BUSINESS MATHEMATICS

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All along a brilliant student Dr. S. M. Shahidul Islam son of Late Tajim Uddin Ahmed was born in 1976 at Kumria, Dinajpur, Bangladesh. At present, he is an Associate Professor in the Department of Mathematics at Hajee Mohammad Danesh Science and Technology University, Dinajpur. His many research papers are appeared in many prestigious journals such as International Journal of Production Economics, International Journal of Operational Research, etc. His book named Linear Programming is followed and appreciated by the teachers and the students of different universities.

Business Mathematics

Dr. S. M. Shahidul Islam

$$F = P(1+i)^n$$

Written in accordance with the new syllabus of the National University, Dhaka University, Chittagong University, Rajshahi University, Asian University of Bangladesh and East West University for the students of the Department of Business administrations, Management, Accounting, Marketing and Finance.

Business Mathematics

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PREFACE

DEDICATED TO MY PARENTS

Bismillahir Rahmanir Rahim.

The book titled *Business Mathematics* has been written with the blessings of Almighty Allah. It is clear that the use of mathematical tools has been on the increase day by day in business arena and now-a-days business mathematics is a compulsory course in all business related disciplines. But the fact is that our students are always afraid of mathematics and they need proper guidance and dependable books. There are text books available in the market on Business Mathematics but most of them are inadequate in meeting the requirements of the students. So in order to gear up the students in business mathematics we have attempted to write the book using easy language and mathematical techniques.

The text is written with the basic objects of introducing students of business administration to the mathematical concepts that help in decision making. An attempt has been made to present explanations in such a way that the underlying mathematical theories are fully exposed and the relation between theory and application thoroughly understood. The text will also cover a large part of *Production & Operations Management* and *Management Science*.

We have included in the book a large number of examples, mostly collected from question papers of different universities. We have verified every question and their solution minutely, and if there is any error yet, we beg apology for that.

I am grateful and indebted to my teachers Professor Dr. A.A.K. Majumdar, Ritsumeikan Asia-Pacific University, Japan; Professor Dr. Nurul Alam Khan and Professor Dr. Makbul Hossen of the department of Mathematics, Jahangirnagar University; Professor Dr. Abdul Awal Khan, Dean, School of Business, Asian University of Bangladesh and many of my colleagues who encouraged and helped me in writing the book. I am also grateful to the authors whose books I have freely consulted in the preparation of the manuscript. The authority of Abir Publication deserves thanks for undertaking the responsibility to publish the book in these days of hardships in publication business.

Finally, special recognition should go to the endless co-operation of my wife without which the book could not have seen the light of the day.

Constructive criticisms, corrections, and suggestions towards the improvement of the book will be thankfully received.

S. M. Shahidul Islam

National University

Subject: Management (Hons.)

Course: NM-201 *Business Mathematics*

1. **Basic Concepts :** Concepts of number system, fractions, exponents, equations, factoring, polynomials, ordered pairs, relations, functions, types of functions.
2. **Set Theory:** Sets, set notation, operations with sets, laws of set operations, Venn diagrams, application of set theory.
3. **Logarithms:** Rules for logarithms, common logarithms, calculation of logarithm of a number, natural logarithm.
4. **Equation system:** Solution of equations, simultaneous equation system, solution of simultaneous equation systems with specific applications to business problems, inequalities.
5. **Geometry:** Cartesian co-ordinate system, distance between two points, straight line-slopes-intercepts, equation of a line, application of linear equations.
6. **Differential Calculus:** Explanation of the concepts of limits and continuity, derivative and differentiation, rules of differentiation, higher order differentiation, chain order differentiation, exponential and logarithmic differentiation, partial differentiation, optimization, rate of growth and decays, business applications.
7. **Integral Calculus:** Meaning of integration, rules of integration, indefinite integral, definite integral, resource depletion, resource accumulation, area between curves, business applications of integration in business decisions.
8. **Matrix Algebra:** Vectors, matrices, laws and operations, transposes, inverses, adjoints, Cramer’s rule, determinants, solution of system of equations, application of matrix algebra in business.
9. **Mathematics of Finance:** Annuities, sinking fund, discount, simple and compound interest, amortization, calculation of present value and future value and future value of annuities.

Asian University of Bangladesh

BBA Elementary Mathematics and Statistics

The topics include number system, set theory, functions & equations, exponential & logarithmic functions, collection of data, presentation of data, measures of central tendency (mean, median & mode).

BBA 2309 Business Mathematics

The topics include coordinate geometry, mathematics of finance, matrix algebra, calculus and linear programming. This course will familiarize students with the more commonly used types of quantitative techniques used in economics and business.

MBA 6127 Business Mathematics

The course includes elements of algebra, number fields, linear and non-linear inequalities, functions, set, analytical geometry, logarithm, limit, differential and integral calculus, matrix and linear programming. The purpose of the course is to help the students learn mathematical tools which are used in business arena.

East West University

MAT 110 : Mathematics For Business and Economics I

Topics include: Set, liner equations and inequalities in one variable, quadratic equations, Cartesian coordinate system and straight lines, function, linear and quadratic functions, exponential and logarithmic functions, system of liner equations, matrices, permutation and combination, binomial theorem, arithmetic and geometric progression.

MAT 311 : Mathematics for Business and Economics II

Topics include: Economic and business models, functions, limits and continuity, concept of derivative, rules of differentiation and integration, and their use. Constrained optimization with Lagrange multiplier, partial derivatives. Theory is presented informally and techniques are related to polynomials, logarithmic and exponential functions.

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01

Chapter

Number System

Highlights:

1.1 Introduction: To fulfill daily human needs, man invented counting numbers at the beginning of the civilization. The successive development of numbers develops the modern mathematics. Mathematics means games of numbers. In business mathematics, we know the use of numbers in business problems. So, to clear the concepts of mathematics or business mathematics, firstly, we have to clear the concepts of number system. Number system means the nature and properties or various types of numbers. In this chapter, we shall learn about the system of numbers. After this chapter, number will mean real number.

1.2 Natural numbers: The positive integer numbers 1, 2, 3, 4, . . . are used for counting. These numbers are known as natural numbers. The set of natural numbers is denoted by English capital letter, N. That is, N = {1, 2, 3, 4, 5, . . .}. Negative numbers, zero and fractions are not natural numbers, that is, – 5, 0 and $\frac{5}{2}$ = 2.5 are not natural numbers.

Properties:

1. Summation or multiplication of any two or more natural numbers must be a natural number. That is, if m, $n \in N$ then $(m + n) \in N$ and $(m \times n) \in N$. As for example $7,10 \in N$; $7+10 = 17 \in N$ and $7 \times 10 = 70 \in N$.
2. Square of any natural number is also a natural number. That is, if $m \in N$ then $m^2 \in N$. As for example $4,5 \in N$ then $4^2 = 16 \in N$ and $5^2 = 25 \in N$.

3. Subtraction and Division of any two natural numbers may not be natural number. That is, if m, n $\in$ N then m – n and $\frac{m}{n}$ may be or may not be natural number. As for example 2, 7, 10 $\in$ N; 10 – 7 = 3 $\in$ N but 7 – 10 = – 3 $\notin$ N, $\frac{10}{2}$ = 5 $\in$ N but $\frac{7}{10}$ = 0.7 $\notin$ N.
4. Square root of a natural number may be or may not be a natural number. That is, if m $\in$ N then $\sqrt{m}$ $\in$ N or $\sqrt{m}$ $\notin$ N. As for example 4, 5 $\in$ N then $\sqrt{4}$ = 2 $\in$ N but $\sqrt{5}$ $\notin$ N.
5. Between two different natural numbers one must be greater than another number. That is, if m, n $\in$ N then m > n or n > m. As for example 5, 7 $\in$ N then here 7 > 5.

1.3 <u>Prime numbers</u>: A natural number other than 1 is a prime number if and only if its only divisors are 1 and the number itself. Such as 2, 3, 5, 7, 11, . . . etc. are prime numbers.

1.4 <u>Integer numbers</u>: The natural numbers, all natural numbers with negative sign before and zero together known as integer numbers. That is, the integers are whole numbers positive, negative and zero. The set of integer numbers is denoted by English capital letter, Z or I. That is, Z = {. . ., –3, –2, –1, 0, 1, 2, 3, . . .}. Fractions are not integer number, that is, $\frac{5}{2}$ = 2.5 is not integer number. It is clear that, all natural numbers are integer number.

Properties:

1. Summation or subtraction or multiplication of any two integer numbers must be an integer number. That is, if m, n $\in$ Z then (m + n) $\in$ Z, (m – n) $\in$ Z and (m $\times$ n) $\in$ Z. As for example 5, – 3 $\in$ Z then 5 +(– 3) = 2 $\in$ Z, 5 – (– 3) = 8 $\in$ Z and 5 $\times$ (– 3) = –15 $\in$ Z
2. Square of any integer number is also an integer number. That is, if m $\in$ Z then m^2 $\in$ Z. As for example 4, –5 $\in$ Z; 4^2 =16 $\in$ Z and $(–5)^2$ = 25 $\in$ Z.
3. Division of any two integer numbers may be or may not be integer number. That is, if m, n $\in$ Z; then $\frac{m}{n}$ may be or may not be integer number. As for example 2, 5, 8 $\in$ Z; then $\frac{8}{2}$ = 4 $\in$ Z but $\frac{5}{2}$ = 2.5 $\notin$ Z.
4. Square root of an integer number may be or may not be an integer number. That is, if m $\in$ Z then $\sqrt{m}$ $\in$ Z or $\sqrt{m}$ $\notin$ Z. As for example 4, 5 $\in$ Z then $\sqrt{4}$ = 2 $\in$ Z but $\sqrt{5}$ $\notin$ Z.
5. Between two different integer numbers one must be greater than another number, that is, if m, n $\in$ Z then m > n or n > m. As for example 5, 7 $\in$ Z; here 7 > 5.

1.5 <u>Even numbers</u>: The integer numbers, which are divisible by 2 are called even numbers. Such as . . ., –6, –4, –2, 0, 2, 4, 6, . . . are even numbers. Generally, an even number is denoted by 2n where $n \in Z$. If addition or subtraction or multiplication or square or square root of even number(s) exists as integer, it must be an even number. But division of two even numbers may or may not be an even number, if it exists as integer.

1.6 <u>Odd numbers</u>: The integer numbers, which are not divisible by 2 are called odd numbers. Such as . . ., –5, –3, –1, 1, 3, 5, . . . are odd numbers. Generally, an odd number is denoted by 2n –1 where $n \in Z$. If multiplication or square or square root of odd number(s) exists as integer, it must be an odd number.

<u>Example</u>: Prove that, if we divide the square of any odd positive integer by 8, remainder will be 1 always.

Proof: Let x be any positive odd integer number, that is, odd natural number. So, we can let

$x = 2n - 1$, where $n \in N$

$\therefore \quad x^2 = (2n-1)^2 = 4n^2 - 4n + 1 = 4n(n-1) + 1$

Here, n and (n –1) are two consecutive natural number, that is, one of them is an even number.

So, n(n –1) is divisible by 2, that is, 4n(n –1) is divisible by $4 \times 2 = 8$.

Therefore, if $x^2 = 4n(n-1) + 1$, for all $n \in N$ is divided by 8, remainder will be 1 always.

1.7 <u>Rational numbers</u>: The numbers which can be expressed in the form $\frac{p}{q}$ where p, q are integers and q is not equal to zero are called rational numbers. The set of rational numbers is denoted by English capital letter, Q. That is, $Q = \{\frac{p}{q} : p, q \in Z \text{ and } q \neq 0\}$. 7, –2 and 2.5 are rational number because $7 = \frac{7}{1}$, $-2 = \frac{-2}{1}$ and $2.5 = \frac{5}{2}$. We know that, $4 = \frac{4}{1}$, $-4 = \frac{-4}{1}$, that is, every integer number can be expressed in $\frac{p}{q}$ form. So, all integer numbers are rational number.

Properties:

1. Summation or subtraction or multiplication or division or square of rational number(s) must be a rational number. That is, if $m, n \in Q$ then $(m + n) \in Q$, $(m - n) \in Q$, $(m \times n) \in Q$, $m/n \in Q$ and $m^2 \in Q$. As for example 7, $10 \in Q$ then $7 + 10 = 17 \in Q$, $7 - 10 = -3 \in Q$, $7 \times 10 = 70 \in Q$, $7/10 = 0.7 \in Q$ and $7^2 = 49 \in Q$.
2. Square root of a rational number may be or may not be a rational number. That is, if $m \in Q$ then $\sqrt{m} \in Q$ or $\sqrt{m} \notin Q$. As for example 4, $5 \in Q$ then $\sqrt{4} = 2 \in Q$ but $\sqrt{5} \notin Q$.

3. Between two different rational numbers one must be greater than another number. That is, if m, n $\in$ Q then m > n or n > m. As for example 5.4, 7 $\in$ Q; here 7 > 5.4.
4. There are infinite rational numbers between any two different numbers. Let us consider any two different numbers 2 and 3, then all of 2.01, 2.001, 2.0001, 2.00001, 2.1, 2.2, 2.0000002, 2.5, 2.45, $2.\dot{5}$, 2.9452, $2.4\dot{4}7\dot{5}$, ... etc. are greater than 2 and less than 3 but all are rational numbers.

1.7.1 <u>Theorem</u>: Division of any two rational numbers (divisor must be non zero) is a rational number.

Proof: Let us consider, any m, n $\in$ Q and let n be divisor, that is, n $\neq 0$.
So, by the definition of rational numbers, we get

m = $\frac{p_1}{q_1}$ and n = $\frac{p_2}{q_2}$; where $p_1, p_2, q_1, q_2 \in Z$ and $p_2, q_1, q_2 \neq 0$

Now division of m and n is $\frac{m}{n} = \frac{\frac{p_1}{q_1}}{\frac{p_2}{q_2}} = \frac{p_1}{q_1} \times \frac{q_2}{p_2} = \frac{p_1 q_2}{p_2 q_1} = \frac{p}{q(\neq 0)} \in Q$.

[We know that, multiplication of integers is an integer. So, let $p_1 q_2$ = p $\in$ Z, $p_2 q_1$ = q $\in$ Z and $p_2 q_1$ = q $\neq 0$ (because of $p_2, q_1 \neq 0$).]

Therefore, Division of any two rational numbers is a rational number. (Proved)

1.7.2 <u>Theorem</u>: Between two different rational numbers, there lie an infinite number of rational numbers.

Proof: Let m, n be any two rational numbers such that m < n.

Now, m < n
⇨ m + n < n + n
⇨ m + n < 2n
⇨ $\frac{1}{2}$(m + n) < n

Again m < n
⇨ m + m < m + n
⇨ 2m < m + n
⇨ m < $\frac{1}{2}$(m + n)

So, m < $\frac{1}{2}$(m + n) < n

By properties of rational numbers, $\frac{1}{2}$(m + n) is a rational number between m and n.

We have thus shown that between two different rational numbers there lies a third rational number. From this it follows that there lie an infinite number of rational numbers between two different rational numbers. [Proved]

1.8 Fraction numbers: The numbers which can be expressed in the form $\frac{p}{q}$ where p$\in$ Z–{0}, q$\in$N–{1} and p, q are prime to each other are called fraction numbers. That is, the rational numbers, which leave a remainder, are called fraction numbers. Such as $\frac{5}{3}$, $\frac{7}{10}$, 2.1 = $\frac{21}{10}$ are fraction numbers. When we write the fraction number in decimal form is called decimal fraction number. Such as 5.3, 2.4, 67.23, $2.4\dot{4}7\dot{5}$ etc. are decimal fraction numbers.

Example: Convert into the rational form: (i) 3.5, (ii) $2.3\dot{5}\dot{2}$

Solution: (i) 3.5 = $\frac{35}{10} = \frac{7}{2}$ (Answer)

(ii) $2.3\dot{5}\dot{2} = \frac{2352-23}{990}$

$$\left[\frac{\text{All the digits (neglecting decimals)} - \text{The digits without recurring decimals}}{9 \text{ for each digits with recurring decimals } \& \text{ } 0 \text{ for each digits with decimal but not with recurring decimals}}\right]$$

$= \frac{2329}{990}$ (Answer)

1.9 Irrational numbers: The numbers which can not be expressed in the form $\frac{p}{q}$ where p, q are integers and q not equal to zero are called irrational numbers, that is, the real numbers which are not rational number are called irrational numbers. The set of irrational numbers is denoted by $Q^{/}$, that is, $Q^{/}$ = {the real numbers, which are not rational}. Such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, Π = 3.1415927... etc. are irrational numbers because we cannot express these numbers in the form $\frac{p}{q}$.

Properties:

1. Summation or subtraction or multiplication or division of different irrational numbers is an irrational number. That is, if m, n$\in Q^{/}$ then (m + n)$\in Q^{/}$, (m – n) $\in Q^{/}$, (m$\times$n)$\in Q^{/}$ and m/n$\in Q^{/}$. As for example $\sqrt{2}$, $\sqrt{3} \in Q^{/}$; ($\sqrt{2}$ + $\sqrt{3}$)$\in Q^{/}$, ($\sqrt{3}-\sqrt{2}$)$\in Q^{/}$, $\sqrt{2} \times \sqrt{3} = \sqrt{6} \in Q^{/}$ and $\sqrt{3}/\sqrt{2} \in Q^{/}$.

2. Square of any irrational number is a rational number. That is, if m $\in Q'$ then $m^2 \in$ Q. As for example $\sqrt{2} \in Q'$ then $\left(\sqrt{2}\right)^2 = 2 \in$ Q.
3. Square root of an irrational number must be an irrational number. That is, if m $\in Q'$ then $\sqrt{m} \in Q'$. As for example $\sqrt{2} \in Q'$; $\sqrt{\sqrt{2}} \in Q'$.
4. Algebraic operations between a rational number and an irrational number make the results irrational numbers. That is, if m $\in$ Q and n $\in Q'$ then (m + n), (m – n), m $\times$ n and m/n are irrational numbers. As for example $2 \in$ Q and $\sqrt{2} \in Q'$ then $(2 + \sqrt{2}) \in Q'$, $(2 - \sqrt{2}) \in Q'$, $2\sqrt{2} = 2\sqrt{2} \in Q'$ and $2/\sqrt{2} = \sqrt{2} \in Q'$.
5. Between two different irrational numbers one must be greater than another number. That is, if m, n $\in Q'$ then m >n or n >m. As for example $\sqrt{2}, \sqrt{3} \in Q'$; here $\sqrt{3} > \sqrt{2}$.
6. There are an infinite irrational numbers between any two different numbers. Let us consider any two different numbers 2 and 3, then all of 2.010564..., 2.00145379..., 2.00014512679..., 2.000017612389..., 2.14367845... etc. are greater than 2 and less than 3 but all are irrational numbers.

Note: All integer numbers, finite decimal numbers and recurring decimal numbers are rational numbers because they can be expressed in the form $\frac{p}{q}$, but all infinite decimal numbers are irrational numbers because they can not be expressed in the form $\frac{p}{q}$. Such as 6, -7, 10, 4.5, 6.04, 10.0001, $0.\dot{5}$, $5.03\dot{4}7\dot{5}$ $\left(= \frac{503475 - 503}{99900} = \frac{125743}{24975}\right)$ and $237.\dot{9}\dot{0}$ are rational numbers, but 1.4142136..., 3.6055513... and 3.1415927... are irrational numbers.

<u>Example</u>: Show that $\sqrt{2}$ is an irrational number. [AUB-2002 BBA]

Proof: $1^2 = 1$, $2^2 = 4$ and $\left(\sqrt{2}\right)^2 = 2$

So, $\sqrt{2}$ is greater than 1 and less than 2, that is, $\sqrt{2}$ is not an integer number.

Thus, let $\sqrt{2} = \frac{p}{q}$, where p, q $\in$ N, q > 1 and p, q is co-prime.

That is, $2 = \frac{p^2}{q^2}$

Or, $\quad 2q^2 = p^2 \quad$ - - - (i)

Here, left hand number is an even number, so right hand number p^2 or p must also be an even number.

Let p = 2n, then from (i) we have

$2q^2 = (2n)^2$

Or, $2q^2 = 4n^2$

Or, $q^2 = 2n^2$

Here, right hand number is an even number, so left hand number q^2 or q must also be an even number.

It means that both p and q are even numbers, that is, p and q have a common factor at least 2, which contradicts the fact that p and q are integers prime to each other (co-prime).

Hence, $\sqrt{2} = \frac{p}{q}$ is absurd. So, $\sqrt{2}$ cannot be a rational number, that is, $\sqrt{2}$ is an irrational number. [Proved]

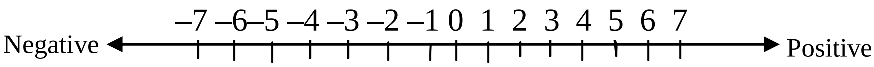
1.10 <u>Real numbers</u>: Rational and irrational numbers together known as real numbers. The set of real numbers is denoted by R; that is, R = {x: x $\in$ Q or x $\in$ Q' }. Such as 2, 5, – 6, 3.54, – 7.8903, $\sqrt{2}$, – $\sqrt{3}$, 45.8769403..., $\frac{5}{7}$ etc. are real numbers. From above discussion, we find the following relation:

$$N \subset Z \subset Q \subset R$$

Thus, natural numbers constitute a proper subset of integers and the integers constitute a proper subset of rational numbers and the latter constitutes a proper subset of real numbers.

Properties:

1. Summation or subtraction or multiplication or division or square of real number(s) must be a real number. That is, if m, n $\in$ R then (m + n) $\in$ R, (m – n) $\in$ R, (m $\times$ n) $\in$ R, m/n $\in$ R and $m^2 \in$ R. As for example 7, 20 $\in$ R then 7 + 20 = 27 $\in$ R, 7–20 = –13 $\in$ R, 7 $\times$ 20 = 140 $\in$ R, 7/20 = 0.35 $\in$ R and $7^2 = 49 \in$ R.
2. Even power of any real number is positive real number. So, in particular, square of any real number is a positive number. That is, if m $\in$ R then $m^2 \in R_+$ (R_+ is the set of positive real numbers.). As for example 3.4 $\in$ R; $(3.4)^2 = 11.56 \in R_+$.
3. Square root of a real number may be or may not be a real number. That is, if m $\in$ R then $\sqrt{m} \in$ R or $\sqrt{m} \notin$ R. As for example 4, – 2 $\in$ R then $\sqrt{4} = 2 \in$ R but $\sqrt{-2} \notin$ R.
4. Between two different real numbers one must be greater than another number. That is, if m, n $\in$ R then m > n or n > m. As for example 5, 7 $\in$ R; here 7 > 5.
5. There are infinite real numbers between any two different real numbers. Let us consider any two different numbers 2 and 3, then all of 2.1, 2.0021, 2.0001, 2.00001, 2.71, 2.02, 2.0000002, 2.35, 2.405, $2.0\dot{5}$, 2.52, $2.0\dot{4}7\dot{5}$, ... etc. are greater than 2 and less than 3 but all are real numbers.
6. We can represent all real numbers by a straight line. This straight line is known as numbers line or directed line. The line is as follows:



Number line
Figure – 1.1

7. If m, n $\in$ R and m $\times$ n = 0, then at least one of m and n is 0. As for example $0 \times 5 = 0$, $5 \times 0 = 0$ and $0 \times 0 = 0$ but $4 \times 5 \neq 0$.

<u>Example</u>: Solve the polynomial equation: $x^2 + 3x - 10 = 0$.
Solution: Given that, $x^2 + 3x - 10 = 0$
Or, $x^2 + 5x - 2x - 10 = 0$ [By middle term break-up method]
Or, $x(x + 5) - 2(x + 5) = 0$
Or, $(x + 5)(x - 2) = 0$ [We know, if $m \times n = 0$ then m = 0 or n = 0]
So, $x + 5 = 0 \therefore x = -5$
Or, $x - 2 = 0 \therefore x = 2$
Therefore, the solution, x = – 5, 2 [Answer]

1.11 <u>Interval</u>: The interval or domain of x is denoted by $a \le x \le b$ or [a, b]; where $a \le b$ and a,b $\in$ R. This means the value of x contains any real number from a to b. This interval is known as closed interval because it starting and ending values are known, that are a and b respectively. No one knows the immediate before real number of a known real number as well as the immediate after real number of that number. So, if a and b be omitted from the interval $a \le x \le b$, it is indicated as $a < x < b$ or (a, b). It means the value of x contains any real number from a to b except a and b. This interval is called opened interval because it starting and ending values are not known.

1.12 <u>Modulus of real number</u>: (Absolute numerical value) The modulus of a real number, a is defined as the real number a or – a according as a is non-negative or negative. The modulus of a real number, a is denoted by $|a|$ and is defined by

$$|a| = \begin{cases} a, \text{ if } a \text{ is non-negative} \\ -a, \text{ if } a \text{ is negative} \end{cases}$$

As for example $|5| = 5$, $|-5| = 5$ and $|0| = 0$.

Properties:

1. The modulus of a real number, a is always non-negative, i.e., $|a| \ge 0$.
2. The modulus of a real number, a is always greater or equal to that number, i.e., $|a| \ge a$.
3. The modulus of a and – a is equal, i.e., $|a| = |-a|$.
4. The modulus of a is the maximum value of a and – a, i.e., $|a| = \max. \{a, -a\}$.
5. The modulus of a is the positive square root of the square of a, i.e., $|a| = \sqrt{a^2}$.

Example: Find the solution set of $|3x+2|<7$ and represent it in the number line.

Solution: Given that $|3x+2|<7$

If (3x + 2) be non-negative, then $|3x+2|$ = 3x + 2

That is, 3x + 2 < 7

Or, 3x + 2 – 2 < 7 – 2 [Subtracting 2 from both sides.]

Or, 3x < 5

$\therefore$ $x < \frac{5}{3}$ [Dividing both sides by 3]

If (3x + 2) be negative, then $|3x+2|$ = – (3x + 2)

That is, – (3x + 2) < 7

[We know, 5 > 3 but –5 < –3. For this inequality sign (>, <) changes when we multiply by a negative number]

Or, 3x + 2 > – 7 [Multiplying both sides by –1]

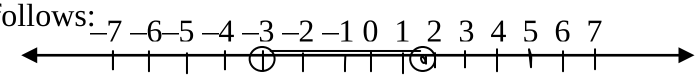
Or, 3x + 2 – 2 > – 7 – 2 [Subtracting 2 from both sides.]

Or, 3x > – 9

$\therefore$ x > – 3 [Dividing both sides by 3]

Therefore, the solutions is $x < \frac{5}{3}$ and x > – 3.

And the solution set, S = {$x \in R$: $x < \frac{5}{3}$ and x > – 3}. The number line representing solution set is as follows:



Number line $\frac{5}{3}$

Figure – 1.2

1.13 <u>Imaginary numbers</u>: Square root of negative numbers is called imaginary numbers, because square of any real number is positive only. As for example $\sqrt{-2}, \sqrt{-3}, \sqrt{-4}, \sqrt{-9}$ etc. are imaginary numbers. Imaginary number $\sqrt{-1}$ is always denoted by i, that is, i = $\sqrt{-1}$. So, $i^2 = -1$ and $i^3 = i^2.i = -1.i = -i$. Therefore, $\sqrt{-9} = \sqrt{(-1).9} = \sqrt{-1}.\sqrt{9} = i3 = 3i$.

1.14 <u>Complex numbers</u>: If 'a' and 'b' are two real numbers then the number of the form a+ib is known as a complex number. It has a real part, 'a' and the imaginary part, 'b'. As for example 4 + i3, 0 + i2, 2 + i0, 3 + i5 and 3 – i5 are complex numbers. The complex number z = (a – ib) is the conjugate of $\bar{z}$ = (a + ib).

We can express every real number as complex number. As for example, complex form of real numbers 3, 4.5 and – 3.2 are 3 + i0, 4.5 + i0 and – 3.2 + i0 respectively.

