lsin. Ushbu funksiya ostidagi soha yuzasi $f(x)$ ning boshlang‘ich funksiyasi bo‘ladi va uni $A(x)$ deb belgilaymiz:

$$\frac{d}{dx}A(x) = f(x).$$

5.5 – bo‘limda $f(x)$ o‘zgarmas, chiziqli funksiya bo‘lgan hollar uchun to‘rtburchak va uchburchak yuzalarini topish geometrik formulalaridan foydalandik. Shuningdek, ixtiyoriy nomanfiy, uzluksiz egri chiziqli funksiya bo‘lganda Riman yig‘indilarini tuzib, soha yuzasini approksimatsiyaladik.

Quyidagi jadvalda geometrik asosda topilgan yuzalarni keltirdik:

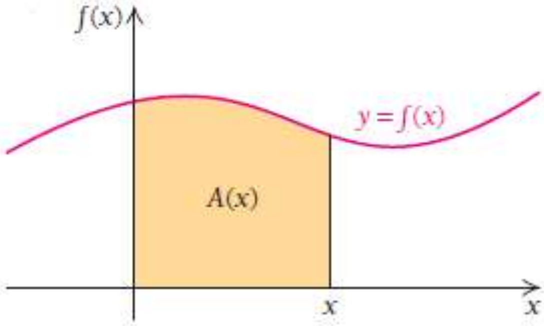
$f(x)$ funksiya	$A(x)$ yuza	Izoh
$f(x) = 3x$	$A(x) = \frac{3}{2}x^2$	496 - bet
$f(x) = k$	$A(x) = kx$	5.5.2- mavzu
$f(x) = mx$	$A(x) = \frac{1}{2}mx$	5.5.2- mavzu

E’tibor qilgan bo‘lsangiz, jadvalda yuza $A(x)$ning hosilasi $f(x)$ ga teng. Har doim shundaymikan?

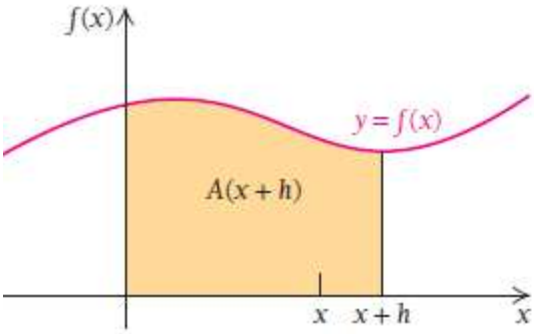
Bu savolga javob olish uchun $[0;\ x]$ kesmada nomanfiy, uzluksiz $f(x)$ funksiya ostidagi yuzadan hosila olamiz:

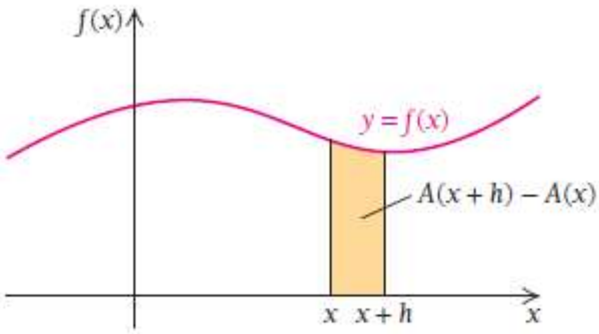
$$A'(x) = \lim_{h\to 0}\frac{A(x+h)-A(x)}{h},$$

bunda $A(x+h)$ yuza $[0;\ x+h]$ kesmadagi $f(x)$ funksiya ostidagi yuza.



Agar $A(x+h)-A(x)$ desak, u holda $f(x)$ funksiya ostidagi $[x;\ x+h]$ yuza kelib chiqadi.





$h\to 0$ ga intilganligi uchun $A(x+h)-A(x)$ yuza asosi h va balandligi $f(x)$ bo‘lgan to‘g‘ri to‘rtburchak yuziga intiladi:

$$A(x+h)-A(x)\approx h\cdot f(x).$$

Bundan
$$\frac{A(x+h)-A(x)}{h}\approx f(x)$$

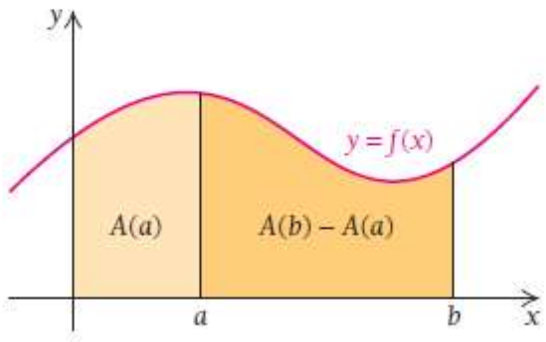
va tenglikni har ikki tomonida $h \to 0$ da limitga o‘tsak,

$$\lim_{h\to 0}\frac{A(x+h)-A(x)}{h}=\lim_{h\to 0} f(x)= f(x)$$

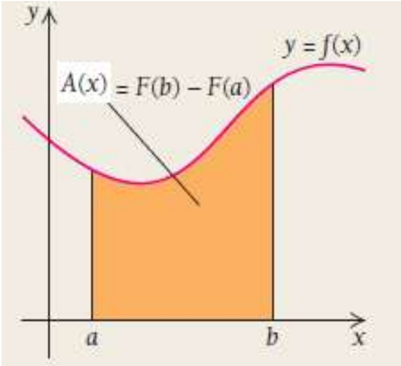
kelib chiqadi. Shunday qilib, $A'(x)= f(x)$ o‘rinli ekanini isbotladik.

Teorema. $f(x)$ funksiya $[0;\ b]$ kesmada aniqlangan nomanfiy, uzluksiz funksiya va $A(x)$ shu $[0;\ x]$, $0 < x < b$ kesmada tepadan $f(x)$ egri chiziq bilan, pastdan Ox o‘qi bilan chegaralangan sohaning yuzasi bo‘lsin. U holda $A(x)$ differensiallanuvchi va $A'(x)= f(x)$ o‘rinli bo‘ladi.

Teoremadan yuqoridagi savolga javob olish mumkin. Ha, har doim yuza $A(x)$ ning hosilasi shu yuzani chegaralagan $f(x)$ qiymatiga teng. Teorema faqat $[0;\ x]$ kesma uchun keltirilgan. Keling teoremani ixtiyoriy $[a;\ b]$ kesma uchun umumlashtiramiz. Chizmadagi $[a;\ b]$ kesmaga mos yuzani hisoblash uchun $[0;\ b]$ kesmaga mos yuzadan $[0;\ a]$ kesmadagi yuzani ayiramiz: $A(b)-A(a)$.



Agar $F(x)$ funksiya $f(x)$ ning boshlang‘ich funksiyasi bo‘lsa, u holda $A(x)= F(x)+C$ tenglik o‘rinli. C- biror o‘zgarmas son. Shunday qilib,



$$A(x) = A(b) - A(a) = F(b) + C - (F(a) + C) = F(b) - F(a)$$

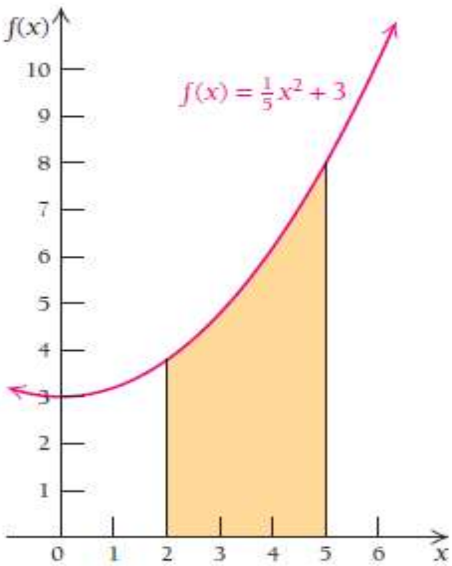
ga ega bo‘lamiz.

1-misol. $f(x) = \frac{1}{5}x^2 + 3$ funksiya grafigi ostidagi [2; 5] kesmaga mos soha yuzasini hisoblang.

Yechilishi: ► $f(x) = \frac{1}{5}x^2 + 3$ funksiyaning boshlang‘ichini topamiz.

$F(x) = \frac{1}{15}x^3 + 3x + C$, soddalik uchun C=0 deb olamiz. [2; 5] kesmadagi yuza bo‘lganligi uchun

$$F(2;5) = F(5) - F(2) = \left(\frac{1}{15}\cdot 5^3 + 3\cdot 5\right) - \left(\frac{1}{15}\cdot 2^3 - 3\cdot 2\right) = 16\frac{4}{5}$$

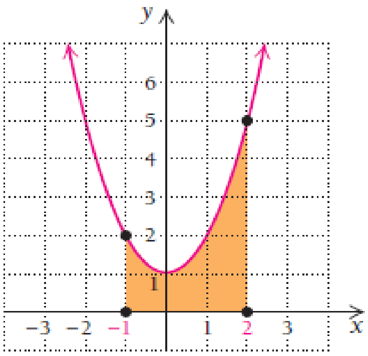


◄

2-misol. $f(x)=x^2+1$ funksiya grafigi ostidagi $[-1;2]$ kesmaga mos soha yuzasini hisoblang.

Yechilishi: ► 1) $f(x)=x^2+1$ ning boshlang'ich funksiyasini topamiz:

$$F(x)=\frac{x^3}{3}+x;$$



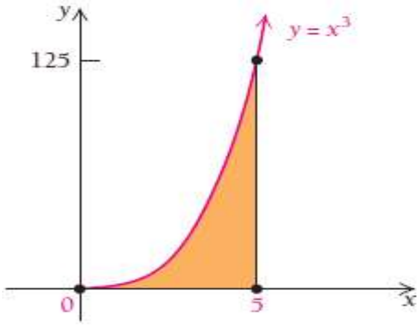
2) $F(2)-F(-1)=\frac{2^3}{3}+2-\left(\frac{(-1)^3}{3}+(-1)\right)=6$ ni hisoblaymiz.

Demak. Yuza 6 kv.birlikka teng ekan. ◄

Ta'rif. $f(x)$ funksiya $[a;b]$ kesmada uzluksiz va $F(x)$ uning boshlang'ich funksiyasi bo'lsin. U holda $f(x)$ funksiyaning a dan b gacha **aniq integrali** $\int_a^b f(x)dx=F(b)-F(a)$ ga teng bo'ladi.

3-misol. Yangi elektrostansiya kuniga $f(x)=x^3$ kW elektr energiyasi ishlab chiqaradi. Bu stansiya 5 kunda qancha elektr energiya ishlab chiqaradi?

Yechilishi: ► $\int_0^5 x^3dx=\left.\frac{x^4}{4}\right|_0^5=\frac{5^4}{4}-0=156\frac{1}{4}$ kW·kun.



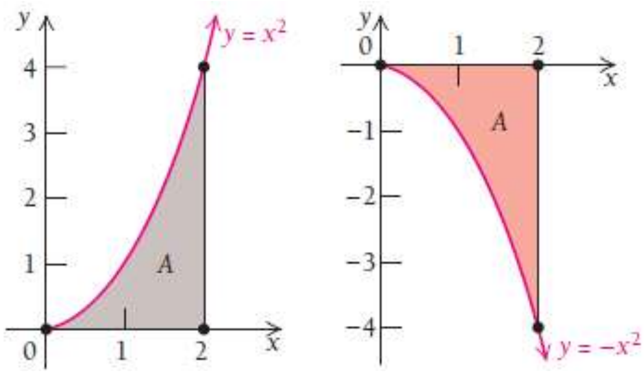
◀

Shu paytgacha siz bilan nomanfiy funksiyalar bilan chegaralangan, ya’ni soha Ox o‘qidan tepada bo‘lgan xollarni qaradik.

Agar funksiya manfiy bo‘lsa-chi, u holda tepadan Ox o‘qi va pastdan shu funksiya bilan chegaralangan yuzani qanday topamiz?

4-misol. Bizga $y = x^2$ va $y = -x^2$ funksiyalar berilgan bo‘lsin.

► Shu funksiyalar, Ox o‘qi va mos kesmalar bilan chegaralangan yuzalarni taqqoslab ko‘ramiz.



Bu funksiyalarning grafiklari Ox o‘qiga nisbatan simmetrik.

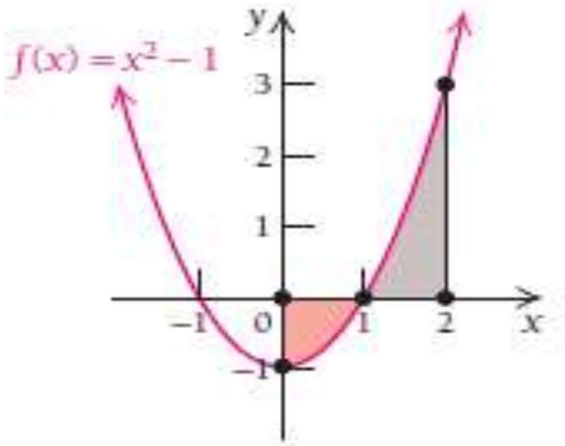
Ox o‘qi va [0; 2] kesmada shu funksiyalar bilan chegaralangan yuzalar nimaga teng:

$$\int_0^2 x^2 dx = \frac{x^3}{3}\bigg|_0^2 = \frac{2^3}{3} = \frac{8}{3} \quad \text{va} \quad \int_0^2 -x^2 dx = -\frac{x^3}{3}\bigg|_0^2 = -\frac{2^3}{3} = -\frac{8}{3}.$$

Integrallar bir-biriga teng, biroq ishoralar har xil. Bundan kelib chiqadiki, manfiy funksiya bilan chegaralangan sohalarni absolyut qiymatini olish kerak ekan. ◄

5-misol. Endi [0; 2] kesmada $y = x^2 - 1$ funksiya bilan chegaralangan yuzani qaraylik.

► Bu funksiya [0; 1] kesmada manfiy, [1; 2] kesmada musbat qiymatlar qabul qiladi. Agar yuzani hisoblash uchun funksiyani
[0; 2] kesmada integrallasak, **noto‘g‘ri qiymat olinadi**:



$$S = \int_0^2 (x^2 - 1)dx = \left(\frac{x^3}{3} - x\right)\Bigg|_0^2 = \left(\frac{2^3}{3} - 2\right) - 0 = \frac{8}{3} - 2 = \frac{2}{3}.$$

Yuza musbat qiymat chiqishiga sabab, Ox o‘qidan yuqoridagi sohaning yuzi pastdagi soha yuzidan katta bo‘lganligidir.

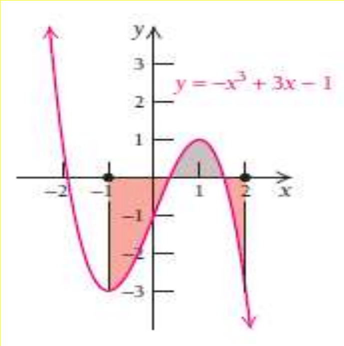
Xato qilib qo‘ymaslik uchun kesmani [0; 1] va [1; 2] bo‘laklarga ajratamiz va yuzalarni qo‘shamiz, 1-bo‘lakda funksiyani absolyut qiymatini olamiz:

$$S_1 = \int_0^2 (x^2 - 1)dx = \int_0^1 |x^2 - 1| dx = \left|\left(\frac{x^3}{3} - x\right)\right|_0^1 = \left|\left(\frac{1^3}{3} - 1\right) - 0\right| = \left|\frac{1}{3} - 1\right| = \left|-\frac{2}{3}\right| = \frac{2}{3}$$

$$S_2 = \int_1^2 (x^2 - 1)dx = \left(\frac{x^3}{3} - x\right)\Bigg|_1^2 = \left(\frac{2^3}{3} - 2\right) - \left(\frac{1}{3} - 1\right) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3},$$

$$S = S_1 + S_2 = \frac{2}{3} + \frac{4}{3} = 2. \blacktriangleleft$$

6-vazifa. [-1; 2] kesmada $y = -x^3 + 3x - 1$ funksiya bilan

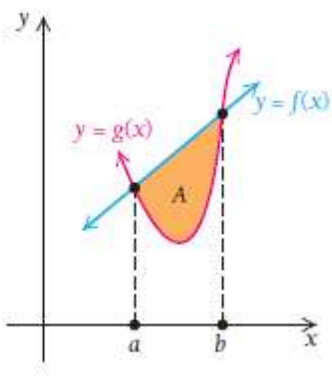


chegaralangan yuzani toping:

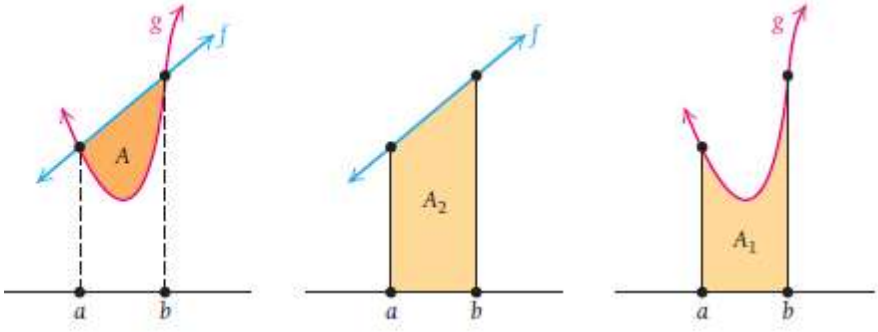
Agar soha tepadan $f(x)$ funksiya bilan, pastdan $g(x)$ funksiya bilan chegaralangan bo‘lsa, u holda

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx \quad \text{yoki} \quad A = \int_a^b [f(x) - g(x)]dx$$

formulalar o‘rinli bo‘ladi.

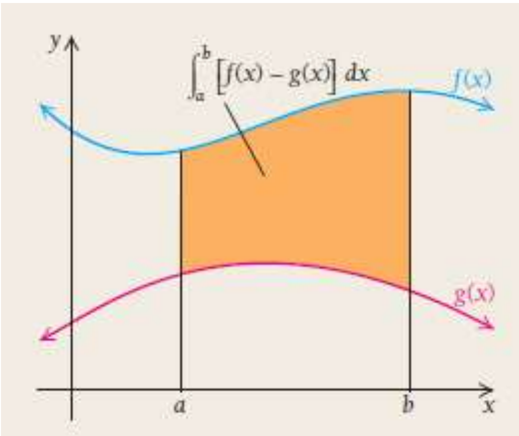


Chunki umumiy soha yuzasi 2 ta soha yuzining ayirmasidan iborat:



Teorema. Agar $f(x)$ va $g(x)$ funksiyalar $[a;b]$ kesmada uzluksiz va $f(x)\geq g(x)$ bo'lsa, u holda bu funksiyalar hamda $x=a$ va $x=b$ chiziqlar bilan chegaralangan sohaning yuzasi

$$A=\int_a^b [f(x)-g(x)]dx$$ ga teng bo'ladi.



6-misol. Ushbu $4y=8x-x^2$, $4y=x+6$ chiziqlar bilan chegaralangan shaklning yuzini toping.

Yechilishi: ► Bu chiziqlardan biri parabola, ikkinchisi to'g'ri chiziq bo'lib, ularning kesishish nuqtasini topamiz:

$$\begin{cases} 4y=8x-x^2 \\ 4y=x+6 \end{cases}$$

$$8x-x^2=x+6$$

$$x^2-7x+6=0$$

$$x_1=1 \quad \text{va} \quad x_2=6,$$

$$y_1=\frac{x+6}{4}=\frac{1+6}{4}=\frac{7}{4} \quad \text{va} \quad y_2=\frac{x+6}{4}=\frac{6+6}{4}=3.$$

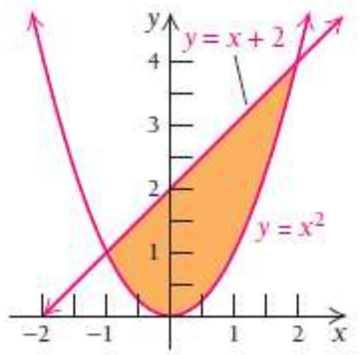
Shunda bu ikki grafik bir-biri bilan A(1;7/4) va B(6;3) nuqtalarda kesishadi. Bunda tepadagi grafik bilan chegaralangan yuzadan pastdagi grafik bilan chegaralangan yuzani ayiramiz:

$$S=\frac{1}{4}\int_1^6[(8x-x^2)-(x+6)]dx=\frac{1}{4}\int_1^6(-x^2+7x-6)dx=$$

$$=\frac{1}{4}\left(-\frac{x^3}{3}+\frac{7x^2}{2}-6x\right)\Bigg|_1^6=5\frac{5}{24} \text{ kv.birl.} \quad ◄$$

7-misol. $y=x^2$ va $y=x+2$ chiziqlar bilan chegaralangan shaklning yuzini toping.

Yechilishi: ► 1) Bu chiziqlarning kesishish nuqtalarini topib olamiz. Buning uchun tenglamalar sistemasini yechish kerak:



$$\begin{cases} y=x^2 \\ y=x+2 \end{cases} \quad \rightarrow \quad x^2=x+2 \quad \rightarrow \quad x^2-x-2=0 \quad \rightarrow \quad x=2, \quad x=-1$$

1) Endi $[-1;\ 2]$ kesmada integralni hisoblaymiz:

$$S=\int_{-1}^{2}(x+2-x^2)dx=\left(\frac{x^2}{2}+2x-\frac{x^3}{3}\right)\Bigg|_{-1}^{2}=$$

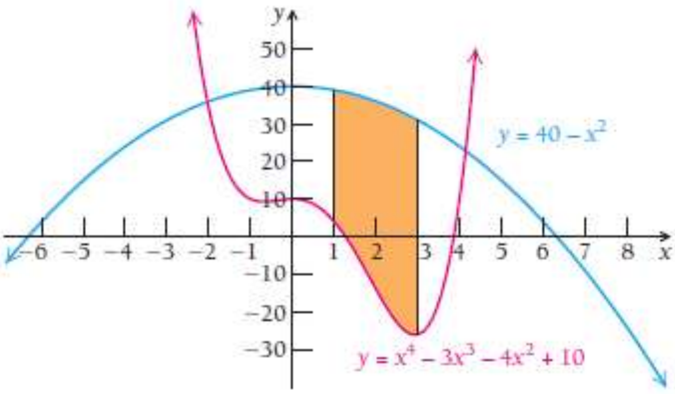
$$=\frac{2^2}{2}+2\cdot2-\frac{2^3}{3}-\left(\frac{(-1)^2}{2}+2\cdot(-1)-\frac{(-1)^3}{3}\right)=\frac{9}{2}$$

$y=x^2$ va $y=x+2$ chiziqlar bilan chegaralangan shaklning yuzini 4.5 kv. birlikka teng chiqdi. ◄

8-misol. $y=x^4-3x^3-4x^2+10$ va $y=40-x^2$ hamda $x=1$ va $x=3$ chiziqlar bilan chegaralangan shaklning yuzini toping.

Yechilishi: ► $S=\int_{1}^{3}[(40-x^2)-(x^4-3x^3-4x^2+10)]dx=$

$$=\int_{1}^{3}-x^4+3x^3+3x^2+30)]dx=\left(-\frac{x^5}{5}+\frac{3x^4}{4}+x^3+30x\right)\Bigg|_{1}^{3}=96.7$$



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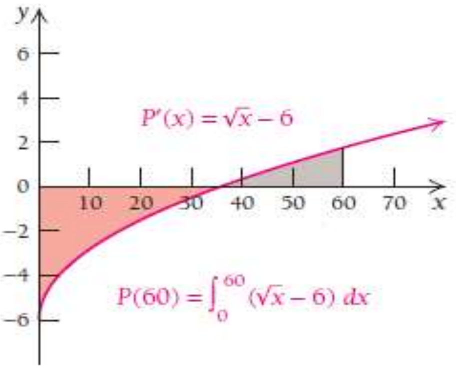
9-misol. Tadbirkorlik. Chegaraviy foydadan umumiy foydani topish. “Uzbekistan Airways” aviakompaniyasining Toshkent-Dubay turizm bo‘yicha sotilgan x biletdan kelib tushadigan chegaraviy foydasi $P'(x)=\sqrt{x}-6$ (yuz dollar hisobida). 60 ta bilet sotilgandagi umumiy foydani toping.