<u>Example</u>: Multiply 3 + i5 and 3 – i5.

Solution: Multiplication: $(3 + i5)(3 - i5) = 3.3 - 3.i5 + i5.3 - i5.i5$
$= 9 - 15i + 15i - 25i^2$
$= 9 - 25(-1)$ [We know, $i^2 = -1$]
$= 9 + 25$
$= 34$, is a real number.

1.15 <u>Number systems in chart</u>: From the above discussion we now present the various number systems in the form of following chart:

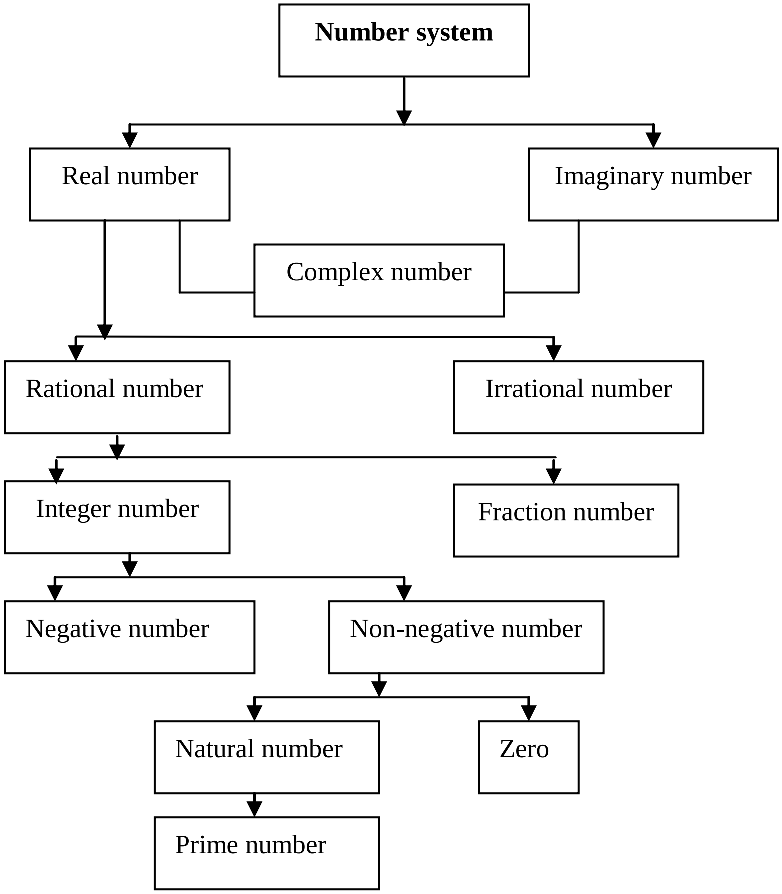


Figure – 1.3

1.16 <u>Exercise</u>:

1. Define with examples: (i) counting number, (ii) decimal fraction number, (iii) co-prime numbers.
2. Is it possible to convert every real number as complex number? And how?
3. Is every integer number rational number? Discuss your opinion.

4. Discuss the differences between: (i) natural and integer numbers;
(ii) rational and irrational numbers;
(iii) imaginary and complex numbers.
5. State whether the following statements are true or false:
   (i) Every rational number is a real number.
   (ii) Every irrational number is a real number.
   (iii) Every real number is a rational number.
   (iv) $2.1\dot{5}$ is a real number.
   (v) 6.4343434343 . . . is an irrational number.
   (vi) The product of two rational numbers is rational.
   (vii) The product of two odd integers is an odd integer.

   [Answer: (i) True, (ii) True, (iii) False, (iv) True, (v) False, (vi) True, (vii) True]
6. Write two rational numbers and two irrational numbers between 3.0001 and 3.0002. [Answer: Rational numbers: 3.00012316767, 3.000143435; Irrational numbers: 3.0001554684672 . . ., 3.00015463. . .]
7. Convert into the rational form: (i) 4.05, (ii) 7, (iii) $0.\dot{3}$, (iv) $2.1\dot{5}$ [Answer: (i) $\frac{81}{20}$, (ii) $\frac{7}{1}$, (iii) $\frac{1}{3}$, (iv) $\frac{97}{45}$]
8. Show that $\sqrt{3}, \sqrt{5}$ *and* $\sqrt{13}$ are irrational numbers.
9. Solve the following equations:
   (i) (x + 1)(x – 3.5) = 0 [Answer: x = – 1, 3.5]
   (ii) $x^2 + 5x + 6 = 0$ [Answer: x = – 2, – 3]
   (iii) $x^2 - 17x + 70 = 0$ [Answer: x = 7, 10]
10. Find the solution sets:
    (i) $|2x+10| = 6$ [Answer: {– 2, – 8}]
    (ii) $|x-2| \leq 7$ [Answer: {x∈R: - 5 ≤ x ≤ 9}]
    (iii) $|5x+2| \geq 5$ [Answer: {x∈R: x ≥ $\frac{3}{5}$ or x ≤ $-\frac{7}{5}$ }]
    (iv) 4 < $|2x|$ < 10 [Answer: {x∈R: 2 < x < 5 or – 2 > x > – 5}]
11. Prove that between two different real numbers, there lie an infinite number of real numbers.
12. Prove that between two different irrational numbers, there lie an infinite number of irrational numbers.
13. Find the square roots with complex number i:
    (i) – 4, (ii) – 8, (ii) – 16 [Answer: (i) 2i, (ii) $2\sqrt{2}\,i$, (iii) 4i]
14. Find the multiplications of (i) (3 + i8)(3 – i8), (ii) (2 + i5)(5 + i9) [Answer: (i) 73, (ii) – 35 + 43i]

02

Chapter

Set Theory

Highlights:

2.1 Introduction: Some simple concepts about groups or collections or sets are core ideas in mathematics. We use braces to indicate a set, and specify the members or elements of the set within the braces. The concepts of sets are used not only in mathematics but also in statistics and many other subjects. Here we shall discuss the concepts of sets and a few applications in business problems.

2.2 Definition of Set: A set is a collection of well-defined and well-distinguished objects. It is almost a convention to indicate sets by capital letters, like A, B, C etc. and to enclose its elements by second brackets, {}.

Example: 1) A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 0}
2) B = {x: x is the student of BBA 15^{th} batch of AUB}
3) C = {x: x be the possible outcomes in the toss of a die}

N.B. The basic characteristic of a set is that it should be well defined and its members or elements should be well distinguished for easy recognition by description.

2.3 Elements of a set: The objects that make up a set are called the elements or members of the set. If a be element of set A then we write $a \in A$ and is read as 'a belongs to A'. And if c is not an element of set A then we write $c \notin A$ and is read as 'c does not belong to A.'
Example: Let a set, A = {1, 2, 3, 4}. Here 1, 2, 3 & 4 are the elements of the set A, that is $1 \in A$, $2 \in A$, $3 \in A$ and $4 \in A$.

2.4 Methods of describing a set: There are two methods of describing a set. First one is Tabular/Roster/Enumeration method and second one is Selector/Property builder/Rule method.

i) Tabular method: In this method we enumerate or list all the elements of the set within second brackets.

Example: A = {a, b, c, d}, B = {0, 1, 2, 3}, etc.

ii) Selector method: In this method the elements are not listed but are indicated by description of their characteristics.

Example: A = {x: x is a vowel in English alphabet}

2.5 Types of sets: There are various types of sets. We describe below a few of them:

1) Finite set: When the elements of a set can be counted by a finite number then the set is called a finite set.

Example: A = {1, 2, 3, 4}
B = {1, 2, 3, . . ., 1000}
C = {x: $2 \le x \le 100$ and x is even integer}

2) Infinite set: If the elements of a set cannot be counted in a finite number, the set is called an infinite set.

Example: N = {1, 2, 3, . . .}
B = {x: x be even integer numbers}

3) Singleton: The set, which contains only one element is called a singleton or a unit set.

Example: A = {1}
B = {∅}
C = {x: x is an integer neither positive nor negative}
D = {x | $6 < x < 8$ and x is an integer number}

4) Null or Empty set: The set, which has no element, is called null or empty or void set. Generally it is denoted by a Greek letter ∅ (phi).

Example: ∅ = { }
A = {x: $6 < x < 7$ and x is an integer number}

5) Equal set: Two sets A and B are said to be equal if every element of A is also an element of B and every element of B is also an element of A.

Example: A ={1, 2, 3}
B = {1, 2, 3}

Here every element of A is also element of B and every element of B is also element of A. So set A and set B is equal set.

6) Equivalent set: If the element of a set can be put into one to one correspondence with the elements of another set, then the two set are called equivalent set. The symbol ' $\equiv$' is used to denote equivalent sets.

Example: $A = \{1, 2, 3, 4\}$

$B = \{a, b, c, d\}$

Here, set $A \equiv B$ because it is possible to make one to one correspondence among the elements of both sets.

***N.B*:** If two sets are infinite sets or contain same number of elements then they are equivalent sets. Example: Let, $A = \{x: x \in N$ and x be even integer$\}$

and $B = \{x: x \in N$ and x be odd integer$\}$

Then $A \equiv B$

7) Disjoint Sets: Two or more sets having no common element are called disjoint sets.

Example: Set $A = \{a, b, c, d\}$ and $B = \{p, q, r, s, t\}$ are disjoint sets because they have no common element.

8) Subsets: If every element of a set B is also an element of a set A then, set B is called subset of A and is written as, $B \subseteq A$ or $B \subset A$ or $A \supseteq B$ or $A \supset B$

Example: Let $A = \{1, 2, 3, 4, 5, 6\}$

And $B =\{2, 4, 6\}$

Then $B \subseteq A$ because of every element of B is an element of A.

***N.B*:** Every set has its 2^n number subsets where n is the number of elements of that set. Two of them are improper subsets and the remaining are proper subsets.

9) Proper Subset: Set B is called proper subset of super set A if each and every element of set B are the elements of the set A and at least one element of super set A is not an element of B.

Example: Let $A = \{1, 2, 3, 4, 5, 6, 7\}$

$B = \{1, 3, 5, 7\}$

So, $B \subset A$

10) Improper Subset: A set itself is a subset of that set and Ø is a subset of every set; these two subsets are called improper subset.

Example: Let us consider $A =\{a, b, c\}$

The subsets of A are {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}, Ø

Here, we have $2^n = 2^3 = 8$ subsets of A. Subsets {a, b, c} and Ø are improper subset of A and the remaining are proper subsets of A.

11) Family of sets: The set which all the elements are sets themselves then it is called a family of sets or set of sets.

Example: A = {{1}, {2}, {1, 2}, Ø} is a family of sets.

12) Power set: If a family of sets contains all subset of a set then this family of sets is called power set of that set.
Example: Let A = {1, 2, 3}
The subsets of A are Ø, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}
Then power set of A is P(A) = {Ø, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

13) Universal set: Our discussing sets are subsets of a big set, this big set is known as universal set of that sets. It is generally denoted by the symbol U.
Example: The set of integers may be considered as a universal set for the set of even integers and the set of odd integers.

2.6 <u>Operations on sets</u>:
1) Intersection of sets: The Intersection of two sets A and B is the set consisting of all elements, which belong to both sets A and B. The intersection of sets A and B is denoted by $A \cap B$ and is read as " A intersection B," or " the intersection of A and B." Symbolically, $A \cap B = \{x: x \in A \text{ and } x \in B\}$
Example: Let, A = {1, 2, 3, 4, 5} and B = {2, 4, 6}
$\therefore A \cap B = \{2, 4\}$

2) Union of sets: The union of two sets A and B is the set consisting of all elements, which belong to either A or B or both. The union of sets A and B is denoted by $A\cup B$ and is read as 'A union B,' or 'the union of A and B.' Symbolically, $A\cup B = \{x: x\in A \text{ or } x\in B\}$
Example: Let, A = {1, 2, 3, 4, 5} and B = {2, 4, 6}
$\therefore A \cup B = \{1, 2, 3, 4, 5, 6\}$

3) Difference of two sets: The Difference of two sets A and B is the set of all those elements, which belong to A and not to B. The difference of sets A and B is denoted by $A - B$ or $A \sim B$ and is read as " A difference B," or " the difference of A and B." Symbolically, $A - B = \{x: x \in A \text{ and } x \notin B\}$
Example: Let, A = {1, 3, 5}, B = {4, 5, 6}
$\therefore A - B = \{1, 3\}$ and $B - A = \{4, 6\}$.

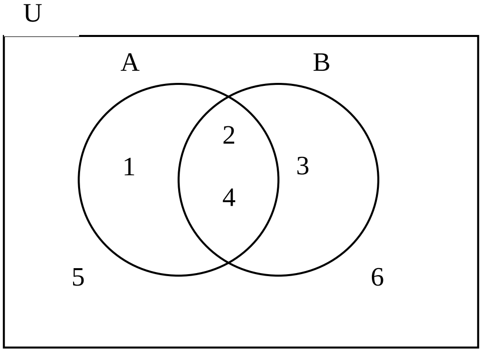
4) Symmetric difference of two sets: If A and B are two sets, then the set $(A-B)\cup(B-A)$ is called the symmetric difference of two sets and is written as $A \Delta B$. That is, the symmetric difference of two sets A and B is the set of all those elements of A and B, which are not common to both A and B. We have to note that $A \Delta B = B \Delta A$.
Example: If A = {a, b, c, d} and B = {b, c, d, e, f} then
$A - B = \{a\}$ and $B - A = \{e, f\}$
So, $A \Delta B = (A - B) \cup (B - A) = \{a, e, f\}$.

5) Complement of a set: The complement of a set is the set of all elements, which do not belong to that set. The compliment of the set A is $A^/$ or $A^c = U - A = \{x: x \in U \text{ and } x \notin A\}$.
Example: Let, $U = \{1, 2, 3, 4, 5, 6\}$ and $A = \{1, 3, 5\}$
Then, A^c or $A^/ = U - A = \{2, 4, 6\}$

6) Cartesian product: If A and B be any two sets, then the set of all ordered pairs whose first element belongs to set A and second element belongs to set B is called the Cartesian product of A and B and is denoted by A × B.
i.e. $A \times B = \{(x, y): x \in A, y \in B\}$
N.B: An **ordered pair** of objects consists of two elements a and b written in parentheses (a, b). The ordered pair (a, b) and (b, a) are not same, i.e., $(a, b) \neq (b, a)$. Two ordered pairs (a, b) and (c, d) will be equal if a = c and b = d.
Example: Let $A = \{1, 2\}$, $B = \{a, b\}$
$\therefore A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$
And $B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$
So, $A \times B \neq B \times A$

2.7 <u>Relation</u>: If A and B be two sets then non empty subset of ordered pairs of Cartesian product, $A \times B$ is called relation of A and B and is denoted by R. If we consider $x \in A$ and $y \in B$ then $(x, y) \in R$.
Example (i): If $A = \{1, 2, 3\}$ and $B = \{3, 5\}$ then $A \times B = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)\}$. So, the relation $x < y$ where $x \in A$ and $y \in B$ is $R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5)\}$.
Example (ii): If A = {\$4, \$7, \$8} is a set of cost of per unit product and B = {\$5, \$8} is the set of selling price of per unit product of a production firm. Find the profitable relation between cost and selling price.
Solution: Here, A×B = {(\$4, \$5), (\$4, \$8), (\$7, \$5), (\$7, \$8), (\$8, \$5), (\$8, \$8)}
A firm becomes profitable if its selling price of per unit product is greater than the cost of per unit product. So, the profitable relation, R = {(\$4, \$5), (\$4, \$8), (\$7, \$8)}. (Answer)

2.8 <u>Venn diagram</u>: The Venn diagrams are named after English Logician John Venn. In this diagram, the universal set U is denoted by a region enclosed by a rectangle and one or more sets are shown by circles or closed curves within this rectangle. These circles or closed curves intersect each other if there are any common elements among them if there are no common elements then they are shown separated. This diagram is useful to illustrate the set relations, the set operations etc.
Example: Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $U = \{1, 2, 3, 4, 5, 6\}$. Various relation and operations are shown bellow by the Venn diagrams:



Venn diagram

Figure – 2.1

From diagram, we get

$A \cap B = \{2, 3\}$
$A \cup B = \{1, 2, 3, 4\}$
$A - B = \{1\}$
$B - A = \{4\}$
$U - (A \cup B) = \{5, 6\}$
$A^c = \{4, 5, 6\}$
$B^c = \{1, 5, 6\}$ etc.

2.9 <u>Number of elements in a finite set</u>: It is very important to find out the number of elements in a finite set in the solution of the practical problems. Generally, if A be a set then n(A) means the number of elements of the set A. Such as if A = {a, b, c, d, e, f, g} then n(A) = 7; if B = {1, 3, 5, 7} then n(B) = 4 and if C = {x: x be even integer numbers between 3 to 15} then n(C) = 6 etc.

Let A and B be two disjoint sets that means they have no common element then total elements of the both sets will be $n(A \cup B)$.

So, $n(A \cup B) = n(A) + n(B)$ - - - (i).

But, if A and B are not disjoint sets that means they have some common elements then total elements of both sets will be

$$\mathbf{n(A \cup B) = n(A) + n(B) - n(A \cap B)} \quad \text{- - - } \mathbf{(ii)}$$

Similarly, if A, B and C be three disjoint sets then,

$n(A \cup B \cup C) = n(A) + n(B) + n(C)$ - - - (iii)

But if A, B and C are not disjoint sets then,

$$\mathbf{n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)} \quad \text{- - - } \mathbf{(iv)}$$

For disjoint cases, each of $n(A \cap B)$, $n(A \cap C)$, $n(B \cap C)$ and $n(A \cap B \cap C)$ is 0. So, formula (ii) and (iv) are main.

2.10 <u>De-Morgan's laws</u>:

1. Complement of a union of two sets is the intersection of complements of that sets.

   i.e. if A and B be any two sets them $(A \cup B)' = A' \cap B'$

2. Complement of an intersection of two sets is the union of the complements of that sets.

   i.e. if A and B be any two sets them $(A \cap B)' = A' \cup B'$

Proof of 1st law: Let x be any element of $(A \cup B)'$.

Then by definition of complement,

$x \in (A \cup B)' \Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cap B'$

i.e. $(A \cup B)' \subseteq A' \cap B'$ --- (i)

Again let y be any element of $A' \cap B'$

Then by definition of intersection,

$y \in A' \cap B' \Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin A \cup B$

$\Rightarrow y \in (A \cup B)'$

i.e. $A' \cap B' \subseteq (A \cup B)'$ --- (ii)

From (i) and (ii), we have $(A \cup B)' = A' \cap B'$ (Proved)

Proof of 2nd law: Let x be any element of $(A \cap B)'$.

Then by definition of complement,

$x \in (A \cap B)' \Rightarrow x \notin A \cap B$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cup B'$

i.e. $(A \cap B)' \subseteq A' \cup B'$ --- (i)

Again let y be any element of $A' \cup B'$

Then by definition of union,

$y \in A' \cup B' \Rightarrow y \in A'$ or $y \in B'$

$\Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow y \notin A \cap B$

$\Rightarrow y \in (A \cap B)'$

i.e. $A' \cup B' \subseteq (A \cap B)'$ --- (ii)

From (i) and (ii), we have $(A \cap B)' = A' \cup B'$ (Proved)

2.11 <u>Some worked out examples</u>:

<u>Example (1)</u>: If A = {5, 10, 12, 15, 19}, B = {3, 10, 15} and C = {7, 10}, write the sets $A \cup B$ and $B \cap C$ by tabular & selector methods.

Solution: Given that A = {5, 10, 12, 15, 19}, B = {3, 10, 15} and C = {7, 10}

So, $A \cup B$ = {5, 10, 12, 15, 19} $\cup$ {3, 10, 15}

= {3, 5, 10, 12, 15, 19} (Answer)

In the selector method, $A \cup B$ = {x: x = 3 or 5 or 10 or 12 or 15 or 19}

And $B \cap C$ = {3, 10, 15}$\cap$ {7, 10}

= {10}

In the selector method, $B \cap C$ = {x: x = 10} (Answer)

<u>Example (2)</u>: Let A ={1, 2, 3, 4}, B ={2, 4, 6, 8} and C = {3, 4, 5, 6}. Determine (i) A–B, (ii) B – C, (iii) B – B [RU-1993 Mgt.]

Solution: Given that A = {1, 2, 3, 4}, B = {2, 4, 6, 8} and C = {3, 4, 5, 6}.

(i) A – B = {1, 2, 3, 4} – {2, 4, 6, 8} = {1, 3}

(ii) B – C = {2, 4, 6, 8} – {3, 4, 5, 6} = {2, 8}

(iii) B – B = {2, 4, 6, 8} – {2, 4, 6, 8} = $\varnothing$

<u>Example (3)</u>: Let A = {a, b, c, d}. Find the power set P(A). [NU-1999 Mgt., AUB-1998 B.B.A, RU-1993 A/C]

Solution: The subsets of A are

{a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d},

{a, c, d}, {b, c, d}, {a, b, c, d}, $\varnothing$.

So, the Power Set P(A) ={{a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d},

{a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, {a, b, c, d}, $\varnothing$}

<u>Example (4)</u>: Let A = {1, 2, 3}, B = {1, 2}, C = {2, 3}, and D = {1, 3}. Prove that P(A) = {A, B, C, D, {1}, {2}, {3}, Ø}.

Solution: Given that A = {1, 2, 3}

B = {1, 2}

C = {2, 3}

D = {1, 3}

The subsets of A are {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}, Ø

$\therefore$ The power set of A, P(A) = {{1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}, Ø}

= {{1, 2, 3}, {1, 2}, {2, 3}, {1, 3}, {1}, {2}, {3}, Ø}

= {A, B, C, D, {1}, {2}, {3}, Ø} (Proved)

<u>Example (5)</u>: If set A = {a, b, c}, set B = {b, c, d} and the universal set U ={a, b, c, d, e} then verify De-Morgan's laws. [AUB-2002 M.B.A]

Solution: Given that, A = {a, b, c}, B = {b, c, d} and U ={a, b, c, d, e}

Then A' = U – A = {a, b, c, d, e}–{a, b, c} = {d, e},

B' = U – B = {a, b, c, d, e}–{b, c, d} = {a, e},

$A \cap B$ = {a, b, c}$\cap${b, c, d} = {b, c},

$A \cup B$ = {a, b, c}$\cup${b, c, d} = {a, b, c, d},

$(A \cup B)'$ = U – $(A \cup B)$ = {a, b, c, d, e} – {a, b, c, d} = {e},

$A' \cap B' = \{d, e\} \cap \{a, e\} = \{e\}$

So, $(A \cup B)' = A' \cap B'$

And $(A \cap B)' = U - (A \cap B) = \{a, b, c, d, e\} - \{b, c\} = \{a, d, e\}$,

$A' \cup B' = \{d, e\} \cup \{a, e\} = \{a, d, e\}$

Therefore, $(A \cap B)' = A' \cup B'$ (Verified)

Example (6): If $A = \{1, 4\}$, $B = \{4, 5\}$, $C = \{5, 7\}$, verify that,

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Solution: Given that $A = \{1, 4\}$

$B = \{4, 5\}$

$C = \{5, 7\}$

$\therefore B \cap C = \{4, 5\} \cap \{5, 7\}$

$= \{5\}$

$\therefore$ L.H.S. $= A \times (B \cap C)$

$= \{1, 4\} \times \{5\}$ [Putting the value of $(B \cap C)$]

$= \{(1, 5), (4, 5)\}$

$\therefore$ R.H.S. $= (A \times B) \cap (A \times C)$

$= (\{1, 4\} \times \{4, 5\}) \cap (\{1, 4\} \times \{5, 7\})$

$= \{(1, 4), (1, 5), (4, 4), (4, 5)\} \cap \{(1, 5), (1, 7), (4, 5), (4, 7)\}$

$= \{(1, 5), (4, 5)\}$

$\therefore$ L.H.S. = R.H.S. (Verified)

Example (7): Let A, B and C be any there sets. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. [RU-1981 A/C]

Solution: Let x be any element of $A \cap (B \cup C)$

Then by the definition of intersection,

$x \in A \cap (B \cup C) \Rightarrow x \in A$ and $x \in (B \cup C)$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

i.e. $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ - - - (i)

Again let y be any element of $(A \cap B) \cup (A \cap C)$

Then by the definition of union,

$y \in (A \cap B) \cup (A \cap C) \Rightarrow y \in (A \cap B)$ or $y \in (A \cap C)$

$\Rightarrow (y \in A$ and $y \in B)$ or $(y \in A$ and $y \in C)$

$\Rightarrow y \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow y \in A$ and $y \in (B \cup C)$

$\Rightarrow y \in A \cap (B \cup C)$

i.e. $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ - - - (ii)

From (i) and (ii), we have, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Proved)

Example (8): If A, B and C be any three sets, then prove that $A - (B \cup C) = (A-B) \cap (A-C)$ [NU-1998 A/C]

Solution: Let x be any element of $A - (B \cup C)$

Then by the definition of difference,

$x \in A-(B \cup C) \Rightarrow x \in A$ and $x \notin (B \cup C)$

$\Rightarrow$ $x \in A$ and ($x \notin B$ and $x \notin C$)

$\Rightarrow$ ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$)

$\Rightarrow$ $x \in (A - B)$ and $x \in (A - C)$

$\Rightarrow$ $x \in (A - B) \cap (A - C)$

i.e. $A - (B \cap C) \subseteq (A - B) \cup (A - C)$ --- (i)

Again let y be any element of $(A - B) \cup (A - C)$

Then by the definition of union,

$y \in (A - B) \cap (A - C) \Rightarrow y \in (A - B)$ and $y \in (A - C)$

$\Rightarrow$ ($y \in A$ and $x \notin B$) and ($y \in A$ and $x \notin C$)

$\Rightarrow$ $y \in A$ and ($y \notin B$ and $y \notin C$)

$\Rightarrow$ $y \in A$ and $y \notin (B \cup C)$

$\Rightarrow$ $y \in A - (B \cup C)$

i.e., $(A - B) \cap (A - C) \subseteq A - (B \cup C)$ --- (ii)

From (i) and (ii), we have, $A - (B \cup C) = (A-B) \cap (A-C)$ (Proved)

Example (9): If A, B and C be any three sets, then prove that

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Solution: Under the definition of Cartesian product,

$$\begin{aligned} A \times (B \cup C) &= \{(x, y): x \in A, y \in (B \cup C)\} \\ &= \{(x, y): x \in A, (y \in B \text{ or } y \in C)\} \\ &= \{(x, y): (x \in A, y \in B) \text{ or } (x \in A, y \in C)\} \\ &= \{(x, y): (x, y) \in (A \times B) \text{ or } (x, y) \in (A \times C)\} \\ &= \{(x, y): (x, y) \in (A \times B) \cup (A \times C)\} \\ &= (A \times B) \cup (A \times C) \end{aligned}$$ (Proved)

Example (10): If $n(U) = 800$, $n(A) = 200$, $n(B) = 300$, $n(A \cap B) = 150$, then find $n(A' \cap B')$ where A, B are two sets and U is the universal set. [RU-1984, 85, NU-1998 A/C]

Solution: Given that, $n(U) = 800$, $n(A) = 200$, $n(B) = 300$, $n(A \cap B) = 150$, $n(A' \cap B') = ?$

From De-Morgan's laws, we know that, $(A \cup B)' = A' \cap B'$.

So, $n(A' \cap B') = n((A \cup B)')$ and $n((A \cup B)') = n(U) - n(A \cup B)$.

We also know, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 200 + 300 - 150$

$= 350$

Therefore, $n(A' \cap B') = n((A \cup B)')$
$= n(U) - n(A \cup B)$
$= 800 - 350 = 450$ (Answer)

Example (11): Dhaka city has a total population of 8000000. Out of it 1800000 are service holders and 1000000 are businessmen while 120000 are in both positions. Indicate how many people are neither service holders nor businessmen. [AUB-2002 M.B.A]

Solution: Let P, S and B denote the sets of total people, service holders and businessmen respectively. So, $n(P) = 8000000$, $n(S) = 1800000$, $n(B) = 1000000$ and $n(S \cap B) = 120000$

So, the number of people who are at least one position is $n(S \cup B)$.

We know that, $n(S \cup B) = n(S) + n(B) - n(S \cap B)$.
$= 1800000 + 1000000 - 120000$
$= 2680000$

So, the number of people who are neither service holder nor businessman
$= n(P) - n(S \cup B)$
$= 8000000 - 2680000$
$= 5320000$. [Answer]

Example (12): A roads and highway construction firm has 33 bulldozer drivers, 22 crane drivers and 35 cement-mixture drivers. Of the drivers 14 persons can drive both mixture and dozer, 10 persons can drive both mixture and crane, 10 persons can drive both dozer and crane and 4 persons can drive all the three machines. Determine the total number of drivers of the firm. [DU-1987 mgt., RU-1984 mgt.]

Solution: Let, the set of bulldozer drivers be A, the set of crane drivers be B and the set of cement-mixture drives be C.

So, $n(A) = 33$, $n(B) = 22$, $n(C) = 35$, $n(A \cap B) = 10$, $n(B \cap C) = 10$, $n(A \cap C) = 14$ and $n(A \cap B \cap C) = 4$.

$\therefore$ the total number of drivers, $n(A \cup B \cup C) = ?$

We know that,

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$
$= 33 + 22 + 35 - 10 - 10 - 14 + 4$
$= 60$

Therefore, the firm has total 60 drivers. (Answer)

Example (13): A company studies the product preferences of 25,000 consumers. It was found that each of the products A, B and C was liked by 8000, 7000 and 6000 respectively and all the products were liked by 1500. Products A and B were liked by 3000, products A and C were liked by 2000 and products B and C were liked by 2200. Prove that the study results are not correct. [AUB-2003 B.B.A]

Solution: Let A, B and C denote the set of consumers who like products A, B and C respectively. The given data means n(A) = 8000, n(B) = 7000, n(C) = 6000, $n(A\cap B)$ = 3000, $n(B\cap C)$ = 2200, $n(A\cap C)$ = 2000, $n(A\cap B\cap C)$ = 1500 and $n(A\cup B\cup C)$ = 25,000.
We know that

$$n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(A\cap C) - n(B\cap C) + n(A\cap B\cap C)$$
$$= 8000 + 7000 + 6000 - 3000 - 2200 - 2000 + 1500$$
$$= 15{,}300 \neq 25{,}000$$

This shows that the study results are not correct.

Example (14): Out of 1200 students of a college, 400 played cricket, 350 played football and 512 played table tennis: of the total 100 played both cricket and football; 142 played football and table tennis; 95 played cricket and table tennis; 50 played all the three games. (i) How many students did not play any game? (ii) How many students played only one game? [AUB-2002 B.B.A]

Solution: Let the C, F and T denote the sets of players playing cricket, football and table tennis respectively. And S represents the set of total students. Now we are given that
n(C) = 400, n(F) = 350, n(T) = 512, $n(C\cap F)$ = 100, $n(F\cap T)$ = 142, $n(C\cap T)$ = 95, $n(C\cap F\cap T)$ = 50 and n(S) = 1200.

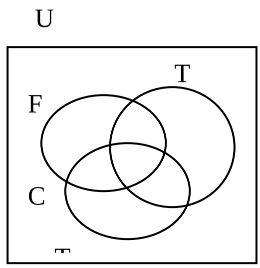


Figure – 2.2

(i) Number of students who played at least one game is given by

$$n(C\cup F\cup T) = n(C) + n(F) + n(T) - n(C\cap F) - n(F\cap T) - n(C\cap T) + n(C\cap F\cap T)$$
$$= 400 + 350 + 512 - 100 - 142 - 95 + 50$$
$$= 975$$

So, number of students who did not play any game = $n(S) - n(C\cup F\cup T)$
= 1200 – 975
= 225 (Answer)

(ii) We know, the number of students who played cricket, n(C) = 400. This contains

a) The students who played cricket only.
b) The students who played cricket and football, $n(C\cap F)$
c) The students who played cricket and table tennis, $n(C\cap T)$
d) The students who played cricket, football and table tennis, $n(C\cap F\cap T)$

Now from the Venn diagram we get:
The number of students who played cricket only
$= n(C) - n(C\cap F) - n(C\cap T) + n(C\cup F\cup T)$

$= 400 - 100 - 95 + 50$
$= 255$

Similarly,

The number of students who played football only

$= n(F) - n(C \cap F) - n(F \cap T) + n(C \cup F \cup T)$
$= 350 - 100 - 142 + 50$
$= 158$

And the number of students who played tennis only

$= n(T) - n(F \cap T) - n(C \cap T) + n(C \cup F \cup T)$
$= 512 - 142 - 95 + 50$
$= 325$

So, the number of students who played only one game $= 255 + 158 + 325$
$= 738$ (Answer)

<u>Example (15)</u>: Demand function of a product D = $\frac{10p}{p-3}$ and supply function S = p^2; where p means the price in dollar of the product per unit. Using set theory determine equilibrium price and quantity. [AUB-2003 M.B.A]

Solution: Substituting p by 1, 2, 3, . . . etc. in D = $\frac{10p}{p-3}$ and S = p^2 we get the following sets of ordered pairs:

$\{(p, D)\} = \{(1, -5), (2, -20), (3, \infty), (4, 40), (5, 25), (6, 20)\}$

(Notice that, using p = 7, 8, 9, . . . etc. we can make a larger set.)

$\{(p, S)\} = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25), (6, 36)\}$

Let, $A = \{(p, D)\} = \{(1, -5), (2, -20), (3, \infty), (4, 40), (5, 25), (6, 20)\}$

And $B = \{(p, S)\} = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25), (6, 36)\}$

So, $A \cap B = \{(5, 25)\}$

Therefore, equilibrium price, p = \$5 and quantity = 25 units.

2.12 <u>Exercise</u>:

1. Define with examples: (i) set, (ii) subset, (iii) power set, (iv) intersection of sets
2. What do you mean by $\varnothing$ and $\{\varnothing\}$?
3. What is difference between union and intersection of sets?
4. Discuss the difference between difference of sets and symmetric difference of sets.
5. State and prove the De-Morgan's laws. [AUB-02, 03, RU-95]
6. What do you mean by ordered pair?
7. Let U = {1, 2, 3, 4, 5, 6, 7}, A = {1, 3, 5, 7}, B = {2, 4, 6} and C = {2, 3, 4, 5} then find the following sets: a) $A \cup B$, b) $A \cap B$, c) $B \cup C$, d) A', e) B', f) $A' \cap B'$, g)

$(A\cup B)'$. [Answer: a){1, 2, 3, 4, 5, 6, 7}, b) ∅, c){2, 3, 4, 5, 6}, d){2, 4, 6}, e){1, 3, 5, 7}, f) ∅, g) ∅]

8. Let A = {5, 10, 15, 20, 25}, B = {5, 15, 21} and C = {10, 20}, write the following sets by the roster and property builder methods: a) $A\cup B$, b) $A\cap B$, c) $A\cup C$, e) $B\cap C$. [Answer: a){5, 10, 15, 20, 21, 25}, {x: x = 5 or 10 or 15 or 20 or 21 or 25} b){5, 15}, {x: x = 5 or 15} c){5, 10, 15, 20, 25}, {x: x = 5 or 10 or 15 or 20 or 25} d){}, {x: x has no value}]
9. Write down three sets A, B and C such that $A \cap B \neq \varnothing$, $A \cap C \neq \varnothing$ and $B \cap C \neq \varnothing$ but $A \cap B \cap C = \varnothing$. [Answer: {0, 1, 2}, {3, 4, 5} and {6, 7, 8}]
10. Let A = {-1, 0, 1, 2}. Find its three subsets, each of which contains 3 elements. [Answer: {-1, 0, 1}, {-1, 0, 2} and {0, 1, 2}]
11. Let A = {1, 2, 3, 4}, find the power set of A, i.e., P(A). [Answer: {{1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}, ∅}]
12. If A = {1, 2} and B = {1, 5, 7} then find $A \times B$ and $A \times A$. [Answer: {(1, 1), (1, 5), (1, 7), (2, 1), (2, 5), (2, 7)} and {(1, 1), (1, 2), (2, 1), (2, 2)}]
13. If A = {a, b, c} and B = {d, e} then prove that $A \times B \neq B \times A$
14. If A = {2, 3, 5} and B = {3, 7} then find the relation, R as $x > y$ where $x \in A$ and $y \in B$, [Answer: {(5, 3)}]
15. If A = {\$3, \$5, \$8} is a set of cost of per unit product and B = {\$5, \$10} is the set of selling price of per unit product of a production firm. Find the profitable relation between cost and selling price. [Answer:{(\$3, \$5), (\$3, \$10), (\$5, \$10), (\$8, \$10)}]
16. If (x + y, 5) = (15, x – y) then find the value of (x, y). [Answer: (10, 5)]
17. If A, B and C be any three sets, prove that: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
18. If A, B and C be any three sets, then prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$
19. For any three sets A, B and C prove that $A \cap (B - C) = (A \cap B) - (A \cap C)$.
20. Let A = {1, 2, 3}, B = {2, 4, 6} and Universal set U = {1, 2, 3, 4, 5, 6}. Verify De-Morgan's laws for these sets.
21. If A = {1, 2, 3}, B = {2, 3, 5}, C = {1, 3, 5} and D = {2, 5, 6} then prove that

$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

22. There are 100 students in a class. Out of them 50 bring books and 30 bring books without khatas in the class. How many students bring both books and khatas in the class? How many students bring khatas without books? [Answer: 20 students, 50 students]
23. In a survey of the Dhaka city, it was found that 65% of the people watched the news on BTV, 40% read a newspaper and 25% read a newspaper and watched the news on BTV. What percent of the people survey neither watched the news on BTV nor read a newspaper? [Answer: 20%]
24. Out of 400 students of a university, 102 studied Business Mathematics, 110 studied Management and 152 studied Business Law: of the total 27 studied Management and Business Law; 36 studied Business Mathematics and Business Law; 18

studied Business Mathematics and Management; 11 studied all the three subjects. (i) How many students did not study any subject? (ii) How many students studied only one subject? [Answer: 106, 235]

25. A town has a total population of 50,000. Out of it 28,000 read Ittefaq and 23,000 read Inclub while 4,000 read both the papers. Determine how people read neither Ittefaq nor Inclub? [Answer: 3,000]
26. A company studies the product preferences of 20,000 consumers. It was found that each of the products A, B and C was liked by 7500, 6500 and 5500 respectively and all the products were liked by 1230. Products A and B were liked by 2500, products A and C were liked by 2300 and products B and C were liked by 2530. Prove that the study results are not correct.
27. A class of 60 students appeared for an examination of Mathematics, Statistics and Economics. 25 students failed in Mathematics, 24 failed in Statistics, 32 failed in Economics, 9 failed in Mathematics alone, 6 failed in Statistics alone; 5 failed in Statistics and Economics only and 3 failed in Mathematics and Statistics only. (i)How many students failed in all three subjects? (ii) How many students passed in all three subjects? [Answer: (i) 10, (ii) 10]
28. Demand function of a product D = $\frac{5p-2}{p-3}$ and supply function S = $p^2 + 2$; where p means the price in taka of the product per unit. Using set theory determine equilibrium price and quantity. [Answer: p = 4 and quantity = 18]

03

Chapter

Progressions (AP & GP)

Highlights:

3.1 Introduction 3.2 Sequence 3.3 Series 3.4 Arithmetic progression (A.P) 3.5 Geometric Progression (G.P),	3.6 Theorem of arithmetic series 3.7 Theorem of geometric series 3.8 Some worked out examples 3.9 Exercise

3.1 Introduction: Arithmetic and geometric series are two special types of series increasing or decreasing by an absolute quantity or a certain ratio. In this chapter we shall discuss the methods of finding different terms and summations of the series with sequences and progressions. We shall also try to show some applications.

3.2 Sequence: A set of numbers that are arranged according to some definite law is called a sequence. Each number of a sequence is called the term so that we have the first term, second term and so on. If the nth term of a sequence be u_n then the sequence is denoted by $\{u_n\}$ or, $< u_n >$ or, $< u_1, u_2, u_3, ..., u_n, ... >$.

Example: i) The sequence of natural number $< 1, 2, 3, ..., n, ... >$ is denoted by $< n >$ or, $\{n\}$

ii) $< n^2 >$ means of sequence $< 1^2, 2^2, 3^2, ... >$ or, $< 1, 4, 9, ... >$

iii) $< (-1)^{n-1}.4^n >$ means the sequence $< 4, -16, 64, ... >$

3.3 Series: A Series is an expression consisting of the sum of the terms in a sequence. The sequence $< u_n >$ forms the series $u_1 + u_2 + u_3 + ... + u_n + ...$ As for example $1 + 2 + 3 + 4 + ...$ is a series.

3.4 Arithmetic progression (A.P): An arithmetic progression is a sequence whose terms increase or decrease by a constant number called the common difference the sequence $<1, 5, 9, 13, 17, . . . >$ is an infinite arithmetic progression with common difference 4. So, the series $(1 + 5 + 9 + 13 + . . .)$ is called arithmetic series.

a + (a +d) + (a +2d) + (a + 3d) + . . . +{a +(n – 1)d}+ . . . is the standard form of arithmetic series where a, d and n are first term, common difference and number of terms respectively.

3.5 Geometric Progression (G.P): A geometric progression is a sequence whose terms increases or decreases by a constant ratio called the common ratio. The sequence < 2, 6, 18, 54, . . . > is an infinite geometric progression whose first term is 2 and the common ratio is 3. This sequence forms the geometries since 2 + 6 + 18 + 54 + . . .
$a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + \ldots$ is the standard form of geometric series where a, r and n are first term, common ratio and number of terms respectively.

3.6 Theorem of arithmetic series: If "a" be the first term and "d" be the common difference of an arithmetic series, then

1) nth term = a + (n – 1)d and
2) sum of first n terms, $S_n = \frac{n}{2}\{2a + (n-1)\,d\}$

Proof: 1) Given that, first term = a
Common difference = d
So, 2^{nd} term = a + d = a + (2 – 1) d
3^{rd} term = a + d + d = a + 2d = a + (3 – 1)d
4^{th} term = a + 2d + d = a + 3d = a + (4 – 1)d
Thus, let mth term = a + (m –1)d
So, (m + 1)th term = a + (m –1)d + d
= a + (m – 1 +1) d
= A + {(m + 1) – 1} d
That is, for all n Є N, the nth term = a + (n – 1)d. (Proved)

2) Since S_n is the sum of first nth terms of the given series,
S_n = a + (a + d) + (a + 2d) + . . . +{a + (n – 2)d}+{a + (n –1)d} . . . (i)
Writing the series in the reverse order, we get
S_n = {a + (n –1)d} + {a + (n – 2)d}+ . . . + (a + 2d) + (a + d) + a . . . (ii)
Adding (i) and (ii), we get
2 S_n = {2a + (n –1)d} + {2a + (n –1)d} + . . . + {2a +(n –1)d}
Or, 2 S_n = n{2a + (n –1)d}
So, $S_n = \frac{n}{2}\{2a + (n-1)d\}$ (Proved)

Note: $S_n = \frac{n(a+l)}{2}$; where l = nth term = a + (n – 1)d

3.7 Theorem of geometric series: If "a" be the first term and "r" be the common ratio of a geometric series, then

1) nth term = a r^{n-1}

2) Sum of first n terms, $S_n = \dfrac{a(r^n - 1)}{r-1}$, when r >1

$= \dfrac{a(1-r^n)}{1-r}$, when r <1

Proof: 1) Given that, first term = a and common ratio = d

So, 2^{nd} term = a ×d = ad = a d^{2-1}

3^{rd} term = ad ×d = a d^2 = ad^{3-1}

4^{th} term = a d^2 × d = ad^3 = ad^{4-1}

Thus, let mth term = ad^{m-1}

So, (m + 1)th term = ad^{m-1} × d

= ad^m

= $ad^{(m+1)-1}$

That is, for all n Є N, the nth term = a r^{n-1} (Proved)

2) Since S_n is the sum of first nth terms of the given series,

$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$ . . . (i)

Multiplying both sides by r, we get

r $S_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$ . . . (ii)

Subtracting (ii) from (i), we get

$S_n - \text{r} S_n = a - ar^n$

Or, (1 – r) $S_n = a(1-r^n)$

So, $S_n = \dfrac{a(1-r^n)}{1-r}$, $r \neq 1$ . . . (iii)

Changing the signs of the numerator and denominator, we can also write

$S_n = \dfrac{a(r^n - 1)}{r-1}$, $r \neq 1$ . . . (iv)

It is convenient to use form (iii) when r < 1 and form (iv) when r >1. So,

Sum of first n terms, $S_n = \dfrac{a(r^n - 1)}{r-1}$, when r >1

$= \dfrac{a(1-r^n)}{1-r}$, when r <1

Note: Sum of infinite terms, $S_\infty = \dfrac{a}{1-r}$, when | r | < 1

= not found, when | r | > 1

3.8 Some worked out examples:

Example (1): Find the sum of the series

$62 + 60 + 58 + . . . + 40$

Solution: The given series is an arithmetic series with first term, a = 62 and common difference, d = 60 – 62 = – 2.

Let n be the number of terms. Then nth term = 40

That is, $a + (n - 1)\, d = 40$

Or, $62 + (n - 1)(-2) = 40$ [Putting value of a and d]

Or, $62 - 2n + 2 = 40$

Or, $-2n + 64 = 40$

Or, $-2n = 40 - 64$

Or, $n = \frac{-24}{-2}$

So, $n = 12$

We know, sum of n terms, $S_n = \frac{n}{2}\{2a + (n - 1)d\}$

So, sum of 12 terms, $S_{12} = \frac{12}{2}\{2 . 62 + (12\text{-}1)(\text{-}2)\}$

$= 6\,(124 - 22)$

$= 6.102$

$= 612$ (Answer)

Example (2): The mth term of an A.P is n and the nth term is m. Show that the rth term is (m + n – r) and the (m + n)th term is 0. [AUB-2003 B.B.A]

Solution: Let a be the first term and d be the common difference of the given series.

So, $n = a + (m - 1)d$ . . . (i)

$m = a + (n - 1)d$ . . . (ii)

Doing (i) – (ii), we have

$n - m = md - nd$

Or, $(n - m) = (m - n)d$

Or, $d = \frac{-(m - n)}{(m - n)}$

So, $d = -1$

Substituting the value of d in (i), we get

$n = a + (m - 1)(-1)$

Or, $n = a - m + 1$

So, $a = m + n - 1$

Therefore, the rth term = a + (r –1)d

$= m + n - 1 + (r - 1)(-1)$ [Putting value of a & d]

$= m + n - 1 - r + 1$

$= m + n - r$

And the (m + n)th terms = a + {(m + n) – 1}d
= m + n –1 + (m + n –1) (–1)
= m + n –1– m – n + 1
= 0 (Proved)

Example (3): If a, b, c are the sums of p, q, r, terms respectively of an A.P, show that

$$\frac{a(q-r)}{p} + \frac{b(r-p}{q} + \frac{c(p-q)}{r} = 0$$

Solution: Let, 1st term = m and the common difference = n. Then

Sum of pth terms, $a = \frac{p}{2}\{2m + (p-1)n\}$

Sum of qth terms, $b = \frac{q}{2}\{2m + (q-1)n\}$

And sum of rth terms, $c = \frac{r}{2}\{2m + (r-1)n\}$

$$\text{L.H.S} = \frac{a(q-r)}{p} + \frac{b(r-p}{q} + \frac{c(p-q)}{r}$$

$$= \frac{p}{2}\{2m + (p-1)n\}\times\frac{(q-r)}{p} + \frac{q}{2}\{2m + (q-1)n\}\times\frac{(r-p)}{r} + \frac{r}{2}\{2m + (r-1)n\}\times\frac{(p-q)}{r}$$

$$= \frac{1}{2}[\{2m + (p-1)n\}\times(q-r) + \{2m + (q-1)n\}\times(r-p) + \{2m + (r-1)n\}\times(p-q)]$$

$$= \frac{1}{2}(2qm + pqn - qn - 2rm - prn + rn + 2rm + qrn - rn - 2pm - pqn + pn +$$
$$2mp + prn - pn - 2qm - qrn + qn)$$

$$= \frac{1}{2}\times 0$$

= 0 = R.H.S

So, L.H.S = R.H.S (Proved)

Example (4): The first term and the last term of an A.P are respectively –4 and 146, and the sum of the A.P = 7171. Find the number of terms of the A.P and also its common difference.

Solution: We have, the first term, a = – 4, the last term, l =146.

Let n be the number of terms of the A.P and d be the common difference.

So, the summation of n terms, $S_n = 7171$.

We know, $S_n = \frac{n(a+l)}{2}$

Or, $7171 = \frac{n(-4+146)}{2}$

Or, $7171 = \frac{n \times 142}{2} = 71n$

Or, 7171 = 71n

Or, $n = \frac{7171}{71}$

So, n = 101

Therefore, the number of terms of this A.P is 101. (Answer)

We also know that,

nth term = a + (n –1)d

Here, last term means the 101st term.

So, 146 = – 4 + (101 –1)d

Or, 146 = – 4 +100 d

Or, 100d = 146 + 4

Or, 100d = 150

So, d = 1.5

Therefore, the common difference is 1.5. (Answer)

<u>Example (5)</u>: A farmer agrees to repay debt of Tk. 6682 in a number of installments, each installment increasing the previous one by Tk. 5. If the first installment be of Tk. 1, find how many installments will be necessary to wipe out the loan completely? [AUB-2000]

Solution: Since every installments (terms) increase by 5 taka, the problem is an A.P of

First installment (term), a = 1 taka

Common difference, d = 5 taka

Let n be the number of installments (terms) to wipe out the loan.

So, summation of n installments means total loan, S_n = 6682 taka

We know that, $S_n = \frac{n}{2}\{2a + (n-1)\, d\}$

Or, $6682 = \frac{n}{2}\{2\times1 + (n-1)\, 5\}$

Or, 13364 = n (2 + 5n – 5)

Or, 13364 = n (5n – 3)

Or, $5n^2 - 3n - 13364 = 0$

Or, $5n^2 - 260n + 257n - 13364 = 0$

Or, 5n(n – 52) + 257(n – 52) = 0

Or, (n – 52) (5n + 257) = 0

So, n – 52 = 0 ∴ n = 52

Or, 5n + 257 = 0 ∴ $n = -\frac{257}{5}$ (Not acceptable)

Hence, the number of installments is 52 (Answer)

Example (6): A man saved \$16500 in ten years. In each year after the first he saved \$100 more than he did in the preceding year. How much did he save in the first year? [AUB-2002 BBA]

Solution: Since each savings increases by \$100, it is an A.P of

Common difference, d = \$100

Number of terms (years), n = 10

Summation of n terms (total savings), S_n = \$16500

First term (savings), a =?

We know that, $S_n = \frac{n}{2}\{2a + (n-1)d\}$

$$\text{Or,}\quad 16500 = \frac{10}{2}\{2a + (10-1)100\}$$
$$\text{Or,}\quad 16500 = 5\{2a + 900\}$$
$$\text{Or,}\quad 10a + 4500 = 16500$$
$$\text{Or,}\quad 10a = 16500 - 4500$$
$$\text{Or,}\quad 10a = 12000$$
$$\text{Or,}\quad a = \frac{12000}{10}$$
$$\text{So,}\quad a = 1200$$

Therefore, he saved \$1200 in the first year. (Answer)

Example (7): Sum to n terms of the series: 5 + 55 + 555 + . . . [AUB-1999 BBA]

Solution: Let S_n be the sum of n terms.

So, $S_n = 5 + 55 + 555 + \ldots + \text{nth term}$

$$= 5(1 + 11 + 111 + \ldots + \text{nth term})$$
$$= \frac{5}{9}(9 + 99 + 999 + \ldots + \text{nth term})$$
$$= \frac{5}{9}[(10-1) + (100-1) + (1000-1) + \ldots + \text{nth term}]$$
$$= \frac{5}{9}[(10 + 100 + 1000 + \ldots + \text{nth term}) - (1 + 1 + 1 + \ldots + \text{nth term})]$$
$$= \frac{5}{9}[(10 + 10^2 + 10^3 + \ldots + \text{nth term}) - n]$$
$$= \frac{5}{9}\left[\frac{10(10^n - 1)}{10-1} - n\right]$$

[(10 + 10^2 + 10^3 + . . . + nth term) is a G.P of 1st term = 10 and common ratio = 10]

$$= \frac{5}{81}[10(10^n - 1) - 9n]$$
$$= \frac{5}{81}(10^{n+1} - 10 - 9n) \quad \text{(Answer)}$$

Example (8): Find the sum of the following series

$$\frac{1}{5}-\frac{2}{5^2}+\frac{4}{5^3}-\frac{8}{5^4}+\ldots \text{ to } \infty$$

Solution: The series is in G.P form, because

$$\frac{-\frac{2}{5^2}}{\frac{1}{5}}=\frac{\frac{4}{5^3}}{\frac{-2}{5^2}}=\frac{-\frac{8}{5^4}}{\frac{4}{5^3}}=\ldots=-\frac{2}{5}$$

Here, 1^{st} term, a = $\frac{1}{5}$

Common ratio, r $=\frac{-\frac{2}{5^2}}{\frac{1}{5}}=-\frac{2}{5^2}\times\frac{5}{1}=-\frac{2}{5}$

And $|r| = |-\frac{2}{5}| = \frac{2}{5} < 1$, so the series has an infinite summation.

We know that the infinite summation,

$$S_\infty=\frac{a}{1-r}$$

$$=\frac{\frac{1}{5}}{1-\left(-\frac{2}{5}\right)}$$

$$=\frac{\frac{1}{5}}{1+\frac{2}{5}}$$

$$=\frac{1}{5}\times\frac{5}{7}$$

$$=\frac{1}{7}$$ (Answer)

Example (9): If the value of a flat depreciated by 25% annually, what will be its estimated value at the end of 5 years if its present value is Tk. 15,00,000? [AUB-2001 BBA]

Solution: It is clear that the value of at the end of first, second, third, fourth and fifth years from a G.P with common ratio, r = (100 – 25)%

= 75%

$= \dfrac{75}{100}$

$= \dfrac{3}{4}$

$= 0{\cdot}75$

The value at the end of 5th year = the value at the beginning of 6th year

Here, the value at the beginning of 1st year, a =15,00,000 taka.

So, the value at the beginning of 6th year = $a\,r^{6-1}$

$= a\,r^{5}$

$= 15{,}00{,}000\,(0{\cdot}75)^5$ taka

$=15{,}00{,}000 \times 0.2373046$ taka

$=3{,}55{,}957{\cdot}03$ taka.

Therefore, the estimated value at the of 5th year is 3,55,957·03 taka. (Answer)

Example (10): If the population of a town increases 2.5% per year and the present population is 26,24,000, what will be the population in three years time? What was it a year ago?

Solution: Since, the population increases by the percentage (ratio) 2.5%, it is a G.P of

The first term (present population), a = 26,24,000

Common ratio, r = (100 + 2.5)%

$= 102.5\%$

$= \dfrac{102.5}{100}$

$= 1.025$

The population in three years = the population at the beginning of 4th year.