Yechilishi: ► Umumiy $P(60)$ foydani topish uchun chegaraviy $P'(x)$ foydadan [0; 60] oraliqda integral olamiz:

$$P(60)=\int_0^{60} P'(x)dx=\int_0^{60}(\sqrt{x}-6)dx=\left(\frac{2}{3}x^{\frac{3}{2}}-6x\right)\Bigg|_0^{60}\approx -50.1613.$$

Demak, agar 60 ta bilet sotilsa, foyda $-5016.13\$$ bo‘ladi. Buning ma’nosi shuki, 60 nafar kishi parvozida aviakompaniya $-5016.13\$$ zarar ko‘radi.



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5.6.1. Egri chiziq yoyi uzunligini hisoblash

Faraz qilaylik, $f(x)$ funksiya $[a;b]$ segmentda aniqlangan va uzluksiz bo‘lib, bu funksiya grafigi 1-chizmada ko‘rsatilgan *AB* egri chiziq yoyini tasvirlasin. $[a;b]$ segmentning biror $P=\{x_{0,}x_1,.........,x_n\}$ (a=x_0<$x_1<x_2<\cdots<x_n=b$) bo‘linishini olib, uning bo‘luvchi x_k $(k=0,n)$ nuqtalari orqali *Oy* o‘qiga parallel to‘g‘ri chiziqlar o‘tkazamiz. Ularning $\widetilde{AB}$ yoy bilan kesishgan nuqtalari

$$A_k=(x_{k,}\,f(x_k))$$

$$\left(A_0=\left(a,f(x)\right),A_n=B=\left(b,f(b)\right),k=\overline{1,n-1}\right)\quad \text{bo'lsin.}$$

$\smile{AB}$ yoydagi $A_k(k = \overline{0,n})$ nuqtalarni bir-biri bilan to‘g‘ri chiziq kesmalari yordamida birlashtirib $\overset{\smile}{AB}$ yoyga chizilgan siniq chiziqni hosil qilamiz. Bu siniq chiziq perimetrini L bilan belgilaylik. Unda tekislikdagi ikki nuqta orasidagi masofa formulasidan foydalanib, $A_k = (x_k, f(x_k))$ va $A_{k+1} = (x_{k+1}, f(x_{k+1}))$ nuqtalar orasidagi masofa

$$|A_k - A_{k+1}| = \sqrt{\left(x_{k+1} - x_{k,}\right)^2 + (f(x_{k+1}) - f(x_k)^2}$$

va L siniq chiziq perimetri

$$L = \sum_{k=0}^{n-1} \sqrt{\left(x_{k+1} - x_{k,}\right)^2 + (f(x_{k+1}) - f(x_k))^2}$$

ga teng bo‘lishini topamiz.

Ravshanki, siniq chiziq perimetri $f(x)$ funksiyaga hamda $[a;b]$ segmentning bo‘linishiga bog‘liq bo‘ladi: $L = L_p(f)$.

P bo‘linishning bo‘luvchi nuqtalar sonini orttirib borilsa, $\overset{\smile}{AB}$ yoydagi siniq chiziqlar shu $\overset{\smile}{AB}$ yoyga yaqinlasha boradi.

Agar $\overset{\smile}{AB}$ yoyga chizilgan siniq chiziq ($[a;b]$ oraliqning ixtiyoriy P bo‘linishida) perimetri

$$L_p = \sum_{k=0}^{n-1} \sqrt{\left(x_{k+1} - x_{k,}\right)^2 + (f(x_{k+1}) - f(x_k))^2} \qquad (1)$$

$\lambda_p \to 0$ da chekli limitga ega bo‘lsa, u holda $\overset{\smile}{AB}$ yoy uzunlikka ega deb ataladi va ushbu limit $\lim\limits_{\lambda_p \to 0} L_p = l$ $\overset{\smile}{AB}$ **yoyning uzunligi** deyiladi.

Qaralayotgan $f(x)$ funksiya $[a;b]$ segmentda uzluksiz bo‘lishi bilan birga u shu segmentda uzluksiz $f'(x)$ hosilaga ham ega bo‘lsin.

Yuqoridagidek, $[a;b]$ segmentning ixtiyoriy P bo‘linishini olib, $\overset{\frown}{AB}$ yoyga chizilgan unga mos siniq chiziqni hosil qilamiz. Bu siniq chiziq perimetri (1) formulaga ko‘ra

$$L_p = \sum_{k=0}^{n-1} \sqrt{\left(x_{k+1} - x_{k,}\right)^2 + (f(x_{k+1}) - f(x_k))^2}$$

bo‘ladi.

$f(x)$ funksiya $[a;b]$ segmentda Lagranj teoremasining shartlarini qanoatlantiradi. Unda bu teoremaga ko‘ra, shunday $\tau_k(x_k \le \tau_k \le x_{k+1})$ nuqta topiladiki, $f(x_{k+1}) - f(x_k) = f'(\tau_k)(x_{k+1} - x_k)$ tenglik o‘rinli bo‘ladi. Natijada

$$L_p = \sum_{k=0}^{n-1} \sqrt{\left(x_{k+1} - x_{k,}\right)^2 + f'^2(\tau_k)(x_{k+1} - x_k)^2} =$$

$$=\sum_{k=0}^{n-1} \sqrt{1 + f'^2(\tau_k)}((x_{k+1} - x_k) =$$

$$\sum_{k=0}^{n-1} \sqrt{1 + f'^2(\tau_k)}\Delta x_k \qquad (2)$$

tenglikka kelamiz.

Ravshanki, $\sqrt{1 + f'^2(x)}$ funksiya $[a;b]$da uzluksiz. Binobarin, u shu segmentda integrallanuvchi. Bu funksiyaning integral yig‘indisi

$$\sum_{k=0}^{n-1} \sqrt{1 + f'^2(\xi_k)} \cdot \Delta x_k$$

bo‘lib uning limiti $[x_k, x_{k+1}]$ oraliqlardan olingan nuqtalarga bog‘liq emas. Demak $\xi_k = \tau_k$ larda

$$\lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} \sqrt{1 + f'^2(\tau_k)} \Delta x_k = \int_a^b \sqrt{1 + f'^2(x)} dx \qquad (3)$$

bo‘ladi.   (2) va (3) munosabatlardan

$$\lim_{\lambda_p \to 0} L_p = \int_a^b \sqrt{1 + f'^2(x)} dx \qquad (3')$$

bo‘lishi kelib chiqadi. Bu esa $\breve{AB}$ yoyning uzunlikka ega va u

$$l = \int_a^b \sqrt{1 + f'^2(x)} dx$$

bo‘lishini bildiradi.

4-misol. Ushbu $f(x) = (x^{\frac{3}{2}})$’ $(0 \le x \le 4)$ funksiyaning egri chiziq yoyi uzunligini toping.

Yechilishi: ►Avvalo berilgan funksiyaning hosilasini hisoblaymiz:

$$f'(x) = (x^{\frac{3}{2}})' = \frac{3}{2} x^{\frac{1}{2}}$$

Unda $\qquad 1 + f'^2(x) = 1 + \frac{9}{4} x, \ \sqrt{1 + f'^2(x)} = \sqrt{1 + \frac{9}{4} x}$

bo‘lib, (3′) formulaga binoan $\ l = \int_0^4 \sqrt{1 + \frac{9}{4} x}\, dx$

bo‘ladi. Keyingi integralda $1 + \frac{9}{4} x = t$ almashtirish bajaramiz.

Unda $\quad dx = \frac{9}{4} dt \quad va \quad 1 \le t \le 10$ bo‘lib,

$$\int_0^4 \sqrt{1+\frac{9}{4}x}\,dx = \frac{9}{4}\int_1^{10} t^{\frac{1}{2}}dt = \frac{8}{27}t^{\frac{3}{2}}\Big|_1^{10} = \frac{8}{27}\left(\sqrt{1000}-1\right)$$

$$= \frac{8}{27}\left(10\sqrt{10}-1\right)$$

bo‘ladi. Demak, yoy uzunligi $l = \frac{8}{27}\left(10\sqrt{10}-1\right)$ ga teng. ◀

5-misol. Ushbu $f(x) = \frac{x^2}{2p}$ $(p > 0)$ parabolaning [0,*a*] oraliqdagi (*a*>0) qismining uzunligini toping.

Yechilishi: ►Avvalo $f(x)$ funksiyaning hosilasini hisoblab, $\sqrt{1+f'^2(x)}$ ni topamiz:

$$f'(x) = \frac{x}{p}, 1+f'^2(x) = \frac{p^2+x^2}{p^2}, \sqrt{1+f'^2(x)} = \frac{1}{p}\sqrt{p^2+x^2}$$

(3’) formulaga ko‘ra qaralayotgan egri chiziqning uzunligi

$$l = \frac{1}{p}\int_0^a \sqrt{p^2+x^2}dx$$

bo‘ladi. Endi ushbu $\int \sqrt{x^2+p^2}dx$ aniqmas integralni hisoblaymiz. Agar bo‘laklab integrallash formulasidan foydalansak, unda

$u = \sqrt{x^2+p^2}, dv = dx$ deb belgilaymiz, $du = \frac{xdx}{\sqrt{x^2+p^2}}$, $v = x$

bo‘lib,

$$\int \sqrt{x^2+p^2}dx = x\sqrt{x^2+p^2} - \int \frac{x^2dx}{\sqrt{x^2+p^2}}$$

bo‘ladi. Bu tenglikning o‘ng tomonidagi integral quyidagicha hisoblanadi:

$$\int\frac{xdx}{\sqrt{x^2+p^2}}=\int\frac{x^2+p^2-p^2}{\sqrt{x^2+p^2}}dx=\int\frac{x^2+p^2}{\sqrt{x^2+p^2}}dx-p^2\int\frac{dx}{\sqrt{x^2+p^2}}=$$

$$=\int\sqrt{x^2+p^2}dx-p^2\int\frac{dx}{\sqrt{x^2+p^2}}=$$

$$=\sqrt{x^2+p^2}dx-p^2\ln\left|x+\sqrt{x^2+p^2}\right|$$

Demak, $\int\sqrt{x^2+p^2}dx=x\sqrt{x^2+p^2}-\int\sqrt{x^2+p^2}dx+p^2\ln\Big|x+$

$$+\sqrt{x^2+p^2}\Big|$$

Bu tenglikdan $2\int\sqrt{x^2+p^2}dx=x\sqrt{x^2+p^2}-p^2\ln\left|x+\sqrt{x^2+p^2}\right|$

bo‘lib,

$$\int\sqrt{x^2+p^2}dx=\frac{x}{2}\sqrt{x^2+p^2}+\frac{p^2}{2}\ln\left|x+\sqrt{x^2+p^2}\right|$$

bo‘lishi kelib chiqadi. Natijada

$$l=\frac{1}{p}\int_0^a\sqrt{x^2+p^2}dx=\frac{1}{p}\left[\frac{x}{2}\sqrt{x^2+p^2}+\frac{p^2}{2}\ln\left|x+\sqrt{x^2+p^2}\right|\right]\begin{matrix}a\\0\end{matrix}=$$

$$=\frac{1}{2p}a\sqrt{a^2+p^2}+\frac{p}{2}\ln\left|a+\sqrt{a^2+p^2}-\frac{p}{2}\ln p\right|$$

bo‘ladi. Demak, $f(x)=\frac{x^2}{2p}$ $(p>0)$ parabolaning [0,a] oraliqdagi (a>0) qismining uzunligi

$$l=\frac{1}{2p}a\sqrt{a^2+p^2}+\frac{p}{2}\ln\left|a+\sqrt{a^2+p^2}-\frac{p}{2}\ln p\right|$$

ga teng ekan. ◄

5.6.2. Parametrik shaklda berilgan funksiyaning yoy uzunligini hisoblash

Faraz qilaylik, $\breve{AB}$ yoy (egri chiziq)

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}, (\alpha \le t \le \beta)$$

tenglamalar sistemasi, ya'ni parametrik shaklda berilgan bo'lib, $x = \varphi(t), y = \psi(t)$ funksiyalar $[\alpha, \beta]$ da aniqlangan, uzluksiz va $\varphi'(t), \psi'(t)$ uzluksiz hosilalarga ega bo'lsin. Bunda $\breve{AB}$ yoy uzunlikka ega bo'lib, uning uzunligi

$$l = \int_{\alpha}^{\beta} \sqrt{\varphi'^2(t) + \psi'^2(t)}\, dt \qquad (4)$$

formula yordamida topiladi.