We know that nth term = $a\,r^{n-1}$

The population will be in three years time = $a\,r^{4-1}$ [4th term = $a\,r^{4-1}$]

$= a\,r^{3}$

$= 26{,}24{,}000 \times (1.025)^3$

$= 2{,}82{,}55{,}761$ (Answer)

The population of a year ago = $a\,r^{0-1}$

$= a\,r^{-1}$

$= 26{,}24{,}000 \times (1.025)^{-1}$

$= 25{,}60{,}000$ (Answer)

3.9 Exercises:

1. Define with examples: (i) sequence, (ii) series
2. Discuss the differences of series and sequence.
3. What is difference between arithmetic and geometric series?
4. Find the sum of the following series:
   (i) 7 + 10 + 13 + . . . up to 40th term. [Answer: 2620]
   (ii) $5+7\frac{1}{2}+10+12\frac{1}{2}+\ ...+25th\ \ term.$ [Answer: 875]
5. How many natural numbers within 1 and 1000, which are divisible by 5? And find their sum. [Answer: 200 and 100500]
6. Find the sum of natural numbers from 1 to 200 excluding those divisible by 5. [Hints: Sum = (1+2+3+ . . . +200) – (5+10+15+ . . . +200)] [Answer: 16000]
7. The 7^{th} and the 9^{th} terms of an A.P are respectively 15 and 27. Find the series of the A.P and the sum of first 50 terms. [Answer: 3 + 6 + 9 +12 + . . . and 3825]
8. The first term of an A.P series is 2, the nth term is 32 and the sum of first n terms is 119. Find the series. [Answer: 2 + 7 + 12 + 17 + . . .]
9. The sum of the first n terms of the A.P series 13 + 16.5 + 20 + . . . is the same as the sum of the first n terms of the A.P series 3 + 7 + 11 + ... Calculate the value of n. [Answer: 41]
10. A man secures an interest free loan of $14500 from a friend and agrees to repay it in 10 installments. He pays $1000 as first installment and then increases each installment by equal amount over the preceding installment. What will be his last installment? [Answer: $1900]
11. A firm produced 500 sets of close circuit TV during its first year and increased its production each year uniformly. The total sum of the productions of the firm at the end of 5 years operation was 4500 sets. (i) Estimate by how many units, production increased each year. (ii) How many TV sets were produced for the 9^{th} year? [Answer: (i) 200, (ii) 2100]
12. The price of each shares of 100.00 taka of a company increases 10 taka every year. A man buys some primary shares. After 10 years, the price of his shares will become 5700.00 taka. What is the price of his primary shares? What will the price of his shares be after 5 years? [Hints: Let, no. of shares = x, so, 1^{st} term = 100x taka and c.d = 10x taka. So, 100x + (10 – 1)10x = 5700 => x = 30] [Answer: 3000taka and 4200 taka]
13. Find the 10^{th} term and sum of first 10 terms of the following series:
    1+ 3 + 9 + 27 + . . . [Answer: 19683 and 29524]
14. If the sum of n terms of a G.P series is 225, the common ratio is 2 and the last term (nth term) is 128. Find the value of n. [Answer: 8]

15. Find the sum of the following series:

(i) $4 + 44 + 444 + \ . \ . \ . \ + \text{nth term}$. [Answer: $\frac{4}{81}(10^{n+1} - 10 - 9n)$]

(ii) $7 + 77 + 777 + . . . + \text{nth term}$. [Answer: $\frac{7}{81}(10^{n+1} - 10 - 9n)$]

16. Sum the following series to infinity:

(i) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ . \ . \ .$ [Answer: 2]

(ii) $\frac{4}{7} - \frac{5}{7^2} + \frac{4}{7^3} - \frac{5}{7^4} + . \ . \ .$ [Answer: $\frac{23}{48}$]

[Hints $= \frac{4}{7} - (\frac{4}{7^2} + \frac{1}{7^2}) + \frac{4}{7^4} - (\frac{4}{7^4} + \frac{1}{7^4}) + . \ . \ .$

$= (\frac{4}{7} - \frac{4}{7^2} + \frac{4}{7^3} - \ . \ . \ . \ ) - (\frac{1}{7^2} + \frac{1}{7^4} + \frac{1}{7^6} + \ . \ . \ .)$]

17. The population of Dinajpur increases 4% per year. How long time is required to be the double people in that town? [Answer: 18.67 years]

18. The cost price of a machine is \$50000. If the price of this machine depreciated by 10% annually, what will be its estimated price at the end of 7 years? [Answer: \$23914.85]

04

Chapter

Permutation and Combination

Highlights:

4.1 Introduction: Permutations mean the different arrangements of things from a given lot taken one or more at a time whereas combinations refer to different sets or groups made out of a given lot, without repeating any element at a time. The difference between permutation and combination will be clear by the following illustration of permutations and combinations made out of a lot of three elements, such as x, y, z.

	Permutations	Combinations
(i) one at a time:	{x}, {y}, {z}	{x}, {y}, {z}
(ii) two at a time:	{x, y}, {y, x}, {x, z}, {z, x}, {y, z}, {z, y}	{x, y}, {x, z}, {y, z}
(iii) three at a time:	{x, y, z}, {x, z, y}, {y, x, z}, {y, z, x}, {z, x, y}, {z, y, x}	{x, y, z}

In the above example we see that every set represents different combination but a set with different arrangements of its elements represents different permutation. And no element appears twice in the sets of permutations or combinations such as {x, x}, {y, y} and {z, z}.

4.2 Permutation: Permutations mean the different arrangements of things from a given lot taken one or more at a time without repetition of any object. Let us consider three letters a, b, c. The different arrangements of these three letters taking three at a time are abc, acb, bac, bca, cab and cba. Thus there are 6 different ways of arranging three distinct objects when each arrangement is of all the three objects without any repetition of objects.

Each of these arrangements is a permutation. We can illustrate the example as follows: there are three places to be filled, the first can be filled in 3 ways, the second in 2 ways while for the third in 1 way. Hence, there are 3.2.1 = 6 ways to arrange the 3 letters at a time.

4.3 <u>Factorial notation</u>: The product of the first n natural numbers, viz., 1, 2, 3, ..., n, is called factorial n or n factorial and is written as $\lfloor n$ or n!.

Thus $\quad n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n$

It follows that $\quad n! = n \times \{(n-1)!\}$

$$= n \times (n-1) \times \{(n-2)!\}$$

$$\ldots \quad \ldots \quad \ldots$$

$$= n(n-1)(n-2) \ldots (n-r+1)\{(n-r)!\}$$

Illustration: $\frac{6!}{4!} = \frac{6.5.4!}{4!} = 6.5 = 30$

4.4 <u>Permutations of n different things</u>: Let the number of permutations of n different things taken r at a time, where r ≤ n is ${}^{n}p_{r}$ or P(n, r).

The number of permutations of n different things taken r at a time is the same as the number of different ways in which r places can be filled up by n things.

The first place can be filled up in n ways, for any one of the n things can be put in it.

$\therefore \quad {}^{n}p_{1} = n$

When the first place has been filled up in any one of the n ways, the second place can be filled up in (n – 1) different ways, for any one of the remaining (n – 1) things can be put in it. Since each way of filling up the first place is associated with each way of filling up the second place, the first two places can be filled up in n(n-1) ways.

$\therefore \quad {}^{n}p_{2} = n(n-1)$

When the first two places have been filled up in any one of the n(n –1) ways, the third place can be filled up in (n – 2) different ways, for any one of the remaining (n – 2) things can be put in it. Since the first two places are associated with the third place, the first three places can be filled up in n(n – 1)(n – 2) ways.

$\therefore \quad {}^{n}p_{3} = n(n-1)(n-2)$

Proceeding in the same way we notice that the number of factors is same as the number of places to be filled up and each factor is less then the former by 1. So, the r places can be filled up in n(n – 1)(n – 2) ... (n – r + 1) ways.

That is, $\quad {}^{n}p_{r} = n(n-1)(n-2) \ldots (n-r+1)$

Remarks:

1. ${}^{n}p_{r} = n(n-1)(n-2) \ldots (n-r+1)$

$$= \frac{n(n-1)(n-2)...(n-r+1)\{(n-r)!\}}{(n-r)!}$$

$$= \frac{n!}{(n - r)!}$$

$$\text{So, } {}^n p_r = \frac{n!}{(n - r)!}$$

2. The number of permutations of n different things taken all at a time is ${}^n p_n$ = n(n – 1)(n – 2) ... 3.2.1 = n!.
3. ${}^n p_n = \frac{n!}{(n - n)!} = \frac{n!}{0!}$, but ${}^n p_n$ = n!
   So, 0! = 1.
   According to the definition, 0! is meaningless. But when we use it as a symbol its value is 1.
4. The number of permutations of n different things taken r things at a time excluding always a particular object is ${}^{n-1} p_r$.
5. The number of permutations of n different things taken r objects at a time including always a particular object is r. ${}^{n-1} p_{r-1}$. (Restricted permutation)
   Illustration: Since a particular object belongs to every permutation, one place of the r places can be filled up in r ways by the particular object. So, the remaining (r – 1) places can be filled up in ${}^{n-1} p_{r-1}$ ways by the remaining (n – 1) objects. Therefore, the r places can be filled up in r. ${}^{n-1} p_{r-1}$ ways. (Restricted permutation)
6. The number of permutations of n different things taken r things at a time in which each object is repeated r times in any permutation is n^r.
   Illustration: The first place can be filled up in n ways, for any one of the n things can be put in it. Since the object that filled up the first place is repeatable, the second place can be filled up in n ways. Thus, the first two places can be filled up in $n \times n = n^2$ ways. Similarly, the third place can be filled up in n ways and the first three places can be filled up in $n \times n^2 = n^3$ ways. Proceeding in the same way we notice that the exponent of n is same as the number of places to be filled up. So, the r places can be filled up in n^r ways.

Example: Find how many three-letter words can be formed out with the letters of the word EQUATIONS (the words may not have any meaning).
Solution: There are 9 different letters, therefore, n is equal to 9 and since we have to find three-letter words, r is 3. Hence the required number of words is

$$^{9}p_{3} = \frac{9!}{(9-3)!}$$
$$= \frac{9.8.7.(6!)}{6!}$$
$$= 9.8.7 = 504$$

Example: Indicate how many 4-digit numbers greater than 8000 can be formed from the digits 5, 6, 7, 8, 9.
Solution: If the digits are to be greater than 8000, then the first digit must be any one of the 8 and 9. Now the first digit can be chosen in $^{2}p_{1}$ = 2 ways and the remaining three digits can be any of the four digits left, which can be chosen in $^{4}p_{3}$ ways. Therefore, the total number of ways

$$= 2 \times {}^{4}p_{3} = 2 \times 4 \times 3 \times 2 = 48$$

Example: Six papers are set in an examination, of which two are Mathematics. In how many different orders can the papers he arranged so that the two statistic papers are not together? [RU-89]
Solution: The total number of arrangements that can be made of 6 papers is 6!.
Now let the mathematics papers be taken together. These taken as one and the remaining 4 can be arranged amongst themselves in 5! ways. The mathematics papers can be arranged between them in 2! ways.
∴ The total number of arrangements in which the mathematics papers can come together is $5! \times 2!$.
∴ The number of arrangements in which the two particular papers are not together is

$$6! - 5! \times 2! = 720 - 240 = 480$$

Example: Find the numbers less than 1000 and divisible by 5 which can be formed with digits 0, 1, 2, 3, 4, 5, 6, 7 such that each digit does not occur more than once in each number.
Solution: The required numbers may be of one digit, two digits or three digits and each of them must end in 5 or 0, except the number of one digit which must end with 5.
The number of one digit ending in 5 is 1.
The number of two digits ending in 5 is $^{7}p_{1} - 1$
(Since, the number having 0 as the first position is to be rejected)
The number of two digits ending in 0 is $^{7}p_{1}$
The number of three digits ending in 5 is $^{7}p_{2} - {}^{6}p_{1}$
(Since, the numbers having 0 as the first position are $^{6}p_{1}$)
The number of three digits ending in 0 is $^{7}p_{2}$

Hence, the total number of required numbers is $1 + ({}^7p_1 - 1) + {}^7p_1 + ({}^7p_2 - {}^6p_1) + {}^7p_2$
$= 1 + 7 - 1 + 7 + 42 - 6 + 42$
$= 92$

4.5 <u>Permutations of n things not all different</u>: Let us consider three letters a, b, c. The different permutations (arrangements) of these three letters taking three at a time are abc, acb, bac, bca, cab and cba. Thus there are 3! = 6 different ways of arranging three distinct objects taking all at a time. If we change c by b we get three letters a, b, b but two of them are each similar to b and the arrangements will be abb, abb, bab, bba, bab and bba in which three arrangements are same to other three arrangements. So, the actual permutations will be abb, bab and bba and the number of permutations is $3 = \frac{3!}{2!}$.

So, by the above discussion we can say that, the number of permutations taking all at a time of n things of which p things are of one kind, q things are of a second kind, r things are of a third kind and all the rest are different is given by

$$\frac{n!}{p! \times q! \times r!}$$

<u>Example</u>: How many numbers greater than 2000000 can be formed with the digits 4, 6, 6, 0, 4, 6, 3?

Solution: Each number must consist of 7 or more digits. There are 7 digits in all, of which there are 2 fours, 3 sixes and the remaining numbers are different.

$\therefore$ The total numbers are $\frac{7!}{2!\,3!} = 420$

Of these numbers, some begin with zero and are less than one million that must be rejected.

The numbers beginning with zero are $\frac{6!}{2!\,3!} = 60$

$\therefore$ The required numbers are $420 - 60 = 360$

<u>Example</u>: (a) Find the number of permutations of the word PERMUTATION.
(b) Find the number of permutations of letters in the word COMBINATION.

Solution: (a) The word PERMUTATION has 11 letters, of which 2 are Ts and the rest are different. Therefore, the number of permutations is

$$\frac{11!}{2!} = \frac{11.10.9.8.7.6.5.4.3.2.1}{2} = 19958400$$

(b) Since the word COMBINATION consists of 11 letters, in which there are 2 Os, 2 Is, 2 Ns and the remaining letters are different, the total number of permutations is

$$\frac{11!}{2!.2!.2!} = \frac{11.10.9.8.7.6.5.4.3.2.1}{2.2.2} = 4989600$$

Example: (i) How many different words can be made out with the letters in the word ALLAHABAD? (ii) In how many of these will the vowels occupy the even places? [AUB-01]

Solutions: (i) The word ALLAHABAD consists of 9 letters of which A is repeated four times, L is repeated twice and the rest all are different.

$$\frac{9!}{4!2!} = 7560$$

(ii) Since the word ALLAHABAD consists of 9 letters, there are 4 even places that can be filled up by the 4 vowels in 1 way only, since all the vowels are similar. Further, the remaining 5 places can be filled up by the 5 consonants of which two are similar which can be filled in $\frac{5!}{2!}$ ways. Hence the required number of arrangements is $1\times\frac{5!}{2!} = 60$.

4.6 Circular permutations: The circular permutations are related with arrangement of objects as in the case of a sitting arrangement of members in a round table conference. In this case, the arrangement does not change unless the order changes. Let us consider the following two arrangements of 4 members:

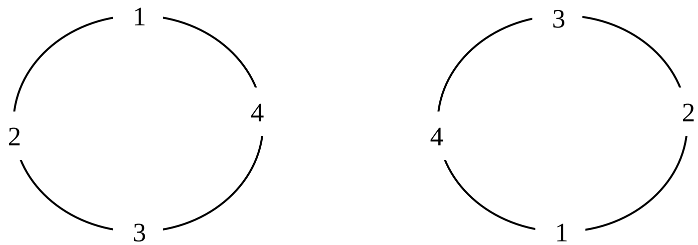


Figure 4.1

In the above figure we see that they have changed their positions but both are same permutation 1234 because their order is not changed.

So, in the circular permutation, the relative position of the other objects depend on the position of the objects placed first. A new permutation is made by the arrangement of the remaining objects. Thus, the number of circular arrangements of n objects will be (n–1)! but not n!. Therefore, the circular arrangement of 4 boys will be in 3! = 3.2.1 = 6 ways.

In the circular permutation, the clockwise and anticlockwise arrangements do not make any difference. If the neighborhood of one or more is restricted, the arrangement will get restricted to that extent. If the restriction is that no two similar objects are close to each other then the number of permutations will be ½{(n – 1)!}. For example if 5 boys are seated around a table so that all of the permutations have not the same neighbors, then the required number of permutations will be ½{(n – 1)!} or ½(4.3.2.1) = 12.

Example: In how many ways can 6 science students and 6 arts students be seated around a table so those no 2 science students are adjacent?

Solution: Let the arts students be seated first. They can sit in 5! ways according to the rule indicated above. Now since the seats for the science students in between arts students are fixed. The option is there for the science students to occupy the remaining 6 seats, there are 6! ways for the science students to fill up the 6 seats in between 6 arts students seated around a table already. Thus, the total number of ways in which both arts and science students can be seated such that no 2 science students are adjacent are $5! \times 6! = 86400$ ways.

4.7 Combination: Combinations refer to different sets or groups made out of a given lot without repeating any object, taking one or more of them at a time neglecting the order. In other words each of the groups which can be made out of n things taking r at a time without repeating and regarding the order of things in each group is termed as combination. It is denoted by ${}^nC_r$ or $C(n, r)$ or $\binom{n}{r}$.

4.7.1 Theorem: The number of combinations of n different things taken r at a time are given by

$$ {}^nC_r = \frac{n!}{r!(n-r)!}, \quad \text{where } (0 \le r \le n) $$

Proof: Let ${}^nC_r$ denote the number of combinations of n different things taken r at a time. So, each of these combinations has r different things.

∴ If the r different things be arranged among themselves in all possible ways, each combination would produce r! permutations.

∴ ${}^nC_r$ combinations would produce ${}^nC_r \times r!$ permutations. But this number is clearly equal to the number of permutations of n different things taken r things at a time.

Hence $\quad {}^nC_r \times r! = {}^nP_r$

Or, $\quad {}^nC_r \times r! = \dfrac{n!}{(n-r)!}$

Or, $\quad {}^nC_r = \dfrac{n!}{r!\,(n-r)!}$

So, $\quad {}^nC_r = \dfrac{n!}{r!\,(n-r)!}$

Example: Find the value of ${}^{10}C_7$.

Solution: $\quad {}^{10}C_7 = \dfrac{10!}{7!\,(10-7)!} = \dfrac{10.9.8.7!}{7!.3!} = 120$

Example: 11 questions are set in the questions paper of Business Mathematics in a year final examination. In how many different ways can an examinee choose 7 questions?
Solution: The number of different choices is evidently equal to the number of combinations of 11 different things taken 7 at a time.

$\therefore$ the required number of ways $= {}^{11}C_7 = \frac{11!}{7!.4!} = \frac{11.10.9.8.7!}{7!.4.3.2.1} = 330$

Example: Show that ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$.

Solution: We know that ${}^nC_r = \frac{n!}{r!\,(n-r)!}$

$$\begin{aligned}
\text{R.H.S} &= {}^nC_r + {}^nC_{r-1}. \\
&= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!} \\
&= \frac{n!}{r\,(r-1)!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)\,(n-r)!} \\
&= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{r} + \frac{1}{(n-r+1)}\right] \\
&= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n+1}{r(n-r+1)}\right] \\
&= \frac{(n+1)!}{r!\,(n-r+1)!} \\
&= \frac{(n+1)!}{r!\,(n+1-r)!} \\
&= {}^{n+1}C_r = \text{L.H.S}
\end{aligned}$$

Example: A cricket team of 11 players is to be formed from 15 players including 3 bowlers and 2 wicket keepers. In how many different ways can a team be formed so that the team contains (i) exactly 2 bowlers and 1 wicket keeper, (ii) at least 2 bowlers and at least 1 wicket keeper.
Solution: (i) A cricket team of 11 players is exactly to contain 2 bowlers and 1 wicket keeper.
2 bowlers can be selected out of 3 in ${}^3C_2 = 3$ ways.
1 wicket keeper can be selected out of 2 in ${}^2C_1 = 2$ ways.
The remaining 8 player can be selected out of remaining 10 players in ${}^{10}C_8 = 45$ ways.
So, the total number of ways in which the team can be formed = $3\times2\times45$ = 270.
(ii) A cricket team of 11 players of which at least 2 bowlers and 1 wicket keeper can be formed in the following ways:

(a) 2 bowlers, 1 wicket keeper and 8 other players.
(b) 2 bowlers, 2 wicket keeper and 7 other players.
(c) 3 bowlers, 1 wicket keeper and 7 other players.
(d) 3 bowlers, 2 wicket keeper and 6 other players.

We now consider the above 4 cases.

(a) 2 bowlers, 1 wicket keeper and 8 other players can be selected in
$^3C_2 \times {}^2C_1 \times {}^{10}C_8 = 3 \times 2 \times 45 = 270$ ways.
(b) 2 bowlers, 2 wicket keeper and 7 other players can be selected in
$^3C_2 \times {}^2C_2 \times {}^{10}C_7 = 3 \times 1 \times 120 = 360$ ways.
(c) 3 bowlers, 1 wicket keeper and 7 other players can be selected in
$^3C_3 \times {}^2C_1 \times {}^{10}C_7 = 1 \times 2 \times 120 = 240$ ways.
(d) 3 bowlers, 2 wicket keeper and 6 other players can be selected in
$^3C_3 \times {}^2C_2 \times {}^{10}C_6 = 1 \times 1 \times 210 = 210$ ways.

Therefore, the total number of different ways $= 270 + 360 + 240 + 210 = 1080$.

4.8 <u>Combinations of n different things taken some or all at a time</u>:
The first thing can be dealt within 2 ways, for it may either be left or taken.
The second thing can also be dealt within 2 ways, for it may either be left or taken.
Since each way of dealing with the first must be associated with the ways of dealing with the second thing, the total number of ways of dealing with these two things is $2 \times 2 = 2^2$.
The third thing can also be dealt within 2 ways, for it may either be left or taken.
So, the total number of ways of dealing with the first three things is $2 \times 2 \times 2 = 2^3$.
Proceeding in this way, the total number of ways for n different things $= 2^n$. But this number includes one case in which none of the things are taken.
Therefore, the number of combinations of n different things taken some or all at a time is $2^n - 1$.

<u>Example</u>: 33% marks have to be secured in each of the 10 subjects in order to pass the S.S.C examination. In how many ways can a student fail?
Solution: Each subject can be dealt within 2 ways, one the student pass in it other he fail in it. So, the 10 subjects can be dealt within $2^{10} = 1024$ ways. But this number includes the case in which the student passes in all subjects. Excluding this case, the number of ways to fail the student is $1024 - 1 = 1023$.

4.9 <u>Combinations of n things not all different taken some or all at a time</u>: Let us consider n things of which p things are of one kind, q things are of a second kind, r things are of a third kind and all the rest are different. Here, $p + q + r \leq n$.
Consider the first kind things. The p things can be dealt in $(p + 1)$ ways, for we may take 1 thing or 2 things or 3 things or . . . or p things or none in any selection. Similarly, the q like things can be dealt in $(q + 1)$ ways and r like things in $(r + 1)$ ways. Since each way of any kind things must be associated with the ways of other kind things, the number of ways is $(p + 1)(q + 1)(r + 1)$.

The remaining {n – (p + q + r)} things are different. So, these things can be dealt in $2^{n-p-q-r}$ ways.
Since every thing is associated to each other, the total number of ways
$= (p + 1)(q + 1)(r + 1)(2^{n-p-q-r})$.
But this number includes the case in which all things are left.
Therefore, the total number of combinations of n things not all different taken some or all at a time $= (p + 1)(q + 1)(r + 1)(2^{n-p-q-r}) - 1$.

<u>Example</u>: Let there are 3 Econo pens, 4 2B pencils, 1 Business Math. book and 1 CD. Find the number of combinations in which at least one thing is present.
Solution: The 3 Econo pens can be dealt in $(3 + 1) = 4$ ways, for we may take 1 pen or 2 pens or 3 pens or none in any selection. Similarly, the 4 2B pencils can be dealt in $(4 + 1) = 5$ ways. So, the pens and the pencils can be dealt in $4\times 5 = 20$ ways.
Business Math. book can be dealt within 2 ways, for taken or left. Similarly, the CD can be dealt within 2 ways, for taken or left. So, the book and the CD can be dealt within $2\times 2 = 2^2 = 4$ ways.
Since every things is associated with others, the total number of ways $= 20\times 4 = 80$.
But this number includes the case in which all things are left. Therefore, the required number of combinations = 80 – 1 = 79. (Answer)
<u>Another way</u>: Using the above formula, we can find the number of combinations very easily as $(3 + 1)(4 + 1)(2^2) - 1 = 4\times 5\times 4 - 1 = 79$.

4.9 <u>Some worked out examples</u>:
<u>Example (1)</u>: Find how many three-letter words can be formed out with the letters of the word LOGARITHM (the words may not have any meaning).
Solution: There are 9 different letters, therefore, n is equal to 9 and since we have to find three-letter words, r is 3. Hence the required number of words is

$$^9p_3 = \frac{9!}{(9-3)!}$$
$$= \frac{9.8.7.(6!)}{6!}$$
$$= 9.8.7 = 504$$

<u>Example (2)</u>: In how many ways can 5 Bengali 3 English and 3 Arabic books he arranged if the books of each different language are kept together.
Solution: The each language book amongst themselves can be arranged in the following ways:

Bengali : 5 books in $^5p_5 = 5!$ ways
English : 3 books in $^3p_3 = 3!$ ways
Arabic : 3 books in $^3p_3 = 3!$ ways

Also arrangement of these groups can be made in ${}^3p_3 = 3!$ ways, hence by the fundamental theorem, the required arrangements are

$$5! \times 3! \times 3! \times 3! = 25920$$

Example (3): Show that ${}^nP_r = n \times {}^{n-1}P_{r-1}$

Solution: R.H.S $= n \times {}^{n-1}P_{r-1}$

$$= n \times \frac{(n-1)!}{\{(n-1)-(r-1)\}!}$$

$$= \frac{n!}{(n-r)!} = {}^nP_r = \text{L.H.S}$$

Example (4): How many arrangements can be made with the letters of the word MATHEMATICS and in how many of them vowels occur together? [RU-88]

Solution: The word MATHEMATICS consists of 11 letters of which 2 are As, 2 Ms, 2 Ts and the rest all different.

$\therefore$ The total number of arrangements are $\frac{11!}{2! \times 2! \times 2!} = 4989600$

The word MATHEMATICS consists of 4 vowels A, A, E and I (two are similar). To find the number of arrangements in which the four vowels occur together, consider the four vowels as tied together and forming one letter. Thus we are left with 8 letters of which 2 are Ms, 2 are Ts, 1 is H, 1 is C, 1 is S and the vowels as 1 letter. These letters can be permuted in $\frac{8!}{2!.2!} = 10080$ ways. The 4 vowels that are tied together can again be permuted among themselves in $\frac{4!}{2!} = 12$ ways (since two of the vowels are similar). Hence the total number of arrangements are $10080 \times 12 = 120960$.

Example (5): In how many ways can the letters of word. 'ARRANGE" be arranged? How many of these arrangements are there in which

(i) the two Rs come together,

(ii) the two Rs do not come together,

(iii) the two Rs and the two As come together ? [AUB–02]

Solution: The word ARRANGE consists of 7 letters of which two are As, two are Rs and the rest all different. Hence they can be arranged amongst themselves in $\frac{7!}{2!.2!} = 1260$ ways

(i) The number of arrangements in which the two Rs come together can be obtained by treating the two Rs as one letter. Thus there are 6 letters of which two (the two As) are similar and so the total number of arrangements = $\frac{6!}{2!.} = 360$.

(ii) The number of arrangements in which the two Rs do not come together can be obtained by subtracting from the total number of arrangements, the arrangements in which the two Rs come together. Thus the required number is 1260 – 360 = 900.

(iii) The number of arrangements in which the two Rs and the two As come together can be obtained by treating the two Rs and the two As a single letter. Thus there are 5 letters that all are different and so the number of arrangements is 5! = 120.

Example (6): In how many ways can 5 Bangladeshis and 5 Pakistanis be seated at a round table so that no two Pakistanis may be together?
Solution: First we let one of the Pakistani in a fixed seat and then the remaining 4 Pakistanis arrange their seats as in the 4! ways. After they have taken their seats in any way, there are five seats for the Bangladeshis each between two Pakistanis. Therefore, the Bangladeshis can be seated in 5! ways.
$\therefore$ Total number of circular permutations is $4! \times 5! = 2880$.

Example (7): Find the value of n when $4 \times {}^{n}p_3 = 5 \times {}^{n-1}p_3$

Solution: We are given that $4 \times {}^{n}p_3 = 5 \times {}^{n-1}p_3$

$$\Rightarrow \quad 4 \times n(n-1)(n-2) = 5 \times (n-1)(n-2)(n-3)$$

[Dividing throughout by (n-l)(n-2), we get]

$$\Rightarrow \quad 4n = 5(n-3)$$
$$\Rightarrow \quad 4n = 5n - 15$$
$$\Rightarrow \quad -n = -15$$
$$\Rightarrow \quad n = 15$$

Example (8): In how many different ways can 7 examination papers be arranged in a line so that the best and worst papers are never together? [DU-85]

Solution: The total number of arrangements that can be made of 7 papers is 7!.
Now let the best and the worst papers be taken together. These taken as one and the remaining 5 can be arranged amongst themselves in 6! ways. The best and the worst papers can be arranged between them in 2! ways.
$\therefore$ The total number of arrangements in which the best and the worst papers can come together is $6! \times 2!$.
$\therefore$ The number of arrangements in which the two particular papers are not together is

$$7! - 6! \times 2! = 5040 - 1440 = 3600$$

Example (9): In how many ways can 5 white and 4 black balls be selected from a box containing 20 white and 16 black balls.
Solution: This is a problem of combinations.

5 out of 20 white balls can be selected in

$$^{20}C_5 = \frac{20!}{5! \times (20-5)!} = \frac{20\times19\times18\times17\times16\times15!}{5\times4\times3\times2\times1\times15!} = 15504 \text{ ways.}$$

4 out of 16 black balls can be selected in $^{16}C_4 = \frac{16\times15\times14\times13}{4\times3\times2\times1} = 1820$ ways.

So, the required number of combinations = 15504×1820 = 28217280.

<u>Example (10)</u>: A question paper contains 5 questions, each having an alternative. In how many ways can an examinee answer one or more questions ? [AUB-02]

Solution: The first question can be dealt with in 3 ways, for the question itself may be answered, or its alternative may be answered or none of them may be answered.

Similarly, the second question also can be dealt with in 3 ways. Hence, the first two questions can be dealt with in 3×3 or 3^2 ways. Proceeding in this way, all the 5 questions may be dealt with in 3^5 ways.

But this number includes one case in which none of the questions is answered.

$\therefore$ The required number of ways $= 3^5 - 1 = 242$.

<u>Example (11)</u>: A committee of 4 boys and 3 girls is to be formed from 7 boys and 8 girls. In how many different ways can the committee be formed if boy-1 and girl-1 refuse to attend the same committee. [AUB-00]

Solution: 3 girls can be selected out of 8 girls in $^8C_3$ ways and 4 boys can be selected out of 7 boys in $^7C_4$ ways.

So, the number of ways of choosing the committee is

$$^8C_3 \times {}^7C_4 = \frac{8!}{3!\,5!} \times \frac{7!}{4!\,3!} = 1960$$

If both boy-1 and girl-1 are members then there remain to be selected 2 girls out of 7 girls and 3 boys from 6 boys. It can be done by

$$^7C_2 \times {}^6C_3 = \frac{7!}{2!\,5!} \times \frac{6!}{3!\,3!} = 420 \text{ ways.}$$

Therefore, the number of ways of forming the committee in which boy-1 and girl-1 are not present together is (1960 – 420) = 1540.

4.10 <u>Exercise</u>:

1. Define permutations and combinations. Illustrate with examples.
2. Distinguish between permutations and combinations.
3. What do you mean by circular permutation?
4. Find the values of (i) $^{10}P_4$ (ii) $^{15}P_4$ [Answer: (i) 5040 (ii) 32760]
5. Find the number of permutations of the word ACCOUNTANT. [Answer: 226800]
6. Prove that $^np_n = {}^np_{n-1}$.
7. Find the value of r if $^7P_r = 60.{}^7P_{r-3}$. [Answer: 3]

8. Find the number of permutations of letters in the word ENGINEERING. [Answer: 1108800]
9. Find the numbers less than 1000 and divisible by 5 which can be formed with digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 such that each digit does not occur more than once in each number. [Answer: 154]
10. How many different numbers of 3 digits can be formed from the digits 1, 2, 3, 4, 5 and 6, if no digits may be repeated? If repetitions are allowed? [Answer: 120, 216]
11. Find how many words can be formed with the letters of the word 'FAILURE', the four vowels always coming together. [Answer: 576]
12. A telegraph post has five arms and each arm is capable of 4 distinct positions including the positions of rest. What is the total number of signals that can be made? [Answer: 1023]
13. A library has 7 copies of one book, 4 copies of each of two books, 6 copies of each of three books and single copies of 9 books. In how many ways can all the books be arranged? [Answer: $\frac{42!}{7!(4!)^2\ (6!)^3}$]
14. Find the values of (i) ${}^7C_2$ (ii) ${}^{20}C_4$ [Answer: (i) 21 (ii) 4845]
15. What is the value of r for ${}^{18}C_r = {}^{18}C_{r+2}$?
16. Prove that the number of combinations of n different things taken r at a time is equal to the number of combinations of n different things taken (n – r) at a time, that is ${}^nC_r = {}^nC_{n-r}$, where $0 \le r \le n$. (Complementary combination)
17. Prove that ${}^nC_r + {}^{n-1}C_{r-1} + {}^{n-1}C_{r-2} = {}^{n+1}C_r$.
18. From 6 boys and 4 girls a committee of 6 is to formed. In how many ways can this be done if the committee contains (i) exactly 2 girls (ii) at least 2 girls. [Answer: (i) 90 (ii) 185]
19. A cricket team consisting of 11 players is to be formed from 16 players of whom 6 persons are bowlers. In how many different ways can a team be formed so that the teams contain at least 4 bowlers? [Answer: 3096]
20. A cricket team consisting of 11 players is to be formed from 16 players of whom 4 can be bowlers and 2 can keep wicket and the rest can neither bowler nor keep wicket. In how many different ways can a team be formed so that the teams contain (i) exactly 3 bowlers and 1 wicket keeper, (ii) at least 3 bowlers and at least 1 wicket keeper. [Answer: (i) 960, (ii) 2472]
21. 36% marks have to be secured in each of the 11 subjects in order to pass the B.Sc examination. In how many ways can a student fail? [Answer: 2047]
22. Let you have 5 Winner classic pens, 7 6B pencils, 4 Business Math. book and 1 calculator. Find the number of combinations in which at least one thing is present. [Answer: 479]
23. A party of 6 is to be formed from 7 boys and 10 girls so as to include 3 boys and 3 girls. In how many ways can the party be formed if 2 particular boys refuse to join the same party? [Answer: 3600]

05

Chapter

Determinant and Matrix

Highlights:

5.1 Introduction: The working knowledge of determinants and matrices is a basic necessity for the students of Mathematics, Business Mathematics, Physics, Statistics, Economics and Engineering. In 1683 a Japanese mathematician Kiowa first devised the idea as well as the notation of determinants and J. J. Sylvester was the first man who introduced the word 'matrix' in 1850. Gabriel Cramer successfully applied determinants in solving systems of linear equations in 1750. At present matrix is a powerful tool of modern mathematics.

5.2 Definition of determinant: A determinant of order n is a square array of n^2 quantities a_{ij} (i, j = 1, 2, 3, ..., n) enclosed between two vertical bars and is generally written in the form given below:

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

which is an ordinary number.

The n^2 quantities a_{ij} (i, j = 1, 2, 3, ..., n) are called the elements of the determinant. The vertical lines of elements are known as columns whereas the horizontal lines of elements are known as rows. Here, a_{ij} represents the element of the i-th row and j-th column of the determinant.

5.3 <u>Value of the determinant</u>: The value of a 2×2 determinant

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is $a_1b_2 - a_2b_1$ and its Sarrus diagram is

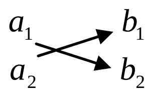


Figure 5.1

The value of a 3×3 determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is

$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$
and its Sarrus diagram is as follows:

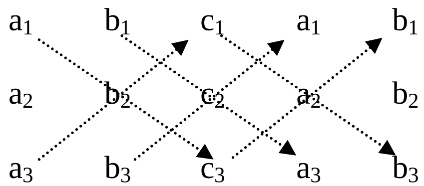


Figure 5.2

To find the value of a determinant we use Sarrus diagram. We multiply the elements joined by arrows. Arrows downwards denote positive sign with the corresponding expression and arrows upwards denote negative sign with the corresponding expression. So, from figure 5.2 we find the value of above 3×3 determinant as follows:

$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$

Or, $a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$ which is same as above.

<u>Example</u>: Using Sarrus diagram find the value of $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$.

Solution: Sarrus diagram of the given determinant as follows:

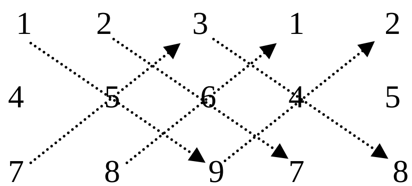


Figure 5.3

From the above Sarrus diagram, we get $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1.5.9+2.6.7+3.4.8-7.5.3-8.6.1-9.4.2$

$= 45+84+96-105-48-72 = 0$ (Answer)

5.4 <u>Minors and co-factors</u>: If the determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Deleting the i-th row and j-th column we will get a new determinant of (n – 1) rows and (n – 1) columns. This new determinant is called the minor of the element a_{ij} and is denoted by M_{ij}. The minor Mij multiplied by $(-1)^{i+j}$ is called the co-factor of the element a_{ij} and is denoted by A_{ij}. So, $A_{ij} = (-1)^{i+j} M_{ij}$.

The determinant D is equal to the sum of the products of the elements of any row or column and their respective co-factors. That is,

$D = a_{11}A_{11} + a_{12}A_{12} + ... + a_{1n}A_{1n}$

As for example let us consider 3×3 determinant

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. So the co-factor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, the co-factor of b_1 is $(-)\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ and the co-factor of c_1 is $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Thus, $D = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 (-)\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

5.5 <u>Fundamental properties of determinants</u>: The followings are fundamental properties of determinants:

1. If all the elements in a row (or in a column) of a determinant are zero, the determinant is equal to zero.

   Let $D = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0(b_2c_3 - b_3c_2) - 0(a_2c_3 - a_3c_2) + 0(a_2b_3 - a_3b_2) = 0$

2. The value of the determinant is not altered when the columns are changed to rows (or the rows to columns).

   Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Interchanging the columns to rows, we get $D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

   Expressing both determinant, we will have $D = D'$.

3. The interchange of any two rows (or any two columns) of a determinant changes the sign of the determinant.

   Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Interchanging first and second rows, we get $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

   So, $D' = a_2(b_1c_3 - b_3c_1) - b_2(a_1c_3 - a_3c_1) + c_2(a_1b_3 - a_3b_1)$
   $= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$
   $= -D.$

4. If two rows (or two column) of a determinant are identical, the determinant is equal to zero.

   Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ which first and second rows are identical.

   Interchanging first and second rows, we get $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and by property-3, its value is – D.

   So, D = – D or, 2D = 0 or, D = 0.

5. If each element in a row (or in a column) is multiplied by any scalar, α, the determinant is multiplied by that scalar α.

   Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

   Multiplying each element in first row by α, we get the following determinant:

   $$D' = \begin{vmatrix} \alpha a_1 & \alpha b_1 & \alpha c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \alpha a_1(b_2c_3 - b_3c_2) - \alpha b_1(a_2c_3 - a_3c_2) + \alpha c_1(a_2b_3 - a_3b_2)$$

   $$= \alpha \{a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\}$$

   $$= \alpha D$$

   So, $\alpha \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} \alpha a_1 & \alpha b_1 & \alpha c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

6. If every element in any column (or in any row) of a determinant is expressed as the sum of two quantities, the determinant can be expressed as the sum of two determinants of the same order.

   If A_1, A_2 and A_3 are co-factors of the elements $(a_1+\alpha_1)$, $(a_2+\alpha_2)$ and $(a_3+\alpha_3)$ respectively of the following determinant, then

$$D = \begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = (a_1+\alpha_1)A_1 + (a_2+\alpha_2)A_2 + (a_3+\alpha_3)A_3$$

$$= (a_1A_1 + a_2A_2 + a_3A_3) + (\alpha_1A_1 + \alpha_2A_2 + \alpha_3A_3)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

7. If we change each element of any column (or any row) by adding to them any constant multiple of the corresponding element of other columns (or other rows), then the value of the determinant will not be altered.

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

   Multiplying column-2 by m and column-3 by n and then adding to column-1, we have the following determinant:

$$D' = \begin{vmatrix} a_1+mb_1+nc_1 & b_1 & c_1 \\ a_2+mb_2+nc_2 & b_2 & c_2 \\ a_3+mb_3+nc_3 & b_3 & c_3 \end{vmatrix}$$

   Using property-6, we have

$$D' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mb_1 & b_1 & c_1 \\ mb_2 & b_2 & c_2 \\ mb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} nc_1 & b_1 & c_1 \\ nc_2 & b_2 & c_2 \\ nc_3 & b_3 & c_3 \end{vmatrix}$$

Or, $$D' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + n\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ [Using property-5]

Or, $D' = D + m.0 + n.0$ [Using property-4]

So, $D' = D$.

8. The sum of the products of the elements of a column (or a row) and respective co-factors of any other column (or any other row) is zero.

Let D = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. So the co-factor of a_1 is $A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$,

the co-factor of a_2 is $A_2 = (-)\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$ and the co-factor of a_3 is $A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

Thus, $D = a_1A_1 + a_2A_2 + a_3A_3$

But, $b_1A_1 + b_2A_2 + b_3A_3$ represents $\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = 0$ [By property-4]

So, $b_1A_1 + b_2A_2 + b_3A_3 = 0$.

5.6 <u>Multiplication of two determinants</u>: To multiply two determinants, we have to make them same order firstly. Then we use one of the following four methods for the multiplication of two determinants. Such as (i) Row by row method, (ii) Row by column method, (iii) Column by row method and (iv) Column by column method. Here, we try to make you understand the row by column method with the following example:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} e & f \\ g & h \end{vmatrix} = \begin{vmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{vmatrix}$$

Note: We can increase the order of a determinant as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$

5.7 <u>Application of determinants (Cramer's rule)</u>: Let us consider the following system of n non-homogeneous linear equations in n unknowns x_1, x_2, x_3, ---, x_n.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}\right\}$$

Then $x_1 = \dfrac{D_1}{D}$, $x_2 = \dfrac{D_2}{D}$, ..., $x_n = \dfrac{D_n}{D}$ where

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0,\ D_1 = \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix},\ \dots,\ D_n = \begin{vmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & b_n \end{vmatrix}$$

Proof: Let us consider the relation

$$\begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & a_{12} & \dots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & a_{22} & \dots & a_{2n} \\ \dots \quad \dots \quad \dots \quad \dots & \dots & \dots & \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

The left hand side determinant can be expressed as the sum of n determinants of which all the determinants are zero except first one, since at least two columns of those determinants are equal. Therefore, from above relation we have

$$x_1 \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Or, $x_1 D = D_1$

So, $x_1 = \dfrac{D_1}{D}$

Similarly, we can find the value of x_2, x_3, ..., x_n as follows:

$$x_2 = \frac{D_2}{D},\ x_3 = \frac{D_3}{D},\ \dots,\ x_n = \frac{D_n}{D}$$

Thus, $x_1 = \dfrac{D_1}{D},\ x_2 = \dfrac{D_2}{D},\ x_3 = \dfrac{D_3}{D},\ \dots,\ x_n = \dfrac{D_n}{D}$

This solution technique of a system of linear equations is called the **Cramer's rule**.

<u>Example</u>: Using Cramer's rule solve the following system of linear equations:

$$\begin{cases} x + 2y - 3z = 0 \\ 2x - y + 4z = 10 \\ 4x + 3y - 4z = 6 \end{cases}$$ [AUB-02 MBA]

Solution: Forming determinant with coefficients of x, y and z, we get

$$D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -4 \end{vmatrix} = 1(4 - 12) - 2(-8 - 16) + (-3)(6 + 4) = -8 + 48 - 30 = 10$$

Forming determinant with constant terms and the coefficients of y and z, we get

$$D_x = \begin{vmatrix} 0 & 2 & -3 \\ 10 & -1 & 4 \\ 6 & 3 & -4 \end{vmatrix} = 0(4 - 12) - 2(-40 - 24) - 3(30 + 6) = 0 + 128 - 108 = 20$$

Forming determinant with coefficients of x, constant terms and coefficients of z, we get

$$D_y = \begin{vmatrix} 1 & 0 & -3 \\ 2 & 10 & 4 \\ 4 & 6 & -4 \end{vmatrix} = 1(-40 - 24) - 0(-8 - 16) - 3(12 - 40) = -64 - 0 + 84 = 20$$

Forming determinant with coefficients of x, y and constant terms, we get

$$D_z = \begin{vmatrix} 1 & 2 & 0 \\ 2 & -1 & 10 \\ 4 & 3 & 6 \end{vmatrix} = 1(-6 - 30) - 2(12 - 40) + 0(6 + 4) = -36 + 56 + 0 = 20$$

So, $x = \frac{D_x}{D} = \frac{20}{10} = 2,$

$y = \frac{D_y}{D} = \frac{20}{10} = 2,$

And $z = \frac{D_z}{D} = \frac{20}{10} = 2.$

Therefore, the solution of the system, (x, y, z) = (2, 2, 2).

Example: Using Cramer's rule solve the following system of linear equations

$2x_1 - x_2 = 2$

$3x_2 + 2x_3 = 16$ [DU-80 Eco.]

$5x_1 + 3x_3 = 21$

Solution: We can rewrite the equations as follows:

$2x_1 - x_2 + 0x_3 = 2$

$0x_1 + 3x_2 + 2x_3 = 16$

$5x_1 + 0x_2 + 3x_3 = 21$

Here, $D = \begin{vmatrix} 2 & -1 & 0 \\ 0 & 3 & 2 \\ 5 & 0 & 3 \end{vmatrix} = 2(3.3 - 0.2) - (-1)(0.3 - 5.2) + 0(0.0 - 5.3)$

$= 2.9 + 1.(-10) + 0 = 18 - 10 = 8$

$$D_1 = \begin{vmatrix} 2 & -1 & 0 \\ 16 & 3 & 2 \\ 21 & 0 & 3 \end{vmatrix} = 2(3.3 - 0.2) - (-1)(16.3 - 21.2) + 0 = 18 + 6 = 24$$

$$D_2 = \begin{vmatrix} 2 & 2 & 0 \\ 0 & 16 & 2 \\ 5 & 21 & 3 \end{vmatrix} = 2(16.3 - 21.2) - 2(0.3 - 5.2) = 12 + 20 = 32$$

$$D_3 = \begin{vmatrix} 2 & -1 & 2 \\ 0 & 3 & 16 \\ 5 & 0 & 21 \end{vmatrix} = 2(3.21 - 0.16) - (-1)(0.21 - 5.16) + 2(0.0 - 5.3) = 126 - 80 - 30 = 16$$

So, $x_1 = \frac{D_1}{D} = \frac{24}{8} = 3$, $x_2 = \frac{D_2}{D} = \frac{32}{8} = 4$ and $x_3 = \frac{D_3}{D} = \frac{16}{8} = 2$.

Therefore, the required solution is $(x_1, x_2, x_3) = (3, 4, 2)$ (Answer)

5.8 <u>Definition of matrix</u>: A rectangular array of $m \times n$ quantities a_{ij} (i = 1, 2, 3, ..., m; j = 1, 2, ..., n) enclosed by a pair of brackets or double vertical bars is known as matrix and is generally written as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

It is not an ordinary number like determinant. The number a_{11}, a_{12}, ..., a_{mn} are called the entries or the elements of the matrix. The above matrix has m rows and n columns and hence it is called an $m \times n$ matrix (read as m by n matrix). And we say the order of the matrix is $m \times n$ or 'm by n'.

5.9 <u>Types of matrices</u>: There are various types of matrices. Here, we shall discuss important types of matrices with examples.

1. **Square matrix:** A matrix with same number of columns and rows is called a square matrix. For examples $\begin{pmatrix} 2 & 6 \\ 4 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 4 & 9 \\ 6 & 5 & 1 \\ 9 & 7 & 3 \end{pmatrix}$ are square matrices.
2. **Row matrix (or row vector):** If a matrix consists of only one row, then it is called a row matrix or row vector. For example $(2 \quad 5 \quad 7)$ is a row matrix of order 1×3.
3. **Column matrix (or column vector):** If a matrix consists of only one column, then it is called a column matrix or column vector. For example $\begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$ is a column matrix of order 3×1.

4. Diagonal matrix: A square matrix whose elements $a_{ij} = 0$ when $i \neq j$ is called a diagonal matrix. For example $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is a diagonal matrix of order 3×3.

5. Scalar matrix: A diagonal matrix whose diagonal elements are all equal is called a scalar matrix. For example $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a scalar matrix of order 3×3.

6. Unit matrix (or identity matrix): A square matrix whose elements $a_{ij} = 0$ when $i \neq j$ and $a_{ij} = 1$ when $i = j$ is called a unit matrix or identity matrix and is denoted by I. For example, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a unit matrix.

7. Null matrix (or zero matrix): The matrix whose each element is 0 (zero) is known as null matrix or zero matrix. For examples $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are null matrices of order 3×2 and 3×3.

8. Triangular matrix: The square matrix in which elements $a_{ij} = 0$ when $i > j$ is called an upper triangular matrix. And the square matrix in which elements $a_{ij} = 0$ for $i < j$ is called a lower triangular matrix. For examples $\begin{pmatrix} 1 & 7 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & -2 & 1 \end{pmatrix}$ are upper and lower triangular matrices respectively of order 3×3.

9. Idempotent matrix: A square matrix A is said to be an idempotent matrix if $A^2 = A$. For example $\begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix}$ is an idempotent matrix.

10. Involutory matrix: A square matrix A is said to be an involutory matrix if $A^2 = I$, where I is the unit matrix. For example, $A = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$ is an involutory matrix.

11. Nilpotent matrix: A square matrix A is said to be a nilpotent matrix of order n if $A^n = 0$ and $A^{n-1} \neq 0$ where 0 is the null matrix and n is a positive integer. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a nilpotent matrix of order 2.

12. Singular matrix: A square matrix A is said to be a singular matrix if the value of the determinant formed by the matrix is 0 (zero). For example,

$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$ is a singular matrix because of $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix} = 0$.

13. Transpose of a matrix: The matrix, which is formed by interchanging the columns to rows (or the rows to columns), is called the transpose matrix of that matrix. Let A be an $m \times n$ order matrix, then the matrix of order $n \times m$ obtained from the matrix A by interchanging its columns to rows (or the rows to columns) is called the transpose of A and is denoted by the symbol A^T. That is, if $A = (a_{ij})$ is an $m \times n$ matrix then $A^T = (a_{ji})$ is a matrix of order $n \times m$. For example, let

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 6 & 7 & 8 \end{pmatrix}, \text{ then } A^T = \begin{pmatrix} 1 & 6 \\ 2 & 7 \\ 5 & 8 \end{pmatrix}.$$

14. Symmetric matrix: A square matrix A is said to be symmetric matrix if $A^T = A$, that is, elements $a_{ij} = a_{ji}$ for all i, j. For example $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 3 \end{pmatrix}$ is a symmetric matrix.

15. Skew-symmetric matrix: A square matrix A is said to be skew-symmetric matrix if $A^T = -A$, that is, elements $a_{ij} = -a_{ji}$ for $i \neq j$ and $a_{ij} = 0$ for $i = j$. For example $\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 7 \\ -3 & -7 & 0 \end{pmatrix}$ is a skew-symmetric matrix.

16. Inverse matrix: If A and B are two non-singular square matrices such that AB = BA = I, where I is the unit matrix, then B is said to be the inverse matrix of A as well as A is the inverse matrix of B. The inverse matrix of A is denoted by A^{-1}. For examples, $A^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the inverse matrix of $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$.

17. Orthogonal matrix: A matrix A is called an orthogonal matrix if $AA^T = A^TA = I$, where I is the unit matrix. That is, $A^T = A^{-1}$. For example,

$$A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$ is an orthogonal matrix.

18. Complex conjugate of a matrix: The complex number $\bar{z}$ = x – iy is the conjugate of the complex number z = x + iy. The conjugate of a matrix A is the matrix whose elements are respectively the conjugates of the elements of A and is denoted by $\bar{A}$. That is, if A = (a_{ij}), then $\bar{A}$ = $(\bar{a}_{ij})$. For example,

$\bar{A} = \begin{pmatrix} 2 & 3+i4 \\ 5-i2 & 3 \end{pmatrix}$ is the complex conjugate of matrix A = $\begin{pmatrix} 2 & 3-i4 \\ 5+i2 & 3 \end{pmatrix}$.

19. Hermitian matrix: The square matrix A = (a_{ij}) of complex numbers is said to be Hermitian matrix if $A^* = \bar{A}^T = A$, that is, $a_{ij} = \overline{a_{ji}}$ for all i, j. For example,

A = $\begin{pmatrix} 2 & 3-i4 \\ 3+i4 & 3 \end{pmatrix}$ is a Hermitian matrix.

20. Skew -Hermitian matrix: The square matrix A = (a_{ij}) of complex numbers is said to be skew-Hermitian matrix if $A^* = \bar{A}^T = -A$, that is, $a_{ij} = -\overline{a_{ji}}$ for all i, j. For example, A = $\begin{pmatrix} 2i & -3+i4 \\ 3+i4 & 0 \end{pmatrix}$ is a skew-Hermitian matrix.

21. Adjoint or adjugate matrix: Let a square matrix A = $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$.

If $A_{11}, A_{12}, A_{13}, \dots, A_{nn}$ are respective co-factors of elements $a_{11}, a_{12}, a_{13}, \dots, a_{nn}$ of the determinant $|A|$ = $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$ then the adjoint matrix of A is

$$\text{Adj } A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

For example, let A = $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & -1 \\ 2 & 1 & 0 \end{pmatrix}$ then Adj A = $\begin{pmatrix} 1 & -2 & -5 \\ 3 & -6 & 3 \\ -11 & 4 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & -11 \\ -2 & -6 & 4 \\ -5 & 3 & 1 \end{pmatrix}$

22. Augmented matrix: Consider the system of linear equations A$\underline{x}$ = $\underline{b}$, where A is an m×n matrix, $\underline{x}$ and $\underline{b}$ are column vectors. Then the m×(n – 1) matrix (A|b), obtained by adjoining the column vector $\underline{b}$ to the matrix A on the right, is known as the augmented matrix of the system A$\underline{x}$ = $\underline{b}$. For example, consider the system

$$\begin{cases} x + 2y - 3z = 6 & \quad . . . \ (1) \\ 2x - y + 4z = 2 & \quad . . . \ (2) \\ 4x + 3y - 4z = 14 & \quad . . . \ (3) \end{cases}$$

We form the following augmented matrix with the coefficients of x, y, z and the constant terms of the considered system.

(A|$\underline{b}$) = $\left\langle \begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -4 & 14 \end{array} \right\rangle$ [Here, A = $\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -4 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix}$]

23. Echelon matrix: A matrix A = (a_{ij}) is an echelon matrix or is said to be in echelon form if the number of zeros preceding the first non-zero entry of a row increases row by row until only zero rows remain. For example,

A = $\begin{pmatrix} 2 & 3 & 0 & 4 \\ 0 & 6 & 5 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is an echelon matrix.