(4) tenglikning o'rinli ekanligini (3') formula yordamida hamda aniq integralda o'zgaruvchini almashtirish formulasidan foydalanib, keltirib chiqarish mumkin.

6-misol. Ushbu $\begin{cases} \varphi(t) = a(1 - \sin t) \\ \psi(t) = a(1 - \cos t) \end{cases}$ $(0 \le t \le \pi)$

tenglamalar sistemasi bilan aniqlangan egri chiziqning (sikloidaning) uzunligini toping.

Yechilishi: ► $\varphi(t) = a(1 - \sin t), \psi(t) = a(1 - \cos t)$

funksiyalarning hosilalarini hisoblaymiz: $\begin{matrix} \varphi'(t) = a(1 - \cos t) \\ \psi(t) = a \cdot sint \end{matrix}$

Unda $\varphi'^2(t) + \psi'^2(t) = a^2(1 - \cos t)^2 + a^2 sin^2 t = a^2 \cdot 2(1 - \cos t)$

bo‘lib, $\sqrt{\varphi'^2(t) + \psi'^2(t)} = a\sqrt{2(1-\cos t)}$ bo‘ladi.

(4) formulaga ko‘ra, egri chiziqning uzunligi $l = \int_0^{2\pi} a\sqrt{2(1-\cos t)}\,dt$ bo‘ladi. Bu tenglikning o‘ng tomonidagi integralni hisoblaymiz:

$$\int_0^{2\pi} a\sqrt{2(1-\cos t)}\,dt = a\int_0^{2\pi} \sqrt{4sin^2\frac{t}{2}}dt =$$

$$= 2a\int_0^{2\pi} sin\frac{t}{2}dt = 4a\int_0^{2\pi} sin\frac{t}{2}d\left(\frac{t}{2}\right) = -4a\cdot\cos\frac{t}{2}\Big|_0^{2\pi} = 8a$$ ◄

5.6.3. Qutb koordinatasida yoy uzunligini hisoblash

Faraz qilaylik, $\breve{AB}$ egri chiziq qutb koordinata sistemasida

$$\rho = \rho(\theta) \quad (\alpha \le 0 \le \beta) \qquad (5)$$

tenglik bilan berilgan bo‘lsin. Bunda $\rho = \rho(\theta)$ funksiya $[a;b]$ da uzluksiz va $\rho'(\theta)$ uzluksiz hosilaga ega.

Avvalo (5) munosabat bilan berilgan egri chiziq tenglamasini parametrik ko‘rinishda ifodalab olamiz:

$$\begin{cases}\varphi(\theta) = \rho(\theta)\cos\theta \\ \psi(\theta) = \rho(\theta)\sin\theta\end{cases} \quad (\alpha \le 0 \le \beta)$$

So‘ng (4) formuladan foydalanib , $\breve{AB}$ egri chiziq yoyining uzunligini topamiz: $l = \int_\alpha^\beta \sqrt{\varphi'^2(\theta) + \psi'^2(\theta)}d\theta = \int_\alpha^\beta \sqrt{(\rho(\theta)\cos\theta)'^2 + (\rho(\theta)\sin\theta)'^2}d\theta$

$$= \int_\alpha^\beta \sqrt{(\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta)^2 + (\rho'(\theta)\sin\theta + \rho(\theta)\cos\theta)^2}d\theta = \int_\alpha^\beta \sqrt{(\rho'^2(\theta) + \rho^2(\theta)}d\theta$$

Demak, (5) munosabat bilan berilgan egri chiziq yoyining uzunligi

$$l=\int_{\alpha}^{\beta}\sqrt{(\rho'^2(\theta)+\rho^2(\theta)}d\theta \qquad (6)$$

bo‘ladi.

7-misol. Ushbu $\rho = 2a(1+cos\theta)$ $(0 \le \theta \le 2\pi)$ egri chiziq (kardioda) yoyining uzunligini toping.

Yechilishi: ► Bu yopiq chiziq bo‘lib, qutb o‘qiga nisbatan simmetrik joylashgan. Shuning uchun egri chiziqning uzunligi, uning qutb o‘qining yuqorisida joylashgan qismi uzunligining ikkilanganiga teng bo‘ladi. (6) formuladan foydalanib topamiz:

$$l=2\int_0^{\pi}\sqrt{(\rho'^2(\theta)+\rho^2(\theta)}d\theta=2\int_0^{\pi}\sqrt{(2a(1+\cos\theta))'^2+(2a(1+\cos\theta))^2}\,d\theta=$$

$$=2\cdot 2a\int_0^{\pi}\sqrt{\sin^2\theta+(1+\cos\theta)^2}\,d\theta=4\sqrt{2}a\int_0^{\pi}\sqrt{1+\cos\theta}\,d\theta=8a\int_0^{\pi}\cos\frac{\theta}{2}d\theta=16a$$

Demak, kardoida yoyining uzunligi $l = 16a$ bo‘ladi. ◄

8-misol. Ushbu $\rho = \frac{3a\cos\varphi\sin\varphi}{sin^3\varphi+cos^3\varphi}$, $\left(0 \le \varphi \le \frac{\pi}{2}\right)$ chiziq bilan chegaralangan shaklning yuzini toping.

Yechilishi: ► Qaralayotgan shaklning yuzini (6) formuladan foydalanib topamiz:

$$S=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[\frac{3a\cos\varphi\sin\varphi}{sin^3\varphi+cos^3\varphi}\right]^2 d\varphi=\frac{9a^2}{2}\int_0^{\pi/2}\frac{cos^2\varphi sin^2\varphi}{(sin^3\varphi+cos^3\varphi)^2}d\varphi$$

Endi bu tenglikning o‘ng tomonidagi integralni hisoblaymiz:

$$\int_0^{\pi/2} \frac{cos^2\varphi sin^2\varphi}{(sin^3\varphi + cos^3\varphi)^2} d\varphi = \int_0^{\pi/2} \frac{tg^2\varphi}{(1+tg^3\varphi)^2} \cdot \frac{d\varphi}{cos^2\varphi}$$

$$\frac{1}{3}\int_0^{\pi/2} (1+tg^3\varphi)^{-2} d(1+tg^3\varphi) = -\frac{1}{3}(1+tg^3\varphi)^{-1}\Big|_0^{\frac{\pi}{2}} = \frac{1}{3}$$

Demak, $S = \frac{9a^2}{2} \cdot \frac{1}{3} = \frac{3a^2}{2} kv. br.$

Odatda, $\rho = \frac{3a \cos\varphi \sin\varphi}{sin^3\varphi + cos^3\varphi}$, $\left(0 \le \varphi \le \frac{\pi}{2}\right)$ chiziq **Dekart yaprog‘i** deyiladi. ◄

5.6.4. Aylanma sirt yuzini hisoblash

$y = f(x)$ funksiya $[a;b]$ kesmada aniqlangan, uzluksiz bo‘lib, $\forall_x \epsilon [a,b]$ uchun $f(x) \ge 0$ bo‘lsin. $f(x)$ funksiya grafigining Ox o‘qi atrofida aylantirishdan aylanma sirt hosil bo‘ladi.

Bu sirt yuzasining aniq integral orqali ifodalanishini ko‘rsatamiz. $[a;b]$ oraliqning ixtiyoriy P=$\{x_0, x_1, \dots, x_n\}$ (a=$x_0 < x_1 < \dots < x_n = b$) bo‘linishni olaylik. P bo‘linishning har bir $x_k = (k = 0,1,\dots,n)$ bo‘luvchi nuqtalari orqali Oy o‘qiga parallel to‘g‘ri chiziqlar o‘tkazib, ularni $\widetilde{AB}$ yoy bilan kesishgan nuqtalarini $A_k(x_k, f(x_k))$ bilan belgilaylik. Bu $A_k\big(x_k, f(x_k)\big)$ $(k = 0,1,\dots,n)$, $A_0 = A, A_n = B$ nuqtalarni o‘zaro to‘g‘ri chiziq kesmalari bilan birlashtirib, $\widetilde{AB}$ yoyga L siniq chiziq chizamiz. $\widetilde{AB}$ yoyni va L chiziqni Ox o‘qi atrofida aylantiamiz. Natijada L ning aylanishidan kesik konus sirtlaridan tashkil topgan sirt hosil bo‘ladi. Bu sirtning yuzi ushbu

$$Q = 2\pi \sum_{k=0}^{n-1} \frac{f(x_k) + f(x_{k+1})}{2} \sqrt{(x_{k+1} - x_k)^2 + [(x_{k+1}) - f(x_k)]^2}$$

(7)-formula bilan ifodalanadi.

P bo‘linishning diamemtri $\lambda_p \to 0$ da $\breve{AB}$ yoyiga chizilgan L siniq chiziq perimetri $\breve{AB}$ yoy uzunligiga intiladi. Demak, $\lambda_p \to 0$ da L siniq chiziqni Ox o‘qi atrofida aylantirishdan hosil bo‘lgan sirtning yuzasi Q ning limiti biz qarayotgan aylanma sirtning yuzasini aniqlaydi. Bu yuzaning aniq integral orqali ifodasini topamiz.

Buning uchun $f(x)$ funksiya $[a;b]$ da uzluksiz $f'(x)$ hosilaga ega deb olamiz. $f(x)$ funksiya $[a;b]$ oraliqda uzluksiz bo‘lganligi uchun $[(x_{k+1}, x_k]$ oraliqda shunday ξ_k nuqta topiladiki,

$$\frac{f(x_k) + f(x_{k+1})}{2} = f(\xi_k), \xi_n \epsilon [(x_k), x_{k+1}]$$

tenglik o‘rinli bo‘ladi. Ikkinchi tomondan, Lagranj teoremasiga ko‘ra, $[(x_k), x_{k+1}]$ oraliqda shunday τ_k nuqta topiladiki

$$f(x_{k+1}) - f(x_k) = f'(\tau_k)(x_{k+1} - x_k), \qquad \tau_n \epsilon [x_k, x_{k+1}]$$

tenglik ham o‘rinli bo‘ladi. Natijada (7) munosabat ushbu

$$Q = 2\pi \sum_{k=0}^{n-1} f(\xi_h) \sqrt{(x_{k+1} - x_k))^2 + f'^2(\tau_{k)}(x_{k+1} - x_k)^2} =$$

$$=2\pi \sum_{k=0}^{n-1} f(\xi_h) \sqrt{1 + f'^2(\tau_k)} \Delta x_k \qquad (8)$$

ko‘rinishni oladi. Bu tenglamaning o‘ng tomonidagi

$$\sum_{k=0}^{n-1} f(\xi_k)\sqrt{1+f'^2(\tau_k)}\,\Delta x \qquad (9)$$

yig‘indi $\qquad f(x)\sqrt{1+f'^2(x)} \qquad (10)$

funksiyaning integral yig‘indisini eslatadi. (10) funksiya integrallanuvchi bo‘lganligi sababli ξ_k nuqta sifatida τ_k ni olish mumkun.

$\lambda_p \to 0$ da (9) tenglikdan topamiz:

$$\lim_{\lambda_p\to 0} Q = \lim_{\lambda_p\to 0} 2\pi \sum_{k=0}^{n-1} f(\tau_k)\sqrt{1+f'^2(\tau_k)}\Delta x_k =$$

$$= 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}dx$$

Shunday qilib, aylanma sirtning yuzi uchun ushbu formula o‘rinli:

$$S = 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}dx.$$

5.6.5. Aniq integral yordamida hajmlarni hisoblash

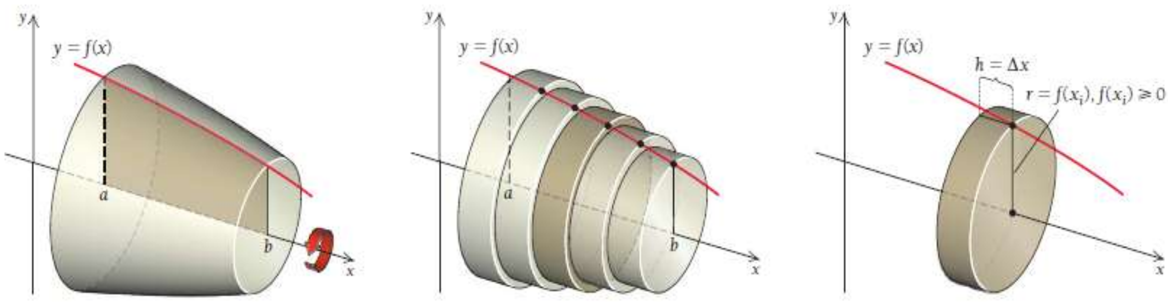
a) Agar jism *Ox* o‘qinig *x* nuqtasiga o‘tkazilgan perpendikulyar tekisliklar bilan kesishishdan hosil bo‘lgan kesim yuzi *S(x)* berilgan bo‘lsa, jism hajmi $V = \int_a^b S(x)dx$ formula bilan topiladi.

Bu yerda a va b lar x ning o‘zgarish chegaralari, *S(x)* funksiya $[a;b]$ kesmada, aniqlangan va uzluksiz deb qaraladi.

b) $y = f(x)$ funksiya grafigini Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jism hajmini topaylik.

1-chizmaga qarang. Agar Ox o‘qidan tepadagi yarim tekislikni Ox o‘qi atrofida aylantirsak, u holda funksiya grafigining har bir nuqtasi aylanma harakat qilib, aylana chizadi va grafikning barcha nuqtasi aylanishidan *aylanma sirt* hosil bo‘ladi.

$y = f(x)$ funksiya grafigi, Ox o‘qi, $x = a,\ x = b$ to‘g‘ri chiziqlar bilan chegaralangan yarim tekislik esa *aylanma jism* hosil qiladi.



Bu jismning hajmini topish uchun (2-chizmaga qarang) qalinligi judayam kichik bo‘lgan chekli sondagi to‘g‘ri silindrlarga yoki disklarga bo‘lib chiqamiz. Bunda $[a;\ b]$ oraliqni har birining uzunligi Δx bo‘lgan qism oraliqlarga ajratamiz. Shunda har bir silindrning balandligi $h = \Delta x$ ga teng bo‘ladi (3-chizma). Silindr radiusini esa bo‘lakning o‘ng tomonidagi x_i nuqtasiga mos funksiya qiymatiga teng deb olamiz: $r = f(x_i)$. Agar $f(x_i)$ manfiy bo‘lsa, uning modulini $|f(x_i)|$ olamiz.

Bizga ma’lumki, to‘g‘ri silindrning hajmi $V = \pi r^2 h$ ga teng. U holda har bir diskning hajmi $V = \pi |f(x_i)|^2 \Delta x$ bo‘ladi.

Bundan aylanma jismning taxminiy hajmi barcha disklarning hajmlari yig‘indisiga teng bo‘lishi kelib chiqadi: $V \approx \sum_{i=1}^{n} \pi |f(x_i)|^2 \Delta x$.

Jismning aniq hajmi esa disklar qalinligi nolga intilgandagi, ya’ni disklar soni cheksizlikka intilgandagi limitga teng bo‘ladi:

$$V \approx \lim_{n\to\infty} \sum_{i=1}^{n} \pi |f(x_i)|^2 \Delta x = \int_a^b \pi [f(x)]^2 dx$$

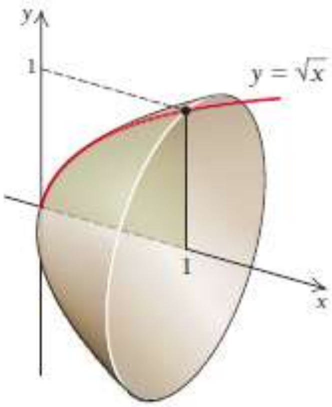
Demak, $y = f(x)$, Ox o‘q va to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning Ox o‘qi atrofida aylanishidan hosil bo‘lgan jism hajmi

$$V = \int_a^b \pi [f(x)]^2 dx$$

formula bilan topiladi.

9-misol. $y = \sqrt{x}$ funksiya grafigini $[0; 1]$ kesmada Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jism hajmini toping.

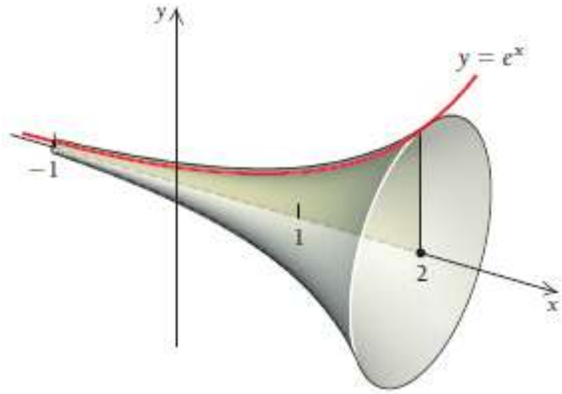
Yechilishi: ► $V = \int_a^b \pi [f(x)]^2 dx$ formuladan foydalanamiz.



$$V = \int_0^1 \pi \left[\sqrt{x}\right]^2 dx = \pi \int_0^1 x dx = \pi \frac{x^2}{2}\Big|_0^1 = \pi \frac{1^2 - 0^2}{2} = \frac{\pi}{2}. \quad ◄$$

10-misol. $y = e^x$ funksiya grafigini $[-1;\ 2]$ kesmada Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jism hajmini toping.

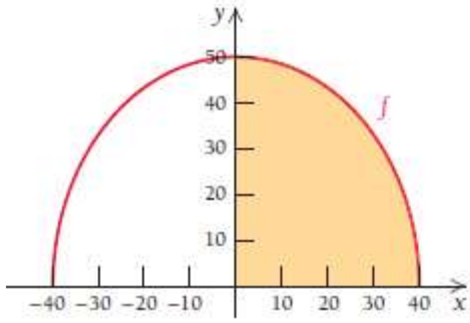
Yechilishi: ► $V = \int_a^b \pi[f(x)]^2 dx$ formuladan foydalanamiz.



$$V = \int_0^1 \pi\left[e^x\right]^2 dx = \pi\int_0^1 e^{2x} dx = \pi\frac{e^{2x}}{2}\bigg|_{-1}^{2} = \pi\frac{e^4 - e^{-2}}{2} \approx 85.55$$ ◄

11-misol. **Tadbirkorlik. Suv rezervuari.** Shaharda suv saqlaydigan rezervuar $[-40;40]$ masofada $f(x) = 50\sqrt{1 - \frac{x^2}{40^2}}$ funksiya grafigini aylantirishdan hosil qilingan aylanma jism shaklida yasalgan. Uning hajmini toping.

Yechilishi: ► Rezervuarning bunday shakliga qisilgan sferoid deyiladi. Chunki uning vertikal diametri (80 ft) gorizontal diametridan (100 ft) kichik. $f(x)$ ni Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jismning hajmini topish uchun chizmadan ko‘rish mumkinki, u $[-40;40]$ oraliqda bo‘lishi kerak. Biz hisonlashni osonroq bajarish maqsadida oraliqning yarmida integralni hisoblab, keyin natijani 2 ga ko‘paytiramiz.



$$V_1 = \pi \int_0^{40} \left[50\sqrt{1-\frac{x^2}{40^2}} \right]^2 dx = 2500\pi \int_0^{40} \left(1-\frac{x^2}{1600}\right) dx = 2500\pi \left(x - \frac{x^3}{4800} \right)\Bigg|_0^{40} =$$

$$= \frac{200\,000\pi}{3};$$

Shunday qilib, rezervuar hajmi $V = 2V_1 = \dfrac{400\,000\pi}{3}$ ga teng.

Ma’lumki, $1\mathrm{ft}^3$ hajmda 7.48 gallon suv bo‘ladi, shunga ko‘ra, ushbu rezervuar sig‘imi 3.13 million gallonga teng ekan. ◄

v) $x = \varphi(y)$ egri chiziq, OY o‘qi va $y = c, y = d$ to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning Oy o‘q atrofida aylanishidan hosil bo‘lgan jism hajmi

$$V = \pi \int_c^d (\varphi(y))^2 dy = \pi \int_c^d x^2 dy$$

formula bilan hisoblanadi.

g) Agar $y = f(x)$, $(a \le x \le b)$ egri chiziq parametrik usulda, ya’ni $\left.\begin{array}{l} x = x(t) \\ y = y(t) \end{array}\right\}, \alpha \le t \le \beta$ bo‘lsa, egri chiziqli trapetsiyaning Ox o‘q atrofida aylanishidan hosil bo‘lgan jismning hajmi

$$V=\pi\int_a^b (y(t))^2 x^1(t)dt$$

formula bilan topiladi.

d) Qutb koordinatalar sistemasida $\rho = f(x)$ tenglama bilan berilgan egri chiziq va $\varphi=\alpha, \varphi=\beta$ radius vektorlar bilan chegaralangan shaklning qutb o‘qi atrofida aylanishidan hosil bo‘lgan jism hajmi

$$V=\pi\int_\alpha^\beta \rho^3 \sin^3\varphi d\varphi$$

formula bilan hisoblanadi.

Agar $0\le\alpha\le\varphi\le\beta\le\pi$ bo‘lsa, $V=\frac{2\pi}{3}\int_\alpha^\beta \rho^3 \sin^3\varphi d\varphi$ formula bilan hisoblanadi.

12-misol. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipsni *Ox* va *Oy* o‘qlari atrofida aylantirish natijasida hosil qilingan jismlarning hajmlarini hisoblang.

Yechilishi: ►Ellips tenglamasidan

$$y^2=\frac{b^2}{a^2}(a^2-x^2); \qquad x^2=\frac{a^2}{b^2}(b^2-y^2)$$

uni *Ox* o‘qi atrofida aylantirishdan hosil bo‘lgan jismning hajmi:

$$V=2V_1=2\pi\int_0^a y^2dx=2\pi\frac{b^2}{a^2}\int_0^a (a^2-x^2)dx=2\pi\frac{b^2}{a^2}(a^2x-\frac{x^3}{3})\Big|_0^a=$$

$$=2\pi\frac{b^2}{a^2}(a^3-\frac{a^3}{3})=\frac{4}{3}\pi ab^2 V=\frac{4}{3}\pi ab^2$$

Ellipsni Oy o‘qi atrofida aylantirishdan hosil bo‘lgan jismning hajmi:

$$V = 2V_1 = 2\pi\int_0^b x^2dy = 2\pi\frac{a^2}{b^2}\int_0^b (b^2 - y^2)dy = 2\pi\frac{a^2}{b^2}(b^2y - \frac{y^3}{3})|_0^b =$$

$$= 2\pi\frac{a^2}{b^2}(b^3 - \frac{b^3}{3}) = \frac{4}{3}\pi a^2 b \quad V = \frac{4}{3}\pi a^2 b \ (kub.birl.)$$ ◄

1-vazifa. Rasmda berilgan aylanma shakl hajmini qanday hisoblash

mumkin?

5.6.6. Integralning fizikaviy tatbiqlari

Biror moddiy jismning tezlanishi ma'lum bo'lsa, aniq integraldan foydalanib, uning tezligi va bosib o'tgan yo'lini hisoblash mumkin.

Chunki $s'(t) = v(t)$ va $s''(t) = v'(t) = a(t)$ bizga ma'lum. Shunga ko'ra,

$$s(t) = \int v(t)dt, \quad v(t) = \int a(t)dt$$

integral formulalar o'rinli.

13-misol. Fizika. O'tilgan yo'l. Faraz qilaylik, $v(t) = 5t^4$ va $s(0) = 9$ ga teng. $s(t)$ ni toping.