In particular, an echelon matrix is called a row reduced echelon matrix if the distinguished elements are

(i) the only nonzero entries in their respective columns;

(ii) each equal to 1.

For example, A = $\begin{pmatrix} 1 & 3 & 0 & 4 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is a row reduced echelon matrix.

5.10 Matrices operations:

1. **Addition of matrices:** Addition of matrices is defined only for the matrices having same order, that is, same number of rows and same number of columns. Let A and B be the two matrices having m rows and n columns. That is,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

Then the sum of A and B is

$$A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}$$

For example, let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 & 9 \\ 6 & 5 & 1 \\ 9 & 7 & 3 \end{pmatrix}$, then

$$A + B = \begin{pmatrix} 1+3 & 2+4 & 3+9 \\ 2+6 & 5+5 & 7+1 \\ 3+9 & 7+7 & 3+3 \end{pmatrix} = \begin{pmatrix} 4 & 6 & 12 \\ 8 & 10 & 8 \\ 12 & 14 & 6 \end{pmatrix}$$

2. **Subtraction of matrices:** Subtraction of matrices is defined only for the matrices having same order, that is, same number of rows and same number of columns. Let A and B be the two matrices having m rows and n columns. That is,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

Then the sum of A and B is

$$A - B = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \dots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \dots & a_{2n}-b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \dots & a_{mn}-b_{mn} \end{pmatrix}$$

For example, let A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & 4 & 9 \\ 6 & 5 & 1 \\ 9 & 7 & 3 \end{pmatrix}$, then

$$A - B = \begin{pmatrix} 1-3 & 2-4 & 3-9 \\ 2-6 & 5-5 & 7-1 \\ 3-9 & 7-7 & 3-3 \end{pmatrix} = \begin{pmatrix} -2 & -2 & -6 \\ -4 & 0 & 6 \\ -6 & 0 & 0 \end{pmatrix}$$

3. **Scalar multiplication of matrix:** The scalar (number) multiplication of a matrix is defined as follows: The product of an $m \times n$ matrix A by a scalar k is denoted by kA or Ak and is the $m \times n$ matrix obtained by multiplying each element of A by k.

Let A = $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ then kA = $\begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix}$. For example,

let A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 3 \end{pmatrix}$ and k = 2, so 2A = $\begin{pmatrix} 2.1 & 2.2 & 2.3 \\ 2.2 & 2.5 & 2.7 \\ 2.3 & 2.7 & 2.3 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 10 & 14 \\ 6 & 14 & 6 \end{pmatrix}$

4. **Multiplication of matrices:** The multiplication of two matrices A and B will be possible if the number of columns in the first matrix is equal to the number of rows of the second matrix. We multiply the rows of first matrix by the columns of the second matrix. Let A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and B = $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then the product of the matrices A and B is given by

$A \times B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$ and $B \times A = \begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix}$.

For example, let A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & 4 & 9 \\ 6 & 5 & 1 \\ 9 & 7 & 3 \end{pmatrix}$, then

$$A \times B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 4 & 9 \\ 6 & 5 & 1 \\ 9 & 7 & 3 \end{pmatrix} = \begin{pmatrix} 1.3+2.6+3.9 & 1.4+2.5+3.7 & 1.9+2.1+3.3 \\ 2.3+5.6+7.9 & 2.4+5.5+7.7 & 2.9+5.1+7.3 \\ 3.3+7.6+3.9 & 3.4+7.5+3.7 & 3.9+7.1+3.3 \end{pmatrix}$$

$$= \begin{pmatrix} 42 & 35 & 20 \\ 99 & 82 & 44 \\ 78 & 68 & 43 \end{pmatrix}$$

Note – 1: Algebraic associative law is applicable for matrix multiplications That is, if A, B and C are three matrices then $A\times (B\times C) = (A\times B) \times C$.
Note – 2: Matrix multiplication may or may not satisfy the commutative law. That is, if A and B are two matrices then $A\times B = B\times A$ or $A\times B \neq B\times A$.

5.11 <u>Process of finding inverse matrix</u>: Let the square matrix A = $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$

and D be the determinant of the matrix A, that is D = $|A|$ = $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$. Find the value of the determinant D. If D = 0 the matrix A is singular and it has no inverse, and if $D \neq 0$ the matrix A is non-singular and the inverse A^{-1} exists. To find the inverse matrix, we have to find the adjoint matrix first. Let A_{11}, A_{12}, A_{13}, ..., A_{nn} are respective co-factors of elements a_{11}, a_{12}, a_{13}, ..., a_{nn} of the determinant D = $|A|$, then the adjoint matrix of A is

Adj A = $\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$. Then the inverse matrix of A is

$A^{-1} = \frac{1}{D} AdjA = \frac{AdjA}{|A|}$. Therefore, $A^{-1} = \begin{pmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \dots & \frac{A_{n1}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \dots & \frac{A_{n2}}{|A|} \\ \dots & \dots & \dots & \dots \\ \frac{A_{1n}}{|A|} & \frac{A_{2n}}{|A|} & \dots & \frac{A_{nn}}{|A|} \end{pmatrix}$.

For example, let A = $\begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix}$. So, the determinant D = $|A|$ = $\begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix}$ = (4 – 0) = 4. Here, co-factors are: $A_{11} = (-)^2|1| = 1$, $A_{12} = (-)^3|2| = -2$, $A_{21} = (-)^3|0| = 0$, $A_{22} = (-)^4|4| = 4$. So,

Adj A = $\begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$

Therefore, inverse matrix, $A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ -1/2 & 1 \end{pmatrix}$

Note: If A and B be any two non-singular square matrices then $(AB)^{-1} = B^{-1}.A^{-1}$ and $\left(A^{-1}\right)^{-1} = A$. These are the properties of the inverse matrix.

5.12 <u>Rank of a matrix</u>: The rank of a matrix can be defined in many equivalent ways. Generally, we use the following two definitions.

(i) Let A be an arbitrary $m \times n$ matrix over a field. The rank is the largest value of r for which there exists at least one $r \times r$ sub-matrix of A which forms a non-vanishing determinant.

(ii) Let A be an $m \times n$ matrix and let A_e be the row reduced echelon form of A. Then number of non-zero rows of A_e is the rank of the matrix A.

For example let A = $\begin{pmatrix} 4 & 5 \\ 0 & 3 \end{pmatrix}$. Then the rank of the matrix A is 2 because of $|A| = \begin{vmatrix} 4 & 5 \\ 0 & 3 \end{vmatrix} = 12 \neq 0$ or the matrix A contains 2 non-zero rows being itself an echelon matrix.

5.13 <u>Use of matrix to solve the system of linear equations</u>: We solve the system of linear equations using matrix in two ways, such as (1) By finding inverse matrix, (2) By making echelon matrix.

(1) By finding inverse matrix: In this method, we use inverse matrix to solve a system of linear equations. Let A be the coefficient matrix, B be the constant term matrix and X be the variable matrix of a system of linear equations, then

$$AX = B$$

So, the solution will be $X = A^{-1}B$.

To illustrate this method, let us consider the following system of linear equations.

$$\begin{cases} x + 2y - 3z = 6 & \ldots (1) \\ 2x - y + 4z = 2 & \ldots (2) \\ 4x + 3y - 4z = 14 & \ldots (3) \end{cases}$$

From the system, we find the following coefficient matrix, constant term matrix and variable matrix.

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -4 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \quad B = \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix}$$

To find the inverse matrix of matrix A, A^{-1} let us consider the following augmented matrix:

$$\left\langle\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & -1 & 4 & 0 & 1 & 0 \\ 4 & 3 & -4 & 0 & 0 & 1 \end{array}\right\rangle$$

Subtracting two times of 1st row from 2nd row and four times of 1st row from 3rd row, we get

$$\left\langle\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -5 & 10 & -2 & 1 & 0 \\ 0 & -5 & 8 & -4 & 0 & 1 \end{array}\right\rangle$$

Subtracting 2nd row from 3rd row, we get

$$\left\langle\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -5 & 10 & -2 & 1 & 0 \\ 0 & 0 & -2 & -2 & -1 & 1 \end{array}\right\rangle$$

Dividing 2nd row by –5 and 3rd row by –2, we get

$$\left\langle\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2/5 & -1/5 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 \end{array}\right\rangle$$

Subtracting 2 times of 2nd row from 1st row and adding 2 times of 3rd row with 2nd row, we get

$$\left\langle\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/5 & 2/5 & 0 \\ 0 & 1 & 0 & 12/5 & 4/5 & -1 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 \end{array}\right\rangle$$

Subtracting 3rd row from 1st row, we get

$$\left\langle\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/5 & -1/10 & 1/2 \\ 0 & 1 & 0 & 12/5 & 4/5 & -1 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 \end{array}\right\rangle$$

So, $A^{-1} = \begin{pmatrix} -4/5 & -1/10 & 1/2 \\ 12/5 & 4/5 & -1 \\ 1 & 1/2 & -1/2 \end{pmatrix}$

Therefore, $X = A^{-1}B$

Or, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4/5 & -1/10 & 1/2 \\ 12/5 & 4/5 & -1 \\ 1 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix}$

Or, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{24}{5}-\frac{1}{5}+7 \\ \frac{72}{5}+\frac{8}{5}-14 \\ 6+1-7 \end{pmatrix}$

Or, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

That is, x = 2, y = 2 and z = 0

So, the solution is (x, y, z) = (2, 2, 0).

(2) By making echelon matrix: In this method, forming the augmented matrix with the coefficients of x, y, z and the constant terms of the system, we try to make it echelon form. Then we check it consistency by the following rules:

If A|B is the row reduced echelon matrix of a system:

i) If rank of matrix A ≠ rank of augmented matrix A|B, then the system is inconsistent.

ii) If rank of matrix A = rank of augmented matrix A|B = number of variables, then the system is consistent and has a unique solution.

iii) If rank of matrix A = rank of augmented matrix A|B < number of variables, then the system is consistent and has many solutions.

To explain the method, let us again consider the system of 3 linear equations with 3 variables x, y and z that we have solved by the previous method:

$$\begin{cases} x + 2y - 3z = 6 & \ldots (1) \\ 2x - y + 4z = 2 & \ldots (2) \\ 4x + 3y - 4z = 14 & \ldots (3) \end{cases}$$

We form the following augmented matrix with the coefficients of x, y, z and the constant terms of the system.

$$A|B = \left\langle \begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -4 & 14 \end{array} \right\rangle$$

Now, our goal is to reach the echelon form of the augmented matrix by row reduced technique. Subtracting 2 times of 1^{st} row from 2^{nd} row and 4 times of 1^{st} row from 3^{rd} row, we get

$$\approx \left\langle \begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 8 & -10 \end{array} \right\rangle$$

Again subtracting 2nd row from 3rd row, we get

$$\approx \left\langle \begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & 0 & -2 & 0 \end{array} \right\rangle$$

Now, dividing 2nd row by –5 and 3rd row by –2, we have

$$\approx \left\langle \begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 \end{array} \right\rangle$$, which is the echelon form.

In the echelon form, we see that there are 3 non-zero rows both in matrix A and augmented matrix A|B. So, the rank of A = the rank of A|B = 3 = number of variables. Therefore, the considered system is consistent and has unique solution. To find the solution, we form the following equations by the rows of the echelon matrix:

$z = 0$
$y - 2z = 2 \quad \Rightarrow \quad y = 2$
$x + 2y - 3z = 6 \quad \Rightarrow \quad x = 2$

Thus, the solution of the system is (x, y, z) = (2, 2, 0)

5.14 <u>Some worked out examples</u>:

<u>Example (1)</u>: Evaluate the determinant D = $\begin{vmatrix} 3 & 5 & 6 \\ 4 & 3 & -7 \\ 8 & 1 & 0 \end{vmatrix}$

Solution: We know that the determinant D is equal to the sum of the products of the elements of any row or column and their respective co-factors.

Thus, D = $3\begin{vmatrix} 3 & -7 \\ 1 & 0 \end{vmatrix} - 5\begin{vmatrix} 4 & -7 \\ 8 & 0 \end{vmatrix} + 6\begin{vmatrix} 4 & 3 \\ 8 & 1 \end{vmatrix}$

= 3{3.0 – 1(-7)} – 5{4.0 – 8(-7)} + 6(4.1 – 8.3)
= 3(0 + 7) – 5(0 + 56) + 6(4 – 24)
= 3.7 – 5.56 + 6.(- 20)
= 21 – 280 – 120
= – 379 (Answer)

<u>Example (2)</u>: If A = $\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix}$ and B = $\begin{vmatrix} 2 & 5 \\ 2 & 8 \end{vmatrix}$ then find $A \times B$.

And also show that (value of A)(value of B) = value of A×B.

Solution: Given that, A = $\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix}$ and B = $\begin{vmatrix} 2 & 5 \\ 2 & 8 \end{vmatrix}$

So, A×B = $\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} \times \begin{vmatrix} 2 & 5 \\ 2 & 8 \end{vmatrix} = \begin{vmatrix} 4.2+1.5 & 2.2+3.5 \\ 4.2+1.8 & 2.2+3.8 \end{vmatrix} = \begin{vmatrix} 13 & 19 \\ 16 & 28 \end{vmatrix}$ (Answer)

Now, value of A = 4.3 – 1.2 = 10,
value of B = 2.8 – 2.5 = 6,
value of A×B = 13.28 – 16.19 = 364 – 304 = 60.

So, value of A × value of B = 10×6 = 60

Thus, (value of A)(value of B) = value of A×B (Shown)

<u>Example (3)</u>: Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ = (a – b)(b – c)(c – a)(ab + bc + ca) [DU-75]

Solution: L.H.S = $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ [Subtracting 2nd column from 1st column and 3rd column from the 2nd column.]

$$= \begin{vmatrix} 0 & 0 & 1 \\ a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$= 1\begin{vmatrix} a^2-b^2 & b^2-c^2 \\ a^3-b^3 & b^3-c^3 \end{vmatrix}$$

$$= \begin{vmatrix} (a+b)(a-b) & (b+c)(b-c) \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} (a+b) & (b+c) \\ (a^2+ab+b^2) & (b^2+bc+c^2) \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} a+b & c-a \\ a^2+ab+b^2 & c^2-a^2+bc-ab \end{vmatrix}$$ [Subtracting 1st column from 2nd column]

$$= (a-b)(b-c)\begin{vmatrix} a+b & c-a \\ a^2+ab+b^2 & (c-a)(a+b+c) \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)\begin{vmatrix} a+b & 1 \\ a^2+ab+b^2 & (a+b+c) \end{vmatrix}$$

= (a – b)(b – c)(c – a)(a^2 + ab + ac + ab + b^2 + bc – a^2 – ab –b^2)
= (a – b)(b – c)(c – a)(ab + bc + ca) = R.H.S (Proved)

Example (4): Solve the following system of linear equations with the help of determinant (Cramer's rule)

$$x + 2y - z = 9$$
$$2x - y + 3z = -2 \qquad \text{[RU-78, AUB-01]}$$
$$3x + 2y + 3z = 9$$

Solution: Forming determinant with coefficients of x, y and z, we get

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & 2 & 3 \end{vmatrix} = 1(-3-6) - 2(6-9) + (-1)(4+3) = -9 + 6 - 7 = -10$$

Forming determinant with constant terms and the coefficients of y and z, we get

$$D_x = \begin{vmatrix} 9 & 2 & -1 \\ -2 & -1 & 3 \\ 9 & 2 & 3 \end{vmatrix} = 9(-3-6) - 2(-6-27) - 1(-4+9) = -81 + 66 - 5 = -20$$

Forming determinant with coefficients of x, constant terms and coefficients of z, we get

$$D_y = \begin{vmatrix} 1 & 9 & -1 \\ 2 & -2 & 3 \\ 3 & 9 & 3 \end{vmatrix} = 1(-6-27) - 9(6-9) + (-1)(18+6) = -33 + 27 - 24 = -30$$

Forming determinant with coefficients of x, y and constant terms, we get

$$D_z = \begin{vmatrix} 1 & 2 & 9 \\ 2 & -1 & -2 \\ 3 & 2 & 9 \end{vmatrix} = 1(-9+4) - 2(18+6) + 9(4+3) = -5 - 48 + 63 = 10$$

So, $\quad x = \dfrac{D_x}{D} = \dfrac{-20}{-10} = 2,$

$\qquad y = \dfrac{D_y}{D} = \dfrac{-30}{-10} = 3,$

And $\quad z = \dfrac{D_z}{D} = \dfrac{10}{-10} = -1.$

Therefore, the required solution (x, y, z) = (2, 3, -1)

Example (5): Let $A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$. Show that $A^2 - 3A + 2I = 0$, where I is the unit matrix of order 2×2 and 0 is the null matrix of order 2×2.

Solution: Given that $\quad A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$

So, $A^2 = A.A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}.\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3.3-2.1 & 3.2+2.0 \\ -1.3-0.1 & -1.2+0.0 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ -3 & -2 \end{pmatrix}$,

$3A = 3.\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3.3 & 3.2 \\ 3.-1 & 3.0 \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ -3 & 0 \end{pmatrix}$

And $2I = 2.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

L.H.S $= A^2 - 3A + 2I$

$= \begin{pmatrix} 7 & 6 \\ -3 & -2 \end{pmatrix} - \begin{pmatrix} 9 & 6 \\ -3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 7-9+2 & 6-6+0 \\ -3+3+0 & -2-0+2 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ = R.H.S (Proved)

Example (6): If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ then prove that AB = – BA and $A^2 = B^2 = I$ [RU-88]

Solution: Given that $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Proof of AB = – BA:

L.H.S = $AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0.0+1.i & 0.-i+1.0 \\ 1.0+0.i & 1.-i+0.0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

R.H.S = $-BA = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0.0-i.1 & 0.1-i.0 \\ i.0+0.1 & i.1+0.0 \end{pmatrix} = -\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

So, AB = – BA (Proved)

Proof of $A^2 = B^2 = I$:

$A^2 = A.A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0.0+1.1 & 0.1+1.0 \\ 1.0+0.1 & 1.1+0.0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$B^2 = B.B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0.0-i.i & 0.-i-i.0 \\ i.0+0.i & -i.i+0.0 \end{pmatrix} = \begin{pmatrix} -i^2 & 0 \\ 0 & -i^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

So, $A^2 = B^2 = I$ (Proved)

Example (7): Find the inverse of the matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{pmatrix}$ [CU-88]

Solution: Let D be the determinant of the matrix, then

$$D = |A| = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{vmatrix} = 2(0+3) + 1(8+3) + 3(12-0) = 6 + 11 + 36 = 53 \neq 0.$$

So the matrix A is non-singular and A^{-1} exists. Now the co-factors of D are

$$A_{11} = \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3,\ A_{12} = (-1)\begin{vmatrix} 4 & -1 \\ 3 & 2 \end{vmatrix} = -11,\ A_{13} = \begin{vmatrix} 4 & 0 \\ 3 & 3 \end{vmatrix} = 12,\ A_{21} = (-1)\begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix} = 11,$$

$$A_{22} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5,\ A_{23} = (-1)\begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix} = -9,\ A_{31} = \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} = 1,\ A_{32} = (-1)\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} = 14,$$

$$A_{33} = \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} = 4. \text{ So, Adj A} = \begin{pmatrix} 3 & -11 & 12 \\ 11 & -5 & -9 \\ 1 & 14 & 4 \end{pmatrix}^T = \begin{pmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{pmatrix}$$

$$\text{Therefore, } A^{-1} = \frac{1}{D} Adj\ A = \frac{1}{53}\begin{pmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{pmatrix} = \begin{pmatrix} 3/53 & 11/53 & 1/53 \\ -11/53 & -5/53 & 14/53 \\ 12/53 & -9/53 & 4/53 \end{pmatrix}$$

Example (8): Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -4 \end{pmatrix}$ by using row canonical form. [JU-93]

Solution: To find the inverse matrix of matrix A^{-1} by using row canonical form, let us consider the following augmented matrix:

$$(AI_3) = \left\langle \begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & -1 & 4 & 0 & 1 & 0 \\ 4 & 3 & -4 & 0 & 0 & 1 \end{array} \right\rangle \quad \begin{array}{l} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 4r_1 \end{array}$$

Subtracting two times of 1st row from 2nd row and four times of 1st row from 3rd row, we get

$$\approx \left\langle \begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -5 & 10 & -2 & 1 & 0 \\ 0 & -5 & 8 & -4 & 0 & 1 \end{array} \right\rangle \quad r_3'' = r_3' - r_2'$$

Subtracting 2nd row from 3rd row, we get

$$\approx \left\langle \begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -5 & 10 & -2 & 1 & 0 \\ 0 & 0 & -2 & -2 & -1 & 1 \end{array} \right\rangle \quad \begin{array}{l} r_2^{///} = r_2^{//}/-5 \\ r_3^{///} = r_3^{//}/-2 \end{array}$$

Dividing 2nd row by –5 and 3rd row by –2, we get

$$\approx \left\langle \begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2/5 & -1/5 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 \end{array} \right\rangle \quad \begin{array}{l} r_1^{iv} = r_1^{///} - 2r_2^{///} \\ r_2^{iv} = r_2^{///} + 2r_3^{///} \end{array}$$

Adding 2 times of 3rd row and 2nd row and subtracting 2 times of 2nd row from 1st row, we get

$$\approx \left\langle \begin{array}{ccc|ccc} 1 & 0 & 1 & 1/5 & 2/5 & 0 \\ 0 & 1 & 0 & 12/5 & 4/5 & -1 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 \end{array} \right\rangle \quad r_1^{v} = r_1^{iv} - r_3^{iv}$$

Subtracting 3rd row from 1st row, we get

$$\approx \left\langle \begin{array}{ccc|ccc} 1 & 0 & 0 & -4/5 & -1/10 & 1/2 \\ 0 & 1 & 0 & 12/5 & 4/5 & -1 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 \end{array} \right\rangle$$

Therefore, the inverse matrix, $A^{-1} = \begin{pmatrix} -4/5 & -1/10 & 1/2 \\ 12/5 & 4/5 & -1 \\ 1 & 1/2 & -1/2 \end{pmatrix}$ (Answer)

<u>Example (9)</u>: Let $A = \begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$, then show that $(AB)^{-1} = B^{-1}.A^{-1}$. [AUB-99]

Solution: Given that $A = \begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

So, $AB = \begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8+0 & 0+0 \\ 4+1 & 0+2 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 5 & 2 \end{pmatrix}$

The determinant formed by the matrix AB is $|AB| = \begin{vmatrix} 8 & 0 \\ 5 & 2 \end{vmatrix} = 16 - 0 = 16$.

The co-factors of the determinant $|AB|$ are: $A_{11} = (-)^2|2| = 2$, $A_{12} = (-)^3|5| = -5$,

$A_{21} = (-)^3|0| = 0$, $A_{22} = (-)^4|8| = 8$. So, Adj AB $= \begin{pmatrix} 2 & -5 \\ 0 & 8 \end{pmatrix}^T = \begin{pmatrix} 2 & 0 \\ -5 & 8 \end{pmatrix}$

Therefore, the inverse matrix, $(AB)^{-1} = \dfrac{1}{16}\begin{pmatrix} 2 & 0 \\ -5 & 8 \end{pmatrix} = \begin{pmatrix} 1/8 & 0 \\ -5/16 & 1/2 \end{pmatrix}$

Now, the determinant formed by the matrix A is $|A| = \begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} = (4 - 0) = 4.$

The co-factors of the determinant $|A|$ are: $A_{11} = (-)^2|1| = 1$, $A_{12} = (-)^3|2| = -2$,

$A_{21} = (-)^3|0| = 0$, $A_{22} = (-)^4|4| = 4$. So, Adj A = $\begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$

Therefore, the inverse matrix, $A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ -1/2 & 1 \end{pmatrix}$

Again, the determinant formed by the matrix B is $|B| = \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = (4 - 0) = 4.$

The co-factors of the determinant $|B|$ are: $A_{11} = (-)^2|2| = 2$, $A_{12} = (-)^3|1| = -1$,

$A_{21} = (-)^3|0| = 0$, $A_{22} = (-)^4|2| = 2$. So, Adj B = $\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}^T = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$

Therefore, the inverse matrix, $B^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/4 & 1/2 \end{pmatrix}$

So, $A^{-1}.B^{-1} = \begin{pmatrix} 1/2 & 0 \\ -1/4 & 1/2 \end{pmatrix} . \begin{pmatrix} 1/4 & 0 \\ -1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1/4 . 1/2 + 0 & 0 + 0 \\ -1/4 . 1/4 - 1/2 . 1/2 & 0 + 1. 1/2 \end{pmatrix} = \begin{pmatrix} 1/8 & 0 \\ -5/16 & 1/2 \end{pmatrix}$

Thus, $(AB)^{-1} = B^{-1}.A^{-1}$ (Proved)

<u>Example (10)</u>: Solve the following system with the help of matrix:

x + y + z = 6

5x – y + 2z = 9 [RU-91]

3x + 6y – 5z = 0

Solution: Expressing the system by matrix, we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & -1 & 2 \\ 3 & 6 & -5 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 0 \end{pmatrix}$$

Or, A.X = B where A = $\begin{pmatrix} 1 & 1 & 1 \\ 5 & -1 & 2 \\ 3 & 6 & -5 \end{pmatrix}$, X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and B = $\begin{pmatrix} 6 \\ 9 \\ 0 \end{pmatrix}$

Or, $X = A^{-1}.B$

Let D be the determinant of the matrix A, then

$$D = |A| = \begin{vmatrix} 1 & 1 & 1 \\ 5 & -1 & 2 \\ 3 & 6 & -5 \end{vmatrix} = 1(5-12) - 1(-25-6) + 1(30+3) = -7 + 31 + 33 = 57 \neq 0.$$

So the matrix A is non-singular and A^{-1} exists. Now the co-factors of D are

$A_{11} = \begin{vmatrix} -1 & 2 \\ 6 & -5 \end{vmatrix} = -7$, $A_{12} = (-1)\begin{vmatrix} 5 & 2 \\ 3 & -5 \end{vmatrix} = 31$, $A_{13} = \begin{vmatrix} 5 & -1 \\ 3 & 6 \end{vmatrix} = 33$, $A_{21} = (-1)\begin{vmatrix} 1 & 1 \\ 6 & -5 \end{vmatrix} = 11$,

$A_{22} = \begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} = -8$, $A_{23} = (-1)\begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} = -3$, $A_{31} = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3$, $A_{32} = (-1)\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} = 3$,

$A_{33} = \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} = -6$. So, Adj A = $\begin{pmatrix} -7 & 31 & 33 \\ 11 & -8 & -3 \\ 3 & 3 & -6 \end{pmatrix}^T = \begin{pmatrix} -7 & 11 & 3 \\ 31 & -8 & 3 \\ 33 & -3 & -6 \end{pmatrix}$

Therefore, $A^{-1} = \frac{1}{D} Adj\ A = \frac{1}{57}\begin{pmatrix} -7 & 11 & 3 \\ 31 & -8 & 3 \\ 33 & -3 & -6 \end{pmatrix}$

Since, $X = A^{-1}.B$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{57}\begin{pmatrix} -7 & 11 & 3 \\ 31 & -8 & 3 \\ 33 & -3 & -6 \end{pmatrix}.\begin{pmatrix} 6 \\ 9 \\ 0 \end{pmatrix} = \frac{1}{57}\begin{pmatrix} -42+99+0 \\ 186-72+0 \\ 198-27+0 \end{pmatrix} = \frac{1}{57}\begin{pmatrix} 57 \\ 114 \\ 171 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

So, x = 1, y = 2 and z = 3 (Answer)

<u>Example (11)</u>: A manufacturer produces three products P, Q and R, which he sells in two markets. Annual sales volumes are indicated as follows:

$$\text{Markets: } \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} \text{Products:} \\ \begin{matrix} P & Q & R \end{matrix} \\ \begin{pmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{pmatrix} \end{matrix}$$

[DU-87, AUB-03]

If unit sale prices of P, Q and R are Tk.2.50, Tk.1.25 and Tk.1.50 respectively, find the total revenue in each market with the help of matrix algebra. And if the unit costs of the above three commodities are Tk.1.80, Tk.1.20 and Tk.0.80, find the gross profit.