Yechilishi: ► Integrallash hosilaga teskari amal bo'lganligi uchun

$$s(t) = \int v(t)dt = \int 5t^4dt = t^5 + C$$

$s(0)=9$ ni tenglikka qo‘yib, C ni topamiz $9=0^5+C$, bundan $C=9$ kelib chiqadi. Demak, bosib o‘tilgan yo‘l $s(t)=t^5+9$ ga teng bo‘ladi. ◀

14-misol. Fizika. O‘tilgan yo‘l. Faraz qilaylik, $a(t)=12t^2-6$ va $v(0)=5$ va $s(0)=10$ ga teng. $s(t)$ ni toping, $a(t),\ v(t),\ s(t)$ funksiyalar grafiklarini chizing.

Yechilishi: ▶ 1) $v(t)$ ni topish uchun $a(t)$ dan integral olamiz:

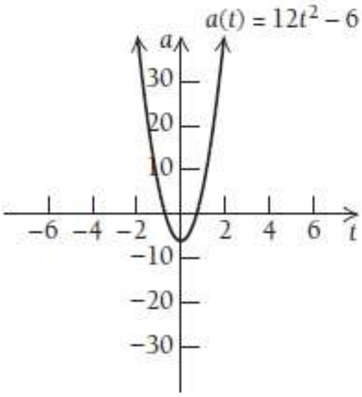
$$v(t)=\int a(t)dt=\int(12t^2-6)dt=4t^3-6t+C_1;$$

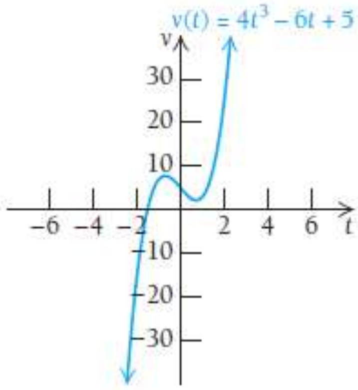
$v(0)=5$ dan C_1 ni topamiz: $5=4\cdot0^3-6\cdot0+C_1 \quad\Rightarrow\quad C_1=5$.

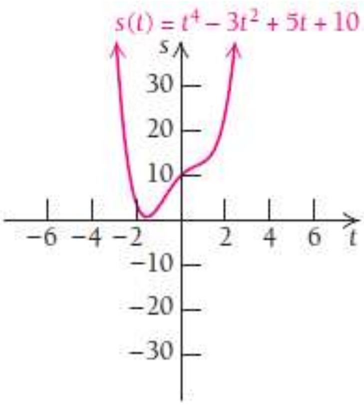
Shunday qilib, $v(t)=4t^3-6t+5$ ni hosil qilamiz.

2) $s(t)=\int v(t)dt=\int(4t^3-6t)dt=t^4-3t^2+C_2$;

$s(0)=10$ dan C_2 ni topamiz: $10=0^4-3\cdot0^2+C_2 \quad\Rightarrow\quad C_2=10$.







◀

15-misol. M moddiy nuqta $v(t)=3t^2+2t+1$ m/s tezlik bilan to‘g‘ri chiziqli harakat qilmoqda. [0, 3] vaqt oralig‘ida o‘tilgan yo‘lni toping.

Yechilishi: ► $S=\int_0^3 (3t^2+2t+1)dt=(t^3+t^2+t)\Big|_0^3=39$ m. ◄

Berilgan kuch ta’sirida bajarilgan ishni hisoblash

Aytaylik, moddiy nuqta Os to‘g‘ri chiziq bo‘ylab $F(s)$ kuch ta’sirida harakatlanayotgan bo‘lsin. Yo‘lning $[a,b]$ qismida bu kuch ta’sirida bajarilgan ish quyidagi formula bilan hisoblanadi:

$$A=\int_a^b F(s)ds\,.$$

16-misol. Agar prujinani 1 smga cho‘zish uchun 1kN kuch sarflansa, shu prujinani 10 sm ga cho‘zish uchun qancha ish bajarish kerak?

Yechilishi: ► Guk qonuniga ko‘ra, prujinani cho‘zish uchun kerak bo‘ladigan F kuch prujinani cho‘zilishiga to‘g‘ri proportsional, ya’ni $F=kx$, bu yerda $x-$prujinaning cho‘zilishi (metrda), $k-$proportsionallik koeffitsiyenti.

Misol shartidan ma’lumki, prujina $F=1$ kN kuch bilan $x=0{,}01$ m cho‘ziladi. Formuladan $1=0{,}01k \quad\longrightarrow\quad k=100$ va $F=100x$ ni topamiz. Shunda bajarilgan ish

$$A=\int_0^{0,1} 100x dx=50x^2\Big|_0^{0,1}=0{,}5 \text{ kJ.}$$ ◄

Tekis shaklning og‘irlik markazini topish

1) Chiziqli zichligi $\delta = \delta(x)$ bo‘lgan $y = f(x)$ funksiya grafigining moddiy AB yoyining og‘irlik markazining koordinatalari $C(x_c, y_c)$ quyidagi formulalar bilan aniqlanadi:

$$x_C = \frac{\int_a^b x\delta(x)\sqrt{1+y'^2}dx}{\int_a^b \delta(x)\sqrt{1+y'^2}dx}, \qquad y_C = \frac{\int_a^b y\delta(x)\sqrt{1+y'^2}dx}{\int_a^b \delta(x)\sqrt{1+y'^2}dx}$$

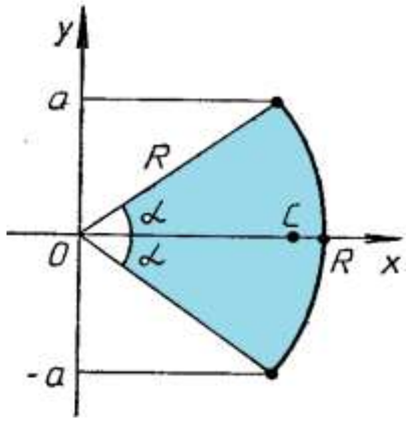
2) Agar shakl pastdan $y = f_1(x)$ va yuqoridan $y = f_2(x)$, shuningdek $[a,b]$ kesmada $f_1(x) \le f_2(x)$ tengsizlik o‘rinli va chiziqli zichlik $\delta = \delta(x)$ bo‘lsa, og‘irlik markazining koordinatalari $C(x_c, y_c)$ quyidagi formulalar bilan aniqlanadi:

$$x_C = \frac{\int_a^b x\delta(x)[f_2(x) - f_1(x)]dx}{\int_a^b \delta(x)[f_2(x) - f_1(x)]dx}, \qquad y_C = \frac{\frac{1}{2}\int_a^b \delta(x)[f_2^2(x) - f_1^2(x)]dx}{\int_a^b \delta(x)[f_2(x) - f_1(x)]dx}$$

17-misol. Radiusi R, markaziy burchagi 2α bo‘lgan bir jinsli sim yoyning og‘irlik markazini toping.

Yechilishi: ► Chizmadan ko‘rinadiki, yoy simmetrik va bir jinsli materialdan qilingan, shuning uchun $y_C = 0$ ga teng. x_C ni quyidagi

formuladan topamiz: $$x_C = \frac{\int_{-a}^{a} x\sqrt{1+y'^2}dx}{\int_{-a}^{a} \sqrt{1+y'^2}dx}.$$



Integralni hisoblash uchun qutb koordinatasiga o‘tamiz.

$\delta = const$, $x = R\cos t$, $y = R\sin t$. U holda Yakobianni topamiz:

$$I = \begin{vmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \cos t & -R\sin t \\ \sin t & R\cos t \end{vmatrix} = R\cos^2 t + R\sin^2 t = R.$$

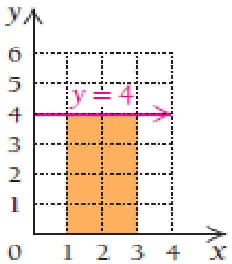
$$x_C = \frac{\int_{-\alpha}^{\alpha} R^2 \cos t dt}{\int_{-\alpha}^{\alpha} R dt} = R\frac{\sin t\Big|_{-\alpha}^{\alpha}}{t\Big|_{-\alpha}^{\alpha}} = R\frac{\sin\alpha}{\alpha}.$$

Shunday qilib, sim yoyining og‘irlik markazi $x_C = R\frac{\sin\alpha}{\alpha}$, $y_C = 0$. ◄

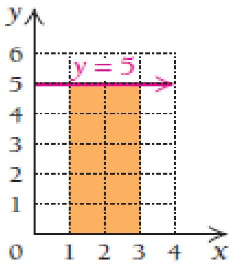
MUSTAQIL YECHISH UCHUN MISOLLAR

1-31 misollarda funksiyalar bilan chegaralangan yuzalarni hisoblang:

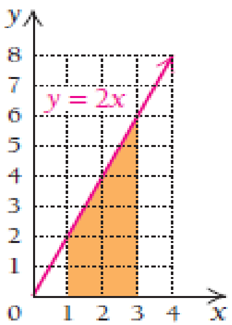
1. $y = 4; \quad [1, 3]$



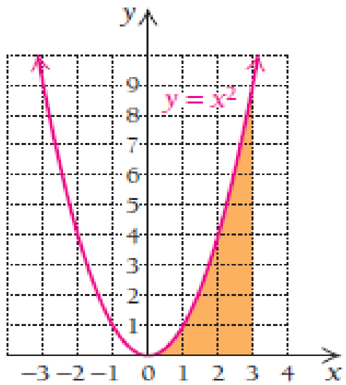
2. $y = 5; \quad [1, 3]$



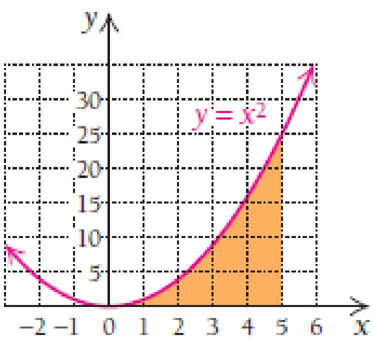
3. $y = 2x; \quad [1, 3]$



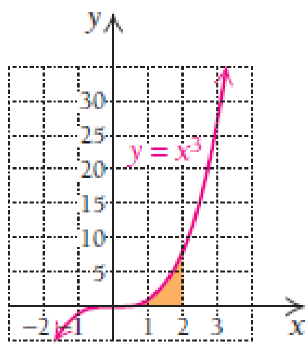
4. $y = x^2; \quad [0, 3]$



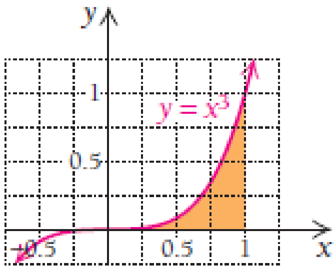
5. $y = x^2; \quad [0, 5]$



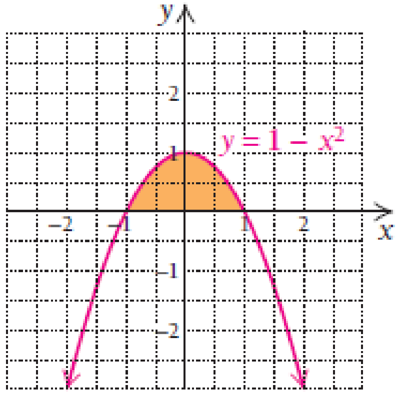
6. $y = x^3; \quad [0, 2]$



7. $y = x^3$; $[0, 1]$



8. $y = 1 - x^2$; $[-1, 1]$



9. $y = 4 - x^2$; $[-2, 2]$

10. $y = e^x$; $[0, 2]$

11. $y = e^x$; $[0, 3]$

12. $y = \frac{2}{x}$; $[1, 4]$

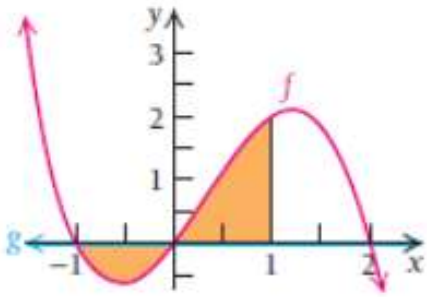
13. $y = \frac{3}{x}$; $[1, 6]$

14. $y = x^2 - 4x$; $[-4, -2]$

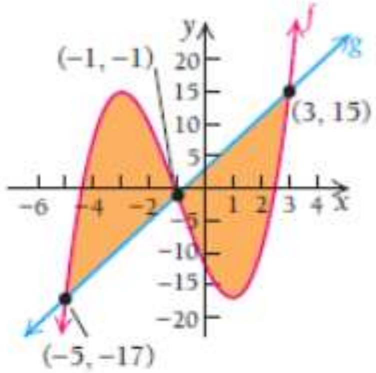
15. $y = \frac{1}{2}x^2 - x + 1$; $y = -\frac{1}{2}x^2 + 3x + 6$.

16. $y = \frac{1}{2}x^2 + x + 7$; $y = -\frac{1}{2}x^2 - 5x + 2$.

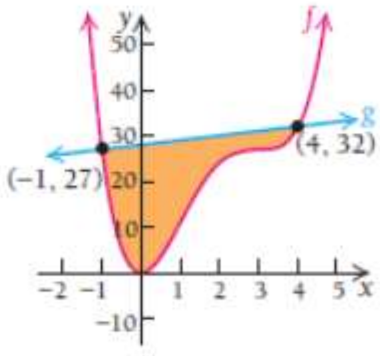
17. $f(x) = 2x + x^2 - x^3$, $g(x) = 0$



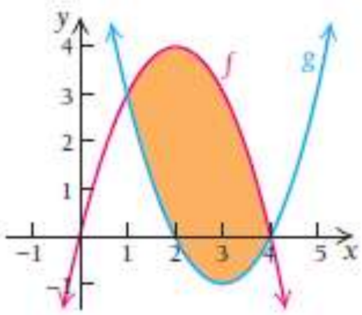
18. $f(x) = x^3 + 3x^2 - 9x - 12, \quad g(x) = 4x + 3$



19. $f(x) = x^4 - 8x^3 + 18x^2, \quad g(x) = x + 28$



20. $f(x) = 4x - x^2, \quad g(x) = x^2 - 6x + 8$



21. $y = x^2 - 5x - 3$; $\quad y = -3x^2 + 2x - 1$.

22. $y = x^2 - 2x - 5$; $\quad y = -x^2 - x + 1$.

23. $y = \frac{1}{4}x^2 - 2x - 5$; $\quad y = -\frac{3}{4}x^2 - x + 1$.

24. $y = \frac{1}{2}x^2 + 3x - 2$; $\quad y = -\frac{1}{2}x^2 - x + 3$.

25. $y = 2x^2 - 6x + 3$; $y = -2x^2 + x + 5$.

26. $y = x^2 - 3x - 4$; $y = -x^2 - x + 8$.

27. $y = \frac{1}{2}x^2 - 1$; $y = -\frac{1}{2}x^2 - x + 2$.

28. $y = 2x^2 + 4x$; $y = -x^2 - x + 1$.

29. $y = 2x^2 + 3x + 1$; $y = -x^2 - 2x + 9$.

30. $y = \frac{1}{3}x^2 - 3x + 2$; $y = -\frac{2}{3}x^2 - 2x + 4$.

31. $y = 2x^2 + 6x - 3$; $y = -x^2 + x + 5$.

32-34 misollarda yoy uzunligini toping:

32. $\begin{cases} x = \cos\left(\frac{t}{2}\right) \\ y = t - \sin t \end{cases}$

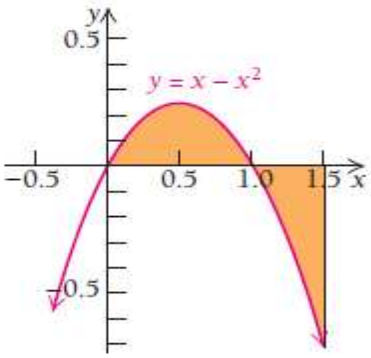
33. $\begin{cases} x = t^3 + 8 \\ y = t^5 + 2t \end{cases}$

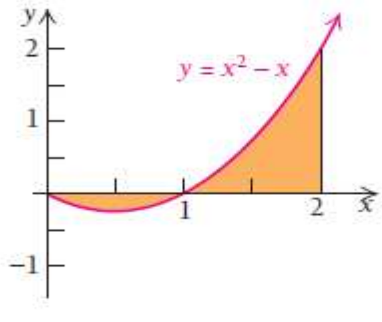
34. $\begin{cases} x = a\cos^2 t \\ y = b\sin^2 t \end{cases}$

35-38 misollarda rangli yuzalar qiymatini hisoblang:

35. $\int_0^{1.5} (x - x^2)\, dx$

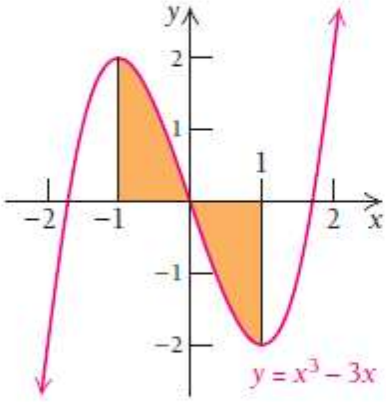
36. $\int_0^{2} (x^2 - x)\, dx$

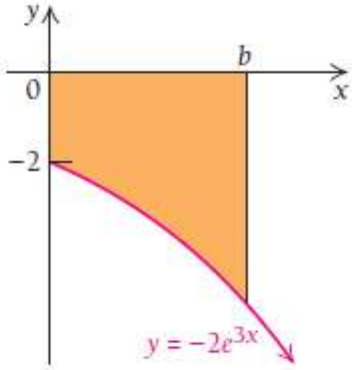




37. $\int_{-1}^{1}(x^3-3x)\,dx$

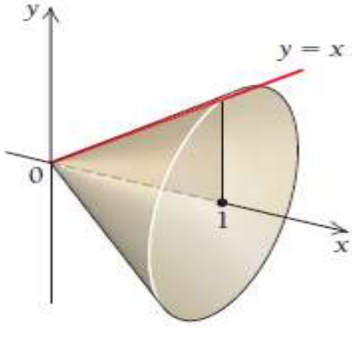
38. $\int_{0}^{b}-2e^{3x}\,dx$





39. Marsni o‘rganish uchun kosmik kemada robot olib borildi. Unga $v(t)=-0.42t^2+2t$ km/ soat tezlik berildi. Birinchi 3 soatda robot qancha masofani bosib o‘tadi?

40. $y=x$ funksiya grafigini $[0;\,1]$ kesmada Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jism hajmini toping.



41. Atom elektr stansiyalarida sovutish mo‘rilari mavjud. Ularning tuzilishi $f(x)=50\sqrt{1+\frac{x^2}{22500}}$ giperbolani $-250\le x\le 250$ masofada Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jism bo‘lib, rasmda keltirilgan. Shu jism hajmini toping.

5.7. I va II tur xosmas integrallar

Aniq integrallarni hisoblashda integral ostidagi ifoda integrallash oralig‘ida aniqlangan va uzluksiz bo‘lishi kerak. Agar funksiya integrallash oralig‘ida uzilishga ega bo‘lsa, yoki funksiyadan cheksiz oraliqda integral olinadigan bo‘lsa, bunday integrallarga **xosmas integral** deyiladi.

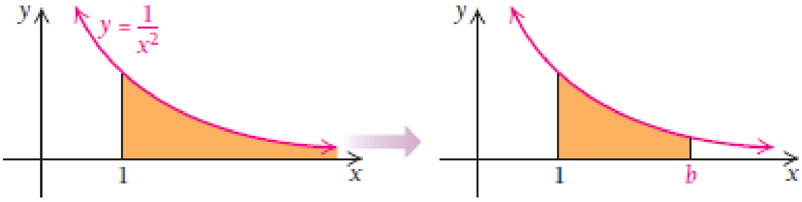
Xosmas integrallarni 2 turga ajratish mumkin:

1. Agar integralning quyi yoki yuqori chegarasi yoki ikkala chegarasi ham cheksiz bo‘lsa, unga **I tur xosmas integral** deyiladi.

2. Agar integral ostidagi funksiya integrallash oralig‘ining quyi yoki yuqori chegarasida yoki oraliq ichidagi biror nuqtada uzilishga ega bo‘lsa, unga **II tur xosmas integral** deyiladi.

5.7.1. I tur xosmas integrallar

Aytaylik, $[1,\ \infty)$ oraliqda $f(x)=\dfrac{1}{x^2}$ funksiya ostidagi soha yuzasini hisoblash so‘ralgan bo‘lsin.



Chizmadan ko‘rish mumkinki, bu soha yuzasi cheksiz. Biz haligacha bunday sohaning yuzasini hisoblab ko‘rmagan edik. Keling egri chiziq ostidagi soha yuzasini 1 dan b gacha oraliqda integrallaymiz.

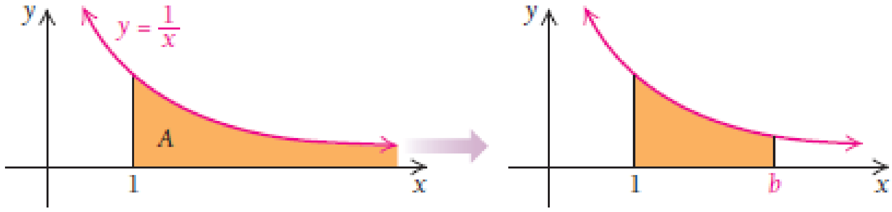
$$\int_1^b \frac{1}{x^2}dx=\left(-\frac{1}{x}\right)\Bigg|_1^b=\left(-\frac{1}{b}\right)-\left(-\frac{1}{1}\right)=1-\frac{1}{b}.$$

So‘ngra yuzani topish uchun $[1,\ b]$ oraliqda $b\to\infty$ da limit olamiz:

$$S(x)=\lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx=\lim_{b\to\infty}\left(1-\frac{1}{b}\right)=1.$$

Demak, so‘ralgan yuza chekli bo‘lib, 1 birlikka teng ekan.

Agar $[1,\ \infty)$ oraliqda $f(x)=\dfrac{1}{x}$ funksiya ostidagi soha yuzasini hisoblash kerak bo‘lsa-chi?



$$A(x) = \lim_{b\to\infty} \int_1^b \frac{1}{x} dx = \lim_{b\to\infty} (\ln x)\Big|_1^b = \lim_{b\to\infty} (\ln b - \ln 1) = \lim_{b\to\infty} \ln b.$$

$y = \ln x$ funksiya grafigini 4.2 bo‘limda ko‘rgan edik, bu funksiya cheksiz o‘suvchi funksiya. Shu sababli $A = \lim_{b\to\infty} \ln b$ yuza ham cheksiz bo‘ladi.

Ta’rif. $[a, \infty)$ oraliqda uzluksiz bo‘lgan funksiyaning xosmas integrali quyidagicha belgilanadi: $\int_a^{\infty} f(x)dx$

va ushbu tenglik bilan aniqlanadi: $\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$.

Agar ushbu formulada o‘ngdagi limit mavjud bo‘lsa, u holda xosmas integral **yaqinlashuvchi** deyiladi. Bu limit integralning qiymati sifatida qabul qilinadi.

Agar limit mavjud bo‘lmasa, xosmas integral **uzoqlashuvchi** deyiladi.

Agar integral ostidagi $f(x)$ funksiya uchun $F(x)$ boshlang‘ich funksiya ma’lum bo‘lsa, u holda xosmas integralning yaqinlashuvchi yoki uzoqlashuvchi ekanini aniqlash mumkin. Nyuton-Leybnits formulasi yordamida quyidagiga ega bo‘lamiz:

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx = \lim_{b\to\infty} F(x)\Big|_a^b = \lim_{b\to\infty} [F(b) - F(a)] = F(\infty) - F(a).$$

Shunday qilib, agar $x \to \infty$ da $F(x)$ boshlang‘ich funksiya ma’lum bo‘lsa, u holda xosmas integral yaqinlashuvchi, agar bu limit mavjud bo‘lmasa, u holda xosmas integral uzoqlashuvchi bo‘ladi.