Solution: Given that,

$$\text{Markets: } \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} \text{Products:} \\ \begin{matrix} P & Q & R \end{matrix} \\ \begin{pmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{pmatrix} \end{matrix}$$

Let sales volume matrix, $S = \begin{pmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{pmatrix}$, Prices (per unit) matrix, $Pr = \begin{pmatrix} 2.50 \\ 1.25 \\ 1.50 \end{pmatrix}$ and Cost (per unit) matrix, $C = \begin{pmatrix} 1.80 \\ 1.20 \\ 0.80 \end{pmatrix}$

We know, Total revenue = Sales volume (S) $\times$ Price per unit (Pr)

$$\text{So, Total revenue} = \begin{pmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{pmatrix} \times \begin{pmatrix} 2.50 \\ 1.25 \\ 1.50 \end{pmatrix}$$

$$= \begin{pmatrix} 10000 \times 2.50 & 2000 \times 1.25 & 18000 \times 1.50 \\ 6000 \times 2.50 & 20000 \times 1.25 & 8000 \times 1.50 \end{pmatrix}$$

$$= \begin{pmatrix} 25000 & 2500 & 27000 \\ 15000 & 25000 & 12000 \end{pmatrix}$$

Hence, total revenue in first market = Tk.(25000 + 2500 + 27000) = Tk.54500,
Total revenue in second market = Tk.(15000 + 25000 + 12000) = Tk.52000
And total revenue in both markets = Tk.(54500 + 52000) = Tk.106500 (Answer)

It is known that, Total cost = Sales volume (S) $\times$ Cost per unit (C)

$$\text{So, Total cost} = \begin{pmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{pmatrix} \times \begin{pmatrix} 1.80 \\ 1.20 \\ 0.80 \end{pmatrix}$$

$$= \begin{pmatrix} 10000 \times 1.80 & 2000 \times 1.20 & 18000 \times 0.80 \\ 6000 \times 1.80 & 20000 \times 1.20 & 8000 \times 0.80 \end{pmatrix}$$

$$= \begin{pmatrix} 18000 & 2400 & 14400 \\ 10800 & 24000 & 6400 \end{pmatrix}$$

We also know that, Total profit = Total revenue – Total cost

$$\text{So, Total profit} = \begin{pmatrix} 25000 & 2500 & 27000 \\ 15000 & 25000 & 12000 \end{pmatrix} - \begin{pmatrix} 18000 & 2400 & 14400 \\ 10800 & 24000 & 6400 \end{pmatrix}$$

$$= \begin{pmatrix} 25000 - 18000 & 2500 - 2400 & 27000 - 14400 \\ 15000 - 10800 & 25000 - 24000 & 12000 - 6400 \end{pmatrix}$$

$$= \begin{pmatrix} 7000 & 100 & 12600 \\ 4200 & 1000 & 5600 \end{pmatrix}$$

Hence, total profit in first market = Tk.(7000 + 100 + 12600) = Tk.19700,
Total profit in second market = Tk.(4200 + 1000 + 5600) = Tk.10800
And total profit in both markets = Tk.(19700 + 10800) = Tk.30500 (Answer)

5.15 Exercise:

1. Define determinant with examples.
2. What is difference between minor and co-factor?
3. What is difference between matrix and determinant?
4. State and prove Cramer's rule for solving a system of linear equations.
5. Define matrix with examples and discuss the uses of matrix.
6. Define with examples (i) Square matrix, (ii) Singular matrix, (iii), Inverse matrix, (iv) Augmented matrix, (v) Echelon matrix, (vi) Hermitian matrix, (vii) Skew-Hermitian matrix.
7. Discuss the difference between adjoint and transpose matrices.
8. Evaluate the following determinants:

   (i) $\begin{vmatrix} 1 & 0 & 3 \\ 6 & 5 & 0 \\ 21 & 3 & 7 \end{vmatrix}$ [Answer: -226] (ii) $\begin{vmatrix} 1 & 2 & -1 & 2 \\ 3 & 0 & 1 & 5 \\ 1 & -2 & 0 & 3 \\ -2 & -4 & 1 & 6 \end{vmatrix}$ [Answer: 90] [AUB-02]

9. Using Sarrus diagram find the value of

   (i) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 2 & 3 & 7 \end{vmatrix}$ [Answer: - 42] (ii) $\begin{vmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{vmatrix}$ [Answer: 4]

10. If A = $\begin{vmatrix} 4 & 2 \\ 3 & 6 \end{vmatrix}$ and B = $\begin{vmatrix} 3 & 7 \\ 2 & 8 \end{vmatrix}$ then find $A \times B$.

    And also show that (value of A)(value of B) = value of $A \times B$.

11. Show that $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0$ [DU-79]

12. Solve the following system of linear equations with the help of determinant (Cramer's rule)

    (i) $2x + 5y = 24$
    $3x + 8y = 38$
    [Answer: (2, 4)]

    (ii) $x - 3y - 8z = -10$
    $3x + y - 4z = 0$
    $2x + 5y + 6z = 13$
    [Answer: (-10, 3, 0)]

    (iii) $-x + 3y + 2z = 24$
    $x + z = 6$
    $5y - z = 8$
    [Answer: (-1, 3, 7)]

    (iv) $3x_1 + x_2 + 2x_3 = 3$
    $2x_1 - 3x_2 - x_3 = -3$
    $x_1 + 2x_2 + x_3 = 4$
    [Answer: (1, 2, -1)]

    (v) $x + 2y + 3z = 14$
    $2x + 3y + 4z = 20$
    $3x + 4y + 6z = 33$
    [Answer: (5, - 6, 7)]

    (vi) $x - 2y + 3z = 11$
    $2x + y + 2z = 10$
    $3x + 2y + z = 9$
    [Answer: (2, 0, 3)]

13. If A = $\begin{pmatrix} 1 & -2 & 3 \\ 5 & 1 & -4 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & -2 \end{pmatrix}$ then find the matrices 2A, A + B and A – B. [Answer: $\begin{pmatrix} 2 & -4 & 6 \\ 10 & 2 & -8 \end{pmatrix}$, $\begin{pmatrix} 3 & 1 & 8 \\ 6 & 5 & -6 \end{pmatrix}$ and $\begin{pmatrix} -1 & -5 & -2 \\ 4 & -3 & -2 \end{pmatrix}$]

14. Write a square matrix and then show that $A + A^T$ is a symmetric matrix.

15. If A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then show that $A\times B = B\times A$.

16. Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & 4 & 9 \\ 6 & 5 & 1 \\ 9 & 7 & 3 \end{pmatrix}$, then prove that $A\times B \neq B\times A$.

17. If A = $\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \end{pmatrix}$, B = $\begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ and C = $\begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 2 & 4 \end{pmatrix}$ then show that $(A\times B)\times C = A\times (B\times C)$.

18. Find the inverse of the matrix (i) A = $\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ (ii) A = $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{pmatrix}$.

[Answer: (i) $\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$, (ii) $\frac{1}{3}\begin{pmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{pmatrix}$, (iii) $\frac{1}{7}\begin{pmatrix} -6 & -1 & 5 \\ 2 & 5 & -4 \\ 3 & -3 & 1 \end{pmatrix}$]

19. Find the inverse of the matrix A = $\begin{pmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{pmatrix}$ by row canonical form.

[Answer: $\begin{pmatrix} 3/53 & 11/53 & 1/53 \\ -11/53 & -5/53 & 14/53 \\ 12/53 & -9/53 & 4/53 \end{pmatrix}$]

20. If A = $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ then prove that $A^3 + A^2 - 21A - 54I = 0$, where I is the unit matrix of order 3×3 and 0 is the null matrix of order 3×3.

21. If A = $\begin{pmatrix} 3 & 6 \\ 2 & 1 \end{pmatrix}$ then show that $\left(A^{-1}\right)^{-1} = A$.

22. Let $A = \begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ & 3 \end{pmatrix}$, then show that $(AB)^{-1} = B^{-1}.A^{-1}$.

23. Solve the following systems of linear equations with the help of matrices:

(i) $5x - 6y + 4z = 15$
$7x + 4y - 3z = 19$
$2x + y + 6z = 46$
[Answer: (3, 4, 6)]

(ii) $x + 2y + 3z = 14$
$2x + 3y + 4z = 20$
$3x + 4y + 6z = 33$
[Answer: (5, - 6, 7)]

(iii) $2x + 3y - z = 1$
$3x + 5y + 2z = 8$
$x - 2y - 3z = -1$
[Answer: (3, – 1, 2)]

24. Solve the following system of linear equations by using row equivalent canonical matrix (by elementary row transformations):

(i) $2x + 3y - z = 1$
$3x + 5y + 2z = 8$
$x - 2y - 3z = -1$
[Answer: (3, – 1, 2)]

(ii) $2x - y + z = 1$
$x + 4y - 3z = -2$
$3x + 2y - z = 0$
[Answer: (0, 1, 2)]

(iii) $2x + 3y - 2z = 5$
$x - 2y + 3z = 2$
$4x - y + 4z = 1$
[Answer: No solution]

25. A manufacturer produces three products P, Q and R, which he sells in two markets. Annual sales volumes are indicated as follows:

Products:

$$\text{Markets: } \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} \quad P \quad\quad Q \quad\quad R \\ \begin{pmatrix} 4000 & 3000 & 2000 \\ 3000 & 2000 & 1000 \end{pmatrix} \end{matrix}$$

[DU-87, AUB-03]

If unit sale prices of P, Q and R are Tk.2.50, Tk.2.00 and Tk.1.50 respectively, find the total revenue in both markets with the help of matrix algebra. And if the unit costs of the above three commodities are Tk.2.00, Tk.1.50 and Tk.1.00, find the gross profit. [Answer: Tk.32000 and Tk.7500]

26. A televisions manufacturer produces three types of TVs A, B and C, which he sells in two markets. Annual sales volumes are indicated as follows:

Products:

$$\text{Markets: } \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} \quad A \quad\quad B \quad\quad C \\ \begin{pmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{pmatrix} \end{matrix}$$

[RU-88]

If unit sale prices of A, B and C types of TVs are Tk.1500, Tk.3000 and Tk.4000 respectively, find the total revenue in both markets with the help of matrix algebra. And if the unit costs of the above three TVs are Tk.1000, Tk.2000 and Tk.3000, find the gross profit. [Answer: Tk.37,50,000 and Tk.11,50,000]

27. The daily cost of operating a hospital(C) is a linear function of the number of inpatients(I) and outpatients(O) plus a fixed cost(x), i.e., $C = x + yO + zI$. Given the following information from 3 days, find the values of x, y and z by setting up a linear system of equations and using the matrix inverse.

Day	Cost(C) in Tk.	No. of inpatients(I)	No. of outpatients(O)
1	7000	50	10
2	6500	40	10
3	7500	50	15

[Answer: x = 3500, y = 100 and z = 50]

28. A special food for athletes is to be developed from two foods: Food X and Food Y. The new food is to be designed so that it contains exactly 16 ounces of vitamin A, exactly 44 ounces of vitamin B and exactly 12 ounces of vitamin C. Each pound of Food X contains 1 ounce of vitamin A, 5 ounces of vitamin B and 1 ounce of vitamin C. On the other hand, each pound of Food Y contains 2 ounces of A, 1 ounce of B and 1 ounce of C. Find the number of pounds of each food to be used in the mixture in order to meet the above requirements. [Answer: 8 pounds of Food X and 4 pounds of Food Y.]

06

Chapter

Functions and Equations

Highlights:

6.1 Introduction: A function can be viewed as an input-output device. The significant relationships in mathematical models typically are represented by functions. It is the purpose of this chapter to introduce this important topic. In business applications, we sometimes are interested in determining whether there are values of variables, which satisfy several attributes. In this chapter, we will also be concerned with the process used to determine whether there are values of variables, which jointly satisfy a set of equations.

6.2 Formula: Computational procedures are described efficiently by formulas employing the symbolism of algebra. Thus, if x is the length and y the width of a rectangle, the area A of the rectangle is expressed by the formula:

$$A = xy$$

If a rectangle has a length of 10 inches and a width of 5 inches, then we compute the area of that rectangle using the above formula, as follows:

Area, A = (10) (5) square inches
= 50 square inches.

6.3 <u>Relation</u>: If A and B be two sets then non empty subset of ordered pairs of Cartesian product, $A \times B$ is called relation of A and B and is denoted by R. If we consider $x \in A$ and $y \in B$ then we get $(x, y) \in R$.

<u>Example (i)</u>: If A = {1, 2, 3} and B = {3, 7} then $A \times B$ = {(1, 3), (1, 7), (2, 3), (2, 7), (3, 3), (3, 7)}. So, the relation x < y where $x \in A$ and $y \in B$ is R = {(1, 3), (1, 7), (2, 3), (2, 7), (3, 7)}.

<u>Example (ii)</u>: If A = {$2, $7, $8} is a set of cost of per unit product and B = {$5, $8} is the set of selling price of per unit product of a production firm. Find the profitable relation between cost and selling price.
Solution: Here, $A \times B$ = {($2, $5), ($2, $8), ($7, $5), ($7, $8), ($8, $5), ($8, $8)}
A firm becomes profitable if its selling price of per unit product is greater than the cost of per unit product. So, the profitable relation, R = {($2, $5), ($2, $8), ($7, $8)}. (Answer)

6.4 <u>Function</u>: If 'f ' is a rule which associates every element of set X with one and only one element of set Y, then the rule 'f ' is said to be the function or mapping from the set X to the set Y. This we write symbolically as

$$f : X \rightarrow Y$$

If y is the element of Y, that corresponds to an element x of X, given by the rule f, we write this as follows:

$y = f(x)$; Here x is independent variable and y is dependent variable.

The set X is known as 'Domain' and the set Y is known as 'Co-domain'. The set formed with those elements of the set Y that are in correspondence with at least one element of X is called the range.

Example: $f : R \rightarrow R;\ f(x) = x^2$
Here, the set of real numbers, R are simultaneously domain and co domain and the set of positive real number, R_+ is the range.

6.5 <u>Types of functions</u>: We shall now introduce some different types of functions, which are particularly useful in different branches of Mathematics.

1. One-one (1-1) Function: If the function f corresponds to the different elements of the set Y for the different elements of set X, then the function is known as one-one (1-1) function.
Example: $f(x) = 2x + 1$

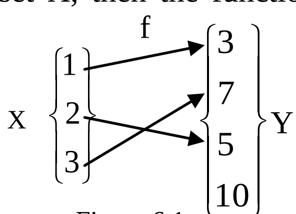


Figure 6.1

2. Onto function: If every element of the co-domain set Y is a correspondence (image) of at least one element of the domain set X, then the function is called onto function.

Example: f: R → R_+ : f(x) = x^2

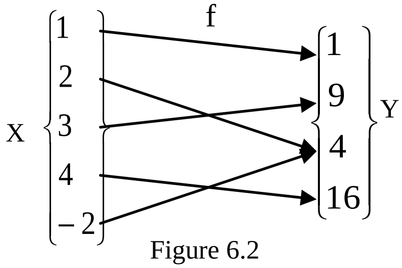


Figure 6.2

3. One-one and onto function: If the function f satisfies the properties of one-one function and onto function then it is known as one-one and into function.

Example: f: X → Y : f(x) = x + 1

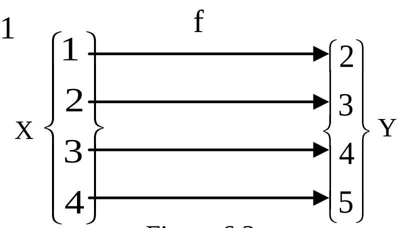


Figure 6.3

4. One valued function: When a function has only one value corresponding to each value of the independent variable, the function is called a one valued function.

Example: If f(x) = x^3, f(x) is a one or single valued function.

5. Many valued function: When a function has several values corresponding to each value of the independent variable, it is called many-valued function or multiple valued function.

Example: If y = f(x) = $\pm\sqrt{x}$, y is a many valued function of x.

6. Explicit function: A function expressed directly in terms of the dependent variable is said to be an explicit function.

Example: y = f(x) = $x^2 + 5x - 4$.

7. Implicit function: The function which is not expressed directly in terms of the dependent variable, there is a mutual relationship between the dependent and the independent variables.

Example: $x^2 + y^2 = 10$ is an explicit function because

$$y = \pm\sqrt{10 - x^2}.$$

8. Algebraic function: When the relation, which involves only a finite number of terms and the variables, are affected only by the operations of addition, subtraction, multiplication, division, powers and roots, the relation is said to be an algebraic function.

Example: $y = f(x) = 2x^3 + 3x^2 - 9$ is an algebraic function.

9. Transcendental function: All the functions of x, which are not algebraic, are called transcendental functions. Thus, $f(x) = e^x + 2x + 1$ is a transcendental function.
We have the following subclasses of transcendental functions:
i) Exponential function: $f(x) = e^{x+1}$
ii) Logarithmic function: $f(x, y) = \log(x + y)$
iii) Trigonometric function: $f(x) = \sin x$
iv) Inverse trigonometric function: $f(x) = \sin^{-1}x$

10. Rational function: Expressions involving x, which consist of a finite number of terms of the form ax^n, in which 'a' is a constant and 'n' a positive integer is called a rational function of x.

Example: $y = f(x) = 4x^4 + 9x - 7$ and $y = f(x) = \frac{x^2 + 6}{3x^3 + 2}$ are rational functions.

11. Irrational function: An expression involving x, which involves root extraction of terms, is called an irrational function.

Example: $y = f(x) = \sqrt{x^2 + 4x + 10} + 9x + 2$ is an irrational functional function.

12. Monotone Function: When the dependent variable increases with an increase in the independent variable, the function is called a monotonically increasing function. And when the dependent variable decreases with an increase in the independent variable, the function is called a monotonically decreasing function.

Example: $y = f(x) = 2x$ is a monotonically increasing function

$y = f(x) = \frac{1}{2x}$ is a monotonically decreasing function

13. Even function: If a function f (x) is such that f (-x) = f(x), then it is called an even function of x.

Example: $y = f(x) = 2x^2$ is an even function.

14. Odd function: If a function f (x) is such that f (-x) = -f (x), then, it is said to be an odd function of x.

Example: $y = f(x) = 2x$ is an odd function of x.

15. Periodic Function: If f (x) = f (x + p) for all value of x, then f (x) is called a periodic function with period p.

Example: $y = f(x) = \sin x$ is a periodic function with period 2π.

16. Linear function: The relationship between y and x expressed by

y = ax + b; a and b are constants.

is called a functional relationship because for each value of x, there is one, and only one corresponding value for y. Notice that the expression states y is in terms of x and so we say y is a function of x. This type of function is known as linear function because it represents straight line in the graph. Here, x is called independent variable and y is called dependent variable because, the value of y depends upon what value we assign to x.
Example: y = f(x) = 3x + 4 is a linear equation.

Note: In coordinate geometry a is called slope and b is called y-intercept of the straight line that is represented by y = ax + b.
Example: The equation of the line that has a slope of 3.2 and y-intercept of 5 is y =3.2x +5

6.6 <u>Polynomial</u>: If n is a positive integer, the expression $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n,\ a_0 \neq 0$ is called a polynomial of degree n. As for example $f(x) = 5x^2 + 3x + 1$ and $f(x) = 2x^4 - 5x + 10$ are polynomial of degree 2 and 4 respectively. Every binomial can be expressed as a polynomial by the following formula: $(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_ny^n$.

6.7 <u>Inequality</u>: When two mathematical expressions become connected by the inequality sign (> or < or ≥ or ≤ or ≠) is known as an inequality.
Example: $3x^2 + 4x + x > 2x + 1$
$3x + 1 > 0$, are inequalities.

<u>Example</u>: In the graph, draw the solution space of the inequality 2x + y ≥ 11.
Solution: To draw the graph, firstly we consider the equation 2x + y = 11.
So, y = 11 – 2x - - - (i)
Substituting x = 1, 3 and 5 in equation (i) we get the following chart:

x	1	2	5
y	9	7	1

Plotting the point (1, 9), (2, 7) and (5, 1) in the following graph we get a straight line.

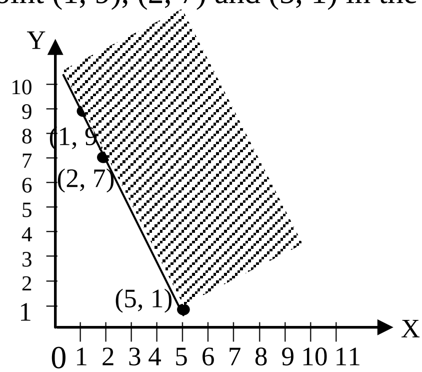


Figure 6.4

The origin (0, 0) lies on the left side of the straight line. If we put (0, 0) in the given inequality we get $0 \geq 11$, which is absurd. So, the points of the opposite side that means all the points of the right side of the line and on the line are the solutions of the given inequality. The shadow region shows the solution space of the given inequality in the graph.

Example: Solve the inequality: $x - 9 > 3x + 1$ and represent it in the number line.

Solution: Given that, $x - 9 > 3x + 1$

Or, $x - 9 + 9 > 3x + 1 + 9$ [Adding 9 to both sides]

Or, $x > 3x + 10$

Or, $x - 3x > 3x + 10 - 3x$ [Subtracting 3x from both sides]

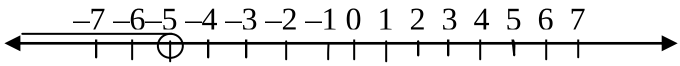
Or, $-2x > 10$

[We know, 5 > 3 but –5 < –3. For this inequality sign (>, <) changes when multiplied or divided by a negative number]

Or, $\frac{-2x}{-2} > \frac{10}{-2}$ [Dividing both sides by – 2]

Or, $x < -5$

Therefore, the required solution: $x < -5$.



Number line

Figure 6.5

6.8 Equation: Equations signify relation between two mathematical (algebraic / trigonometric / transcendental) expressions symbolized by the sign of equality (=). However, the equality is true only for certain value or values of the variable or variables symbolized generally by x, y, z, u, v, w etc. and these values are known as the solution of the equation.

Example:

$3x + 7 = 8x - 3$ [Linear equation]

$9x + 12 = 0$

$2x^2 + 3x + 2 = 0$ [Quadratic equation]

$3e^x + 4\cos x + 5x + 9 = 0$ [Transcendental equation]

6.9 Degree of an equation: The highest power of variable or variables of an equation is called the degree of that equation. Such as

$2x + 5 = x$

$3x^2 + 2y = 2x + 1$

$4x + y^3 = 1$

The highest power of variable or variables in the 1^{st}, 2^{nd} and 3^{rd} equations are respectively 1, 2 and 3. So, they are respectively 1^{st} degree, 2^{nd} degree and 3^{rd} degree equations. Generally, 1^{st} degree equations are known as linear equations. Of one variable, second degree equation, $ax^2 + bx + c = 0$ is known as quadratic equation, third degree equation, $ax^3 + bx^2 + cx + d = 0$ is known as cubic equation and fourth degree equation, $ax^4 + bx^3 + cx^2 + dx + e = 0$ is known as bi-quadratic equation.

6.10 Quadratic equation: The second degree equations of one variable are known as quadratic equations. The standard form of quadratic equation is $ax^2 + bx + c = 0$.
$2x^2 – 5x = 6x + 10$, $x^2 – 8x + 15 = 0$ are quadratic equations.
Every quadratic equation has two roots (solutions). One of the most common solution techniques is as follows:

The solution of standard form $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Here $b^2 - 4ac$ is called discriminant and (1) if $b^2 - 4ac > 0$, there will be two real roots; (2) if $b^2 - 4ac = 0$, there will be one real root; & (3) if $b^2 - 4ac < 0$, there will be no real roots.

Example: Find the roots of the quadratic equation: $2x^2 – 5x = 6x + 10$ [AUB-02]
Solution: Converting the given equation in the standard, we get

$$2x^2 + x – 10 = 0$$

Comparing this equation to the standard form of quadratic equation, we get a = 2, b = 1 and c = –10.

So, the value of $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Or, $x = \dfrac{-1 \pm \sqrt{(1)^2 - 4.2.(-10)}}{2.2}$

Or, $x = \dfrac{-1 \pm 9}{4}$

So, $x = \dfrac{-1+9}{4} = 2$ and $x = \dfrac{-1-9}{4} = -2.5$

Therefore, the roots of the given equation are 2 and – 2.5.

Example: Solve: $\dfrac{x+3}{x+2} + \dfrac{x-3}{x-2} = \dfrac{2x-3}{x-1}$ [RU-92 A/C]

Solution: Given that $\dfrac{x+3}{x+2} + \dfrac{x-3}{x-2} = \dfrac{2x-3}{x-1}$

Or, $\dfrac{(x+3)(x-2) + (x-3)(x+2)}{(x+2)(x-2)} = \dfrac{2x-3}{x-1}$

Or, $\dfrac{x^2 + x - 6 + x^2 - x - 6}{x^2 - 4} = \dfrac{2x-3}{x-1}$

Or, $\frac{2x^2-12}{x^2-4}=\frac{2x-3}{x-1}$

Or, $2x^3-2x^2-12x+12=2x^3-3x^2-8x+12$

Or, $x^2-4x=0$

Or, x(x – 4) = 0

So, x = 0, x – 4 = 0 or x = 4

Therefore, the required solution x = 0 or 4 (Answer)

6.11 Formation of quadratic equation: If given that α and β are two roots of a quadratic equation then we can form the equation as $x^2 - (\alpha + \beta)x + \alpha\beta = 0$,

That is, x^2 – (sum of the roots) x + product of the roots = 0

3 and 5 are two roots of the quadratic equation which is $x^2 - (3 + 5)x + 3.5 = 0$;

Or, $x^2 - 8x + 15 = 0$

Example: If a and b be two roots of $2x^2 - 4x + 1 = 0$, then find the quadratic equation whose roots are $a^2 + b$ and $a + b^2$. [RU-93 A/C]

Solution: Given that a and b are roots of $2x^2 - 4x + 1 = 0$ or $x^2 - 2x + \frac{1}{2} = 0$,

So, a + b = 2 and ab = $\frac{1}{2}$

Now, sum of the roots of the required equation is $\{(a^2 + b) + (a + b^2)\}$

$= (a^2 + b^2) + (a + b)$

$= \{(a + b)^2 - 2ab\} + (a + b)$

$= 2^2 - 2.\frac{1}{2} + 2$

$= 5$

And the product of the roots of the required equation is $(a^2 + b)(a + b^2)$

$= ab + a^2b^2 + a^3 + b^3$

$= ab + (ab)^2 + (a + b)^3 - 3ab(a + b)$

$= \frac{1}{2} + (\frac{1}{2})^2 + 2^3 - 3.\frac{1}{2}.2$

$= \frac{23}{4}$

We know that, the required equation is

x^2 – (sum of the roots) x + product of the roots = 0

$x^2 - 5x + \frac{23}{4} = 0$

$4x^2 - 20x + 23 = 0$ (Answer)

6.12 Identity: When an equation holds true whatever be the value of its variables, it is called an identity.

Example: $(x + 2)^2 = x^2 + 4x + 4.$
$2(x^2 + y) = 2x^2 + 2y$
$x^2 + 2x = x(x + 2)$, are identities

N.B: The following identities are used as formula in various problems. So, students should memorize these identities.