1-misol. $[0, \infty)$ oraliqda $f(x) = e^{-kx}$ funksiyaning xosmas integralini hisoblang.

Yechilishi: ► Berilgan funksiya uchun $F(x) = -\frac{e^{-kx}}{k}$ boshlang‘ich funksiya bo‘ladi. Nyuton-Leybnits formulasini qo‘llaymiz:

$$\int_0^\infty e^{-kx}dx = \lim_{b\to\infty}\left(-\frac{e^{-kx}}{k}\Big|_0^\infty \right) = -\frac{1}{k}\lim_{b\to\infty}(e^{-kb} - 1).$$

Agar $k > 0$ bo‘lsa, $\int_0^\infty e^{-kx}dx = \frac{1}{k}$ integral yaqinlashuvchi.

Agar $k \le 0$ bo‘lsa, $\int_0^\infty e^{-kx}dx = \infty$ integral uzoqlashuvchi. ◄

Xosmas integral $(-\infty, \mathrm{b}]$ **oraliqda ham shunga o‘xshash aniqlanadi:**

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx = \lim_{a\to-\infty} F(x)\Big|_a^b = \lim_{a\to-\infty}[F(b) - F(a)] = F(b) - F(-\infty).$$

Agar $f(x)$ funksiya butun sonlar o‘qida uzluksiz bo‘lsa, u holda umumlashgan xosmas integral quyidagi formula bilan aniqlanadi:

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$

bu yerda c –ixtiyoriy tayinlangan nuqta.

Agar bu formulada o‘ng tomondagi ikkala integral ham yaqinlashuvchi bo‘lsa, u holda xosmas integral ham yaqinlashuvchi bo‘ladi.

2-misol. Ushbu integralning yaqinlashuvchiligini tekshiring:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Yechilishi: ► Yuqoridagi formulada c= 0 deb faraz qilib, quyidagini hosil qilamiz:
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

Tenglikning o‘ng qismidagi xosmas integrallar yaqinlashuvchi bo‘ladi, chunki
$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = arctgx\Big|_{-\infty}^{0} = arctg\,0 - \lim_{x\to-\infty} arctgx = \frac{\pi}{2};$$

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = arctgx\Big|_{0}^{\infty} = \lim_{x\to\infty} arctgx - arctg\,0 = \frac{\pi}{2}$$

Shunga ko‘ra,
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Demak, integral yaqinlashuvchi va uning qiymati π ga teng. ◄

1-vazifa. Hisoblang: a) $\int_{1}^{\infty} \frac{1}{x^3}dx$; b) $\int_{0}^{\infty} 4e^{-2x}dx$; c) $\int_{2}^{\infty} \frac{2}{x^3}dx$.

5.7.2. II tur xosmas integrallar

Ta'rif. $(a,b]$ oraliqda uzluksiz va $x=a$ da aniqlanmagan yoki **II** tur uzilishga ega bo'lgan $f(x)$ funksiyaning xosmas integrali quyidagicha belgilanadi: $\int_a^b f(x)dx$ va ushbu tenglik bilan aniqlanadi:

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^b f(x)dx.$$

Agar ushbu formulada o'ngda turgan limit mavjud bo'lsa, u holda xosmas integral **yaqinlashuvchi** deyiladi. Agar ko'rsatilgan limit mavjud bo'lmasa, u holda xosmas integral **uzoqlashuvchi** deyiladi.

Agar integral ostidagi $f(x)$ funksiya uchun $F(x)$ boshlang'ich funksiya ma'lum bo'lsa, u holda Nyuton-Leybnits formulasini qo'llash mumkin: $\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^b f(x)dx = \lim_{\varepsilon\to 0}[F(b)-F(a+\varepsilon)]$

Shunday qilib, agar $x\to a$ da $F(x)$ boshlang'ich funksiyaning limiti mavjud bo'lsa, u holda xosmas integral yaqinlashuvchi, agarda bu limit mavjud bo'lmasa, u holda xosmas integral uzoqlashuvchi bo'ladi.

$[a,\ b)$ intervalda uzluksiz va $x=b$ da aniqlanmagan yoki **II** tur uzilishga ega bo'lgan $f(x)$ funksiyaning xosmas integrali ham shunga o'xshash aniqlanadi:

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)dx.$$

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)dx = \lim_{\varepsilon\to 0}[F(b-\varepsilon)-F(a)].$$

Agarda $f(x)$ funksiya $[a,\ b]$ kesmaning biror-bir $x = c$ oraliq nuqtasida uzilishga ega yoki aniqlanmagan bo‘lsa, u holda xosmas integral quyidagi integral bilan aniqlanadi:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Agar formulaning o‘ng tomonida turgan integrallardan aqalli bittasi uzoqlashuvchi bo‘lsa, u holda xosmas integral uzoqlashuvchi bo‘ladi.

Agar uning o‘ng tomonidagi ikkala integral yaqinlashuvchi bo‘lsa, u holda xosmas integral ham yaqinlashuvchi bo‘ladi.

3-misol. Ushbu integralning yaqinlashuvchiligini tekshiring.

$$\int_0^4 \frac{dx}{\sqrt{x}}$$

Yechilishi. ► $x \to 0$ da $f(x) = \frac{1}{\sqrt{x}} \to \infty$. $x = 0$ nuqta $[0, 4]$ kesmaning chap oxirida yotadi. Shuning uchun quyidagiga ega bo‘lamiz:

$$\int_0^4 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0}\int_\varepsilon^4 \frac{dx}{\sqrt{x}} = 2\lim_{\varepsilon\to 0}\sqrt{x}\Big|_\varepsilon^4 = 2\cdot 2 = 4$$

Integral yaqinlashuvchi. ◄

Ishorasi almashinuvchi funksiyalarning xosmas integrallarini hisoblashda nomanfiy funksiya bo‘lgan holga olib kelinadi:

Agar $\int_a^\infty |f(x)|dx$ integral yaqinlashuvchi bo‘lsa, u holda $\int_a^\infty f(x)dx$ integral ham yaqinlashuvchi bo‘ladi.

Bunda oxirgi integral **absolyut yaqinlashuvchi integral** deyiladi.

Agar $\int_a^{\infty} f(x)dx$ integral yaqinlashuvchi, $\int_a^{\infty} |f(x)|dx$ integral esa uzoqlashuvchi bo‘lsa, u holda $\int_a^{\infty} f(x)dx$ integral **shartli yaqinlashuvchi integral** deyiladi.

4-misol. Ushbu integrallarning yaqinlashuvchiligini tekshiring.

$$\int_0^{\infty} \frac{\cos xdx}{1+x^2} \quad \text{va} \quad \int_0^{\infty} \frac{\sin xdx}{1+x^2}$$

Yechilishi: ► Integral ostidagi funksiyalar ushbu shartlarni qanoatlantiradi:

$$\left|\frac{\cos x}{1+x^2}\right| \le \frac{1}{1+x^2}, \qquad \left|\frac{\sin x}{1+x^2}\right| \le \frac{1}{1+x^2}.$$

$$\int_0^{\infty} \frac{dx}{1+x^2} = arctgx\Big|_0^{\infty} = \lim_{x\to\infty} arctgx - arctg0 = \frac{\pi}{2}$$

integral yaqinlashuvchi, shuning uchun

$$\int_0^{\infty} \left|\frac{\cos x}{1+x^2}\right| dx \quad \text{va} \quad \int_0^{\infty} \left|\frac{\sin x}{1+x^2}\right| dx$$

integrallar ham yaqinlashuvchi bo‘ladi. ◄

MUSTAQIL YECHISH UCHUN MISOLLAR

1-48 misollarda xosmas integrallarni hisoblang yoki uzoqlashuvchi ekanligini ko‘rsating:

1. $\int_0^1 \frac{dx}{\sqrt{x}}$ 2. $\int_0^2 \frac{dx}{x}$ 3. $\int_0^1 \frac{dx}{x^p}$

4. $\int_0^3 \frac{dx}{(x-1)^2}$ 5. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ 6. $\int_1^\infty \frac{dx}{x}$

7. $\int_1^\infty \frac{dx}{x^2}$ 8. $\int_1^\infty \frac{dx}{x^p}$ 9. $\int_{-\infty}^\infty \frac{dx}{1+x^2}$

10. $\int_{-\infty}^\infty \frac{dx}{x^2+4x+9}$ 11. $\int_0^\infty \sin x\,dx$ 12. $\int_0^{1/2} \frac{dx}{x\ln x}$

13. $\int_0^{1/2} \frac{dx}{x\ln^2 x}$ 14. $\int_a^\infty \frac{dx}{x\ln x}$ $(a>1)$ 15. $\int_a^\infty \frac{dx}{x\ln^2 x}$ $(a>1)$

16. $\int_0^{\pi/2} ctg\,x\,dx$ 17. $\int_0^\infty e^{-kx}\,dx$ $(k>0)$ 18. $\int_0^\infty \frac{arctg\,x}{x^2+1}dx$

19. $\int_2^\infty \frac{dx}{(x^2-1)^2}$ 20. $\int_0^\infty \frac{dx}{x^3+1}$ 21. $\int_0^\infty \frac{xdx}{16x^2+1}$

22. $\int_0^1 \frac{dx}{\sqrt{2-4x}}$ 23. $\int_1^\infty \frac{16xdx}{16x^4-1}$ 24. $\int_{-1}^3 \frac{2x-3}{\sqrt[3]{x^2}}dx$

25. $\int_0^\infty \frac{x^3dx}{\sqrt{16x^4+1}}$ 26. $\int_0^1 \frac{e^{1/x}}{x^2}dx$ 27. $\int_1^3 \frac{dx}{\sqrt[3]{2-4x}}$

28. $\int_{-\infty}^0 \frac{xdx}{\sqrt{(x^2+1)^3}}$ 29. $\int_{1/2}^2 \frac{\ln(2x-1)}{2x-1}dx$ 30. $\int_{1/4}^1 \frac{dx}{20x^2-9x+1}$

31. $\int\limits_0^{1/3}\frac{\cos x dx}{\sqrt[7]{\sin^2 x}}$ 32. $\int\limits_4^{\infty}\frac{xdx}{\sqrt{x^2-4x+1}}$ 33. $\int\limits_0^{1}\frac{xdx}{1-x^4}$

34. $\int\limits_0^{1}\frac{2xdx}{\sqrt{1-x^4}}$ 35. $\int\limits_1^{\infty}\frac{xdx}{x^2-4x+5}$ 36. $\int\limits_0^{\pi/6}\frac{\cos 3x}{\sqrt[3]{(1-\sin 3x)^2}}dx$

37. $\int\limits_1^{\infty}\frac{arctg\,2x}{4x^2+5}dx$ 38. $\int\limits_{1/2}^{1}\frac{dx}{(1-x)\ln^2(1-x)}$ 39. $\int\limits_0^{2/3}\frac{\sqrt[3]{\ln(2-3x)}}{2-3x}dx$

40. $\int\limits_0^{\infty}\frac{dx}{4x^2+4x+5}$ 41. $\int\limits_0^{\infty}\frac{xdx}{9x^2+6x+5}$ 42. $\int\limits_{4/5}^{1}\frac{5dx}{\sqrt[3]{4-5x}}$

43. $\int\limits_0^{\infty}\frac{(x+3)dx}{\sqrt[3]{x^2+6x+5}}$ 44. $\int\limits_0^{\pi/2}\frac{e^{tgx}dx}{\cos^2 x}$ 45. $\int\limits_0^{\infty}\frac{3-x^2}{x^2+4}dx$

46. $\int\limits_0^{1}\frac{e^{\pi-\arcsin x}dx}{\pi\sqrt{1-x^2}}$ 47. $\int\limits_1^{2}\frac{3dx}{\sqrt[3]{4x-x^{2}-4}}$ 48. $\int\limits_0^{\infty}\frac{3+\sqrt{arctg\,3x}dx}{9x^2+1}$.

49. $y=\frac{1}{x}$ funksiya grafigini $[1;\ \infty]$ kesmada Ox o‘qi atrofida aylantirishdan hosil bo‘lgan jism hajmini toping.