1. $(x + y)^2 = x^2 + 2xy + y^2$
$= (x - y)^2 + 4xy$
2. $(x - y)^2 = x^2 - 2xy + y^2$
$= (x + y)^2 - 4xy$
3. $x^2 - y^2 = (x + y)(x - y)$
4. $x^2 + y^2 = (x + y)^2 - 2xy$
$= (x - y)^2 + 2xy$
5. $2(x^2 + y^2) = (x + y)^2 + (x - y)^2$
6. $4xy = (x + y)^2 - (x - y)^2$
7. $xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$
8. $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$
9. $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
$= x^3 + y^3 + 3xy(x + y)$
10. $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$
$= x^3 - y^3 - 3xy(x - y)$
11. $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
$= (x + y)^3 - 3xy(x + y)$
12. $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
$= (x - y)^3 + 3xy(x - y)$

6.13 Linear equation: $a_1x_1 + a_2x_2 + \ ... \ + a_nx_n = b;$ a_i $(i = 1, 2, 3, ..., n)$, $b \in R$ is a linear equation that means, the equation, which does not contain higher degree or multiple terms of variables, is called linear equation. A linear equation always represents a straight line in the graph.

For example: $3x + 4y + 5z = 1$ is a linear equation. Here x, y, z are variables; 3, 4, 5 are coefficients of x, y & z; 1 is constant term.

6.14 System of linear equations: The m linear equations in the n unknowns- $x_1, x_2, ..., x_n$:

$a_{11}\ x_1 + a_{12}\ x_2 + . . . + a_{1n}\ x_n = b_1$
$a_{12}\ x_1 + a_{22}\ x_2 + . . . + a_{2n}\ x_n = b_2$
. . . .
$a_{m1}\ x_1 + a_{m2}\ x_2 + . . . + a_{mn}\ x_n = b_m$

where a_{ij}, $b_i \in R$; is a system of linear equations or simultaneous linear equation. That is, if more than one linear equation makes a problem together, this problem is called a system of linear equations.

If at least one b_i (1, 2, 3, ..., m) is non - zero, the system is <u>non- homogeneous</u> and if otherwise, the system is <u>homogeneous</u>. That is

$a_{11}\ x_1 + a_{12}\ x_2 + \ldots + a_{1n}\ x_n = 0$

$a_{12}\ x_1 + a_{22}\ x_2 + \ldots + a_{2n}\ x_n = 0$

$\ldots \quad \ldots \quad \ldots$

$a_{m1}\ x_1 + a_{m2}\ x_2 + \ldots + a_{mn}\ x_n = 0$

is a system of homogeneous equations.

A homogeneous system always has a solution namely the zero n-tuple, 0 = (0, 0, . . ., 0) called the zero or trivial solution. Any other solution if it exists, is called a non-zero or non - trivial solution.

A non-homogeneous system may or may not have solution.

Example: $x + 2y - 3z = 6$

$2x - y + 4z = 2$

$4x + 3y - 4z = 14$

is example of non-homogeneous system.

And $x + 2y - 3z = 0$

$2x - y + 4z = 0$

$4x + 3y - 4z = 0$

is example of homogeneous system.

The following chart shows the types solutions of a system of linear equation.

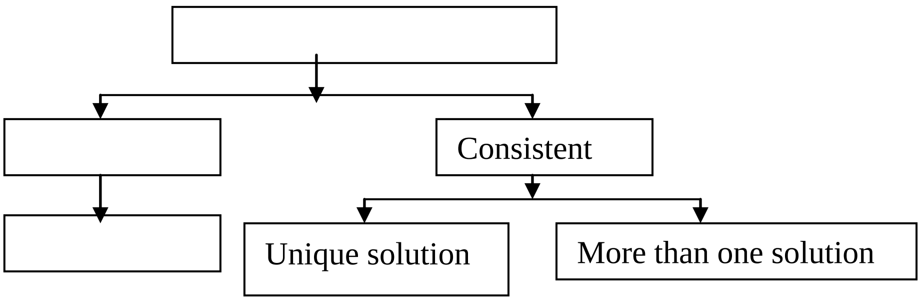


Figure 6.6

6.15 <u>Solution methods of a system of linear equations</u>: There are many solution techniques of a system of linear equations. Such as

1. Substitution method.
2. Gauss elimination method.
3. Cramer's rule (Using determinant)
4. Matrix method: i) Finding inverse matrix
   ii) Making echelon matrix.

1. Substitution method: In this method, finding the value of a variable in terms of other variables and substitute it in other equations. To illustrate this method let us consider the following problem:

$$\begin{cases} x + 2y - 3z = 6 & \quad ... \quad (1) \\ 2x - y + 4z = 2 & \quad ... \quad (2) \\ 4x + 3y - 4z = 14 & \quad ... \quad (3) \end{cases}$$

From equation (1), we have value of x in terms of variables y and z as follows:

$x = 6 - 2y + 3z \quad ... \quad (4)$

Substituting the value of x in equation (2), we get

$2(6 - 2y + 3z) - y + 4z = 2$

Or, $12 - 4y + 6z - y + 4z = 2$

Or, $-5y + 10z = -10$

Or, $y - 2z = 2$ [Dividing by –5]

Or, $y = 2z + 2 \quad ... \quad (5)$

Again substituting the value of x in equation (3), we get

$4(6 - 2y + 3z) + 3y - 4z = 14$

Or, $24 - 8y + 12z + 3y - 4z = 14$

Or, $-5y + 8z = 14 - 24$

Or, $-5(2z + 2) + 8z = -10$ [Substituting $y = 2z + 2$]

Or, $-10z - 10 + 8z = -10$

Or, $-2z = 0$

So, $z = 0$

Substituting $z = 0$ in equation (5), we get $y = 2$

Again substituting $z = 0$ and $y = 2$ in equation (4), we get $x = 2$

Therefore, the solution of the system is $(x, y, z) = (2, 2, 0)$

2. Gauss elimination method: In this method, we always try to eliminate one or more variables from the equations to solve the system. This method will be clear by the following solution.

Let us consider the problem:

$$\begin{cases} x + 2y - 3z = 6 & \quad ... \quad (1) \\ 2x - y + 4z = 2 & \quad ... \quad (2) \\ 4x + 3y - 4z = 14 & \quad ... \quad (3) \end{cases}$$

To eliminate variable x from equation (1) and (2), we multiply equation (1) by 2 and then subtract from equation (2), we get

$-5y + 10z = -10$

Or, $y - 2z = 2 \quad ... \quad (4)$

Again to eliminate variable x from equation (2) and (3), we multiply equation (2) by 2 and then subtract from equation (3), we get

$$5y - 12z = 10 \quad \ldots \quad (5)$$

Again to eliminate variable y from equation (4) and (5), we multiply equation (4) by 5 and then subtract from equation (5), we get

$$-2z = 0$$

So, $z = 0$

And again to eliminate variable z from equation (4) and (5), we multiply equation (4) by 6 and then subtract from equation (5), we get

$$-y = -2$$

So, $y = 2$

Putting the value of y & z in equation (1), we get $x = 2$.

So, the solution is $(x, y, z) = (2, 2, 0)$

3. Cramer's rule: In this method, we use determinant to solve a system of linear equations. To make the method understand we use the same problem as follows:

$$\begin{cases} x + 2y - 3z = 6 & \ldots \ (1) \\ 2x - y + 4z = 2 & \ldots \ (2) \\ 4x + 3y - 4z = 14 & \ldots \ (3) \end{cases}$$

Forming determinant with coefficients of x, y and z, we get

$$D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -4 \end{vmatrix} = 1(4 - 12) - 2(-8 - 16) - 3(6 + 4) = -8 + 48 - 30 = 10$$

Forming determinant with constant terms and the coefficients of y and z, we get

$$D_x = \begin{vmatrix} 6 & 2 & -3 \\ 2 & -1 & 4 \\ 14 & 3 & -4 \end{vmatrix} = 6(4 - 12) - 2(-8 - 56) - 3(6 + 14) = -48 + 128 - 60 = 20$$

Forming determinant with coefficients of x, constant terms and coefficients of z, we get

$$D_y = \begin{vmatrix} 1 & 6 & -3 \\ 2 & 2 & 4 \\ 4 & 14 & -4 \end{vmatrix} = 1(-8 - 56) - 6(-8 - 16) - 3(28 - 8) = -64 + 144 - 60 = 20$$

Forming determinant with coefficients of x, y and constant terms, we get

$$D_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & -1 & 2 \\ 4 & 3 & 14 \end{vmatrix} = 1(-14 - 6) - 2(-28 - 8) + 6(6 + 4) = -20 - 40 + 60 = 0$$

So, $x = \dfrac{D_x}{D} = \dfrac{20}{10} = 2,$

$$y = \frac{D_y}{D} = \frac{20}{10} = 2,$$

And $$z = \frac{D_z}{D} = \frac{0}{10} = 0.$$

Therefore, the solution of the system is $(x, y, z) = (2, 2, 0)$.

4. Matrix method finding inverse matrix: In this method, we use inverse matrix to solve a system of linear equations. Let A be the coefficient matrix, B be the constant term matrix and X be the variable matrix of a system of linear equations, then

$$AX = B$$

So, the solution will be $X = A^{-1}B$.

To illustrate this method, let us consider the following system of linear equations.

$$\begin{cases} x + 2y - 3z = 6 & \quad ...\ (1) \\ 2x - y + 4z = 2 & \quad ...\ (2) \\ 4x + 3y - 4z = 14 & \quad ...\ (3) \end{cases}$$

From the system, we find the following coefficient matrix, constant term matrix and variable matrix.

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -4 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \quad B = \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix}$$

To find the inverse matrix of matrix A, A^{-1} let us consider the following augmented matrix:

$$\left\langle \begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & -1 & 4 & 0 & 1 & 0 \\ 4 & 3 & -4 & 0 & 0 & 1 \end{array} \right\rangle$$

Subtracting two times of 1st row from 2nd row and four times of 1st row from 3rd row, we get

$$\left\langle \begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -5 & 10 & -2 & 1 & 0 \\ 0 & -5 & 8 & -4 & 0 & 1 \end{array} \right\rangle$$

Subtracting 2nd row from 3rd row, we get

$$\left\langle \begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -5 & 10 & -2 & 1 & 0 \\ 0 & 0 & -2 & -2 & -1 & 1 \end{array} \right\rangle$$

Dividing 2nd row by –5 and 3rd row by –2, we get

$$\left\langle\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2/5 & -1/5 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 \end{array}\right\rangle$$

Adding 2 times of 3rd row from 2nd row and subtracting 2 times of 2nd row from 1st row, we get

$$\left\langle\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/5 & 2/5 & 0 \\ 0 & 1 & 0 & 12/5 & 4/5 & -1 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 \end{array}\right\rangle$$

Subtracting 3rd row from 1st row, we get

$$\left\langle\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/5 & -1/10 & 1/2 \\ 0 & 1 & 0 & 12/5 & 4/5 & -1 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 \end{array}\right\rangle$$

So, $A^{-1} = \begin{pmatrix} -4/5 & -1/10 & 1/2 \\ 12/5 & 4/5 & -1 \\ 1 & 1/2 & -1/2 \end{pmatrix}$

Therefore, $X = A^{-1}B$

Or, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4/5 & -1/10 & 1/2 \\ 12/5 & 4/5 & -1 \\ 1 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix}$

Or, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{24}{5} - \frac{1}{5} + 7 \\ \frac{72}{5} + \frac{8}{5} - 14 \\ 6 + 1 - 7 \end{pmatrix}$

Or, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

That is, x = 2, y = 2 and z = 0
So, the solution is (x, y, z) = (2, 2, 0).

5. Matrix method making echelon matrix: In this method, forming the augmented matrix with the coefficients of x, y, z and the constant terms of the system, we try to make it echelon form. Then we check it consistency by the following rules:

If A|B is the row reduced echelon matrix of a system:

iv) If rank of matrix A ≠ rank of augmented matrix A|B, then the system is inconsistent.

v) If rank of matrix A = rank of augmented matrix A|B = number of variables, then the system is consistent and has a unique solution.

vi) If rank of matrix A = rank of augmented matrix A|B < number of variables, then the system is consistent and has many solutions.

To explain the method, let us again consider the system of 3 linear equations with 3 variables x, y and z that we have solved by the previous four methods:

$$\begin{cases} x + 2y - 3z = 6 & \ldots (1) \\ 2x - y + 4z = 2 & \ldots (2) \\ 4x + 3y - 4z = 14 & \ldots (3) \end{cases}$$

We form the following augmented matrix with the coefficients of x, y, z and the constant terms of the system.

$$A|B = \left\langle \begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -4 & 14 \end{array} \right\rangle \qquad \text{[Here, matrix A means } \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -4 \end{pmatrix}\text{]}$$

Now, our goal is to reach the echelon form of the augmented matrix by row reduced technique. Subtracting 2 times of 1st row from 2nd row and 4 times of 1st row from 3rd row, we get

$$\approx \left\langle \begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 8 & -10 \end{array} \right\rangle$$

Again subtracting 2nd row from 3rd row, we get

$$\approx \left\langle \begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & 0 & -2 & 0 \end{array} \right\rangle$$

Now, dividing 2nd row by –5 and 3rd row by –2, we have

$$\approx \left\langle \begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 \end{array} \right\rangle$$, which is the echelon form.

In the echelon form, we see that there are 3 non-zero rows both in matrix A and augmented matrix A|B. So, the rank of A = the rank of A|B = 3 = number of variables. Therefore, the considered system is consistent and has unique solution. To find the solution, we form the following equations by the rows of the echelon matrix:

z = 0

y – 2z = 2 => y = 2

x + 2y –3z = 6 => x = 2

Thus, the solution of the system is (x, y, z) = (2, 2, 0)

Example: Solve the following system of two linear equations and express it by a graph.

2x + y – 11 = 0 [RU-81 A/C]

3x + 5y –27 = 0

Solution: We solve it by substitution method and try to express it by graphical method.

Given that, 2x + y – 11 = 0 --- (1)

3x + 5y –27 = 0 --- (2)

From equation (1) we get, y = 11 – 2x --- (3)

Using equation (3) in equation (2) we get, 3x + 5(11 – 2x) – 27 = 0

Or, 3x + 55 – 10x – 27 = 0

Or, – 7x = – 28

Or, x = 4

Substituting x = 4 in equation (3) we get, y = 11 – 2×4 = 3

So, the required solution: x = 4, y = 3.

Substituting x = 1 and 4 in equation (1) we get y = 9 and 3. That is, from equation (1) we get the following coordinates (1, 9) and (4, 3).

Substituting x = –1 and 4 in equation (2) we get y = 6 and 3. That is, from equation (2) we get the following coordinates (–1, 6) and (4, 3).

Plotting the coordinates (1, 9), (4, 3) and (–1, 6), (4, 3) in the graph we get two straight lines AB and CD respectively. They intersect each other at the point (4, 3). It means the solution of the system: x = 4 and y = 3.

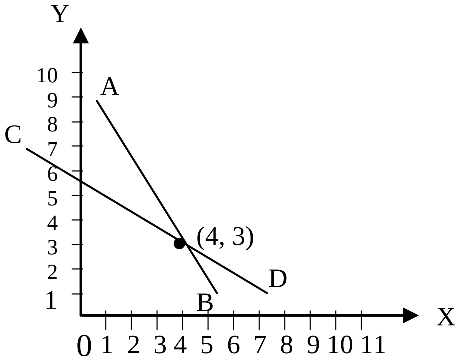


Figure 6.7

Example: Solve the system: $27^x = 9^y$ and 81 = 243(3x) [NU-98 A/C]

Solution: We solve the system by substitution method.

Given that, $27^x = 9^y$ - - - (1)

81 = 243(3x) - - - (2)

From equation (2) we get, 81 = 279x

$$\text{Or, } x = \frac{81}{279}$$

$$\text{So, } x = \frac{1}{9}$$

Now from equation (1) we get, $\left(3^3\right)^x = \left(3^2\right)^y$

$$\text{Or, } 3^{3x} = 3^{2y}$$

Or, 3x = 2y

$$\text{Or, } 3\times\frac{1}{9} = 2y \quad [\text{Substituting the } x = \frac{1}{9}]$$

$$\text{Or, } 2y = \frac{1}{3}$$

$$\text{So, } y = \frac{1}{6}$$

Therefore, the required solution: $x = \frac{1}{9}$ and $y = \frac{1}{6}$ (Answer)

Example: Solve the system: x + y = 5 [RU-95 A/C]

$x^2 + y^2 = 25$

Solution: Given that, x + y = 5 - - - (i)

$x^2 + y^2 = 25$ - - - (ii)

From equation (i) we get, y = 5 – x - - - (iii)

Using equation (iii) in equation (ii) we get, $x^2 + (5 - x)^2 = 25$

Or, $x^2 + 5^2 - 2.5.x + x^2 = 25$

Or, $2x^2 + 25 - 10x - 25 = 0$

Or, $2x^2 - 10x = 0$

Or, 2x(x – 5) = 0

Or, x(x – 5) = 0

So, x = 0, 5

From equation (iii), y = 5 when x = 0

y = 0 when x = 5

Therefore, the solution: (x, y) = (0, 5), (5, 0) (Answer)

Example: An investor plan to invest 10000 taka in the bonds of two companies to receive 820 taka interest. First company pay 7% interest while second company pay 10% interest. How much should be invested in each company? [JU-95]

Solution: Let the investor invest Tk. x in the first company and Tk. y in the second company in order to receive total of Tk. 820 as interest. So, we find the following system of linear equations:

$$\begin{cases} x + y = 10000 \\ x(7\%) + y(10\%) = 820 \end{cases}$$

Or, $$\begin{cases} x + y = 10000 & \ldots \quad (i) \\ 0.07x + 0.1y = 820 & \ldots \quad (ii) \end{cases}$$

Multiplying equation (i) by 0.1 and then subtracting from equation (ii), we get

– 0.03x = –180

So, x = 6000

Substituting x = 6000 in equation (i), we get

6000 + y = 10000

So, y = 4000

Therefore, the investor should invest Tk. 6000 in the first company and Tk. 4000 in the second company.

__Example__**:** (Market equilibrium for two competing products) Given supply and demand functions for a product, market equilibrium exists if there is a price at which the quantity demanded equals the quantity supplied. Suppose that the following demand and supply functions have been estimated for two competing products.

$$\begin{cases} q_{d1} = f_1(p_1, p_2) = 100 - 2p_1 + 3p_2 \\ q_{s1} = h_1(p_1) = 2p_1 - 4 \end{cases}$$

$$\begin{cases} q_{d2} = f_2(p_1, p_2) = 150 + 4p_1 - p_2 \\ q_{s2} = h_2(p_2) = 3p_2 - 6 \end{cases}$$

where q_{d1} = quantity demanded of product 1

q_{s1} = quantity supplied of product 1

q_{d2} = quantity demanded of product 2

q_{s2} = quantity supplied of product 2

p_1 = price per unit of product 1

p_2 = price per unit of product 2

q_1 = equilibrium quantity of product 1

q_2 = equilibrium quantity of product 2

Notice that the quantity demanded of a given product depends not only on the price of the product but also on the price of the competing product. The quantity supplied of a product depends only upon the price of that product.

Market equilibrium would exist in this two-product marketplace if prices were offered such that

$q_{d1} = q_{s1}$

and $q_{d2} = q_{s2}$

Supply and demand are equal for product-1 when

$100 - 2p_1 + 3p_2 = 2p_1 - 4$

Or, $4p_1 - 3p_2 = 104$ . . . (i)

Supply and demand are equal for product-2 when

$150 + 4p_1 - p_2 = 3p_2 - 6$

Or, $-4p_1 + 4p_2 = 156$ . . . (ii)

Solving equations (i) and (ii) simultaneously, we get the equilibrium prices

$p_1 = 221$ and $p_2 = 260$

This result suggests that if the products are priced accordingly, the quantities demanded and supplied will be equal for each product. And the equilibrium quantities will be

$q_1 = 438$ and $q_2 = 774$

Example: Demand and supply equations are (q + 20)(p + 10) = 400 and q = 2p – 7 respectively, where p stands for price and q for quantity. Find the equilibrium price and quantity and show it by a graph. [NU-2000 A/C]

Solution: Given that, Demand: (q + 20)(p + 10) = 400 --- (1)

Supply: q = 2p – 7 --- (2)

Using equation (2) in equation (1), we get

$(2p - 7 + 20)(p + 10) = 400$

Or, $(2p + 13)(p + 10) = 400$

Or, $2p^2 + 33p + 130 - 400 = 0$

Or, $2p^2 + 33p - 270 = 0$

Or, $2p^2 + 45p - 12p - 270 = 0$

Or, $p(2p + 45) - 6(2p + 45) = 0$

Or, $(2p + 45)(p - 6) = 0$

Or, $p = 6$ & $p = \frac{-45}{2}$ [Not acceptable]

So, $p = 6$

Replacing p = 6 in equation (2), we get

$q = 5$

Therefore, the equilibrium price is 6 and quantity is 5. This is shown in following graph:

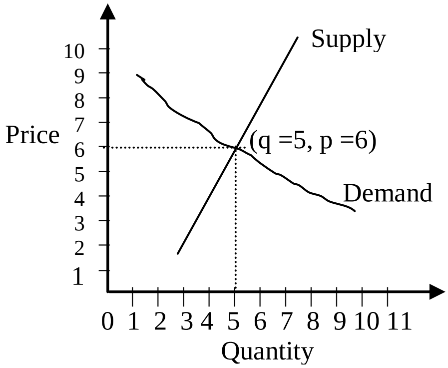


Figure 6.8

6.16 Break-Even point: Of specific concern in break-even analysis is identifying the level of operation or level of output of a business that would result in a zero profit. This level of operations or output is called the break-even point. The break-even point is a useful reference point in the sense that it represents the level of operation at which total revenue equals total cost. Any changes from this level of operation will result in either a profit or a loss. Break-even analysis is valuable particularly as a planning tool when firms are contemplating expansions such as offering new products or services. Similarly, it is useful in evaluating the pros and cons of beginning a new business venture. In each instance the analysis allows for a projection of profitability.

6.17 Break-Even interpretation: In this section we shall consider the case of a manufacturer who produces q units of a product and sells the product at a price of Tk. p per unit. The symbols to be used are:

C = Total cost of producing and selling q units

q = Number of units produced and sold.

v = Variable cost per unit made, assumed to be constant.

F = Fixed cost, a constant

p = Selling price per unit.

R = Total revenue received, which is the same as the volume of sales.

The cost function then is given by,

$C = vq + F \qquad \ldots \text{ (i)}$

And Revenue = (Price per unit) × (Number of units sold)

$R = pq \qquad \ldots \text{ (ii)}$

If the manufacturer is to achieve break-even on operations, neither incurring a loss nor earning a profit, revenue (ii) must be equal to cost (i). That is, at beak-even,

$pq = vq + F \qquad \ldots \text{ (iii)}$

We now may solve (iii) for the production volume, q

$pq - vq = F$

Or, $q(p - v) = F$

$\therefore \quad q = \dfrac{F}{p-v}$, this is the break-even quantity.

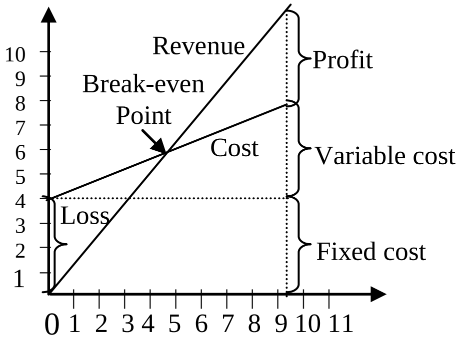


Figure 6.9

<u>Example</u>: A manufacturer of cassette tapes has a fixed cost of Tk. 60,000 and variable cost is Tk. 4 per cassette produced. Selling price is Tk. 7 per cassette.

a) Write the revenue and cost equations.

b) At what number of units will break-even occur?

c) At what sales (revenue) volume will break-even occur?

d) How much profit will be earned if 25000 cassettes are produced? [AUB-02]

Solution: Here, Fixed cost, F = Tk. 60,000,

Selling price per cassette, p = Tk. 7

Variable cost per cassette, v = Tk. 4

Let R means total revenue, C means total cost and q means the number of cassettes produced and sold.

a) The equations are

Revenue: $R = 7q$

Cost: $C = 4q + 60,000$

b) We know that break-even quantity:

$q = \dfrac{F}{p-v}$ [Here, F = Tk. 60000, P = Tk. 7 and V = Tk. 4]

$= \dfrac{60000}{7-4}$

= 20,000 units

c) At break-even 20,000 cassettes would be produced and sold at Tk. 7 each. So, the break-even sales volume would be: R = Tk. (7) (20,000)

= Tk. 1,40,000

d) If 25,000 cassettes produced and sold, then profit = Tk. $\{7\times25000 - (4\times25000 + 60000)\}$

= Tk. 15000

6.17.1 **<u>Another interpretation of break-even</u>:** If cost = \$ c, retail price = \$ p,

Then, mark-up = \$ (p – c) and Margin, M = $\frac{p-c}{p}$

So, cost = (1 – M)p = (1 – $\frac{p-c}{p}$)p = $\left(\frac{p-p+c}{p}\right)p = \frac{c}{p}p$ = c

Generally, we let, fixed expense = \$ F
Now, if total sales of a firm be \$ x, then
Total cost, y = (1 – M)x + F
When break-even occurs, total cost must be equal to the total selling price.
So, from (1), we get.

x = (1 – M)x + F
Or, x – (1 – M)x = F
Or, x – x + Mx = F
Or, Mx = F
Or, x = $\frac{F}{M}$

Thus, break-even occurs at sales, x = $\frac{F}{M}$

<u>Example</u>: Margin is to be 33% of retail price and other variable cost is estimated at \$ 0.13 per dollar of sales. Fixed cost is estimated at \$ 4000.
a) What is the break-event point?
b) Estimate profit of sales for \$ 50000.

Solution: Here, net margin, M = 33% - 0.13
= 0.33 - 0.13
= 0.20

And fixed cost F = \$ 4000
Total cost equation: y = (1 – M)x + F
Or, y = (1 – 0.20)x + 4000
Or, y = 0.80x + 4000

a) Break-even point is $\frac{F}{M} = \frac{\$4000}{0.20}$ = \$ 20000

b) Here, total sales, x = \$ 50000
Hence, total cost, y = (0.80)(50000) + 4000 dollars
= 40000 + 4000 dollars
= 44000 dollars

Therefore, estimated profit = (50000 - 440000) dollars
= 6000 dollars

6.18 Exercises:

1. Define identity and inequality with example.
2. What is difference between equation and identity?
3. Define linear equation and system of linear equations. Give a example of a system of linear equation and solve it.
4. Define degree of an equation. Write an equation of 5 degree.
5. What do you mean by break-even quantity? Interpret the break-even quantity.
6. Write the equation of the line that has a slope of –3.1 and a y-intercept of 10. [Answer: y = -3.1x + 10]
7. If A = {2, 3, 5} and B = {3, 7} then find the relation, R as x < y where $x \in A$ and $y \in B$, [Answer: {(2, 3), (2, 7), (3, 7), (5, 7)}]
8. If A = {$3, $7, $8} is a set of cost of per unit product and B = {$5, $8} is the set of selling price of per unit product of a production firm. Find the profitable relation between cost and selling price. [Answer: {($3, $5), ($3, $8), ($7, $8)}]
9. Write two polynomials of degree 6 and 10 respectively. [Answer: Many answers, in particular, $f(x) = 3x^6 + 7x^2 - x + 1$, $f(x) = 5 - 6x + 4x^7 - 2x^{10}$]
10. Solve: (i) $4(3x + 2) \geq 2x$, (ii) $5(3 - 2x) < 3(4 - 3x)$ [Answer: (i) $x \geq -\frac{4}{5}$, (ii) $x > 3$]
11. A student bought x pencils per pencil $5 and bought (x + 4) pens per pen $8. At best how many pencils did he buy, if his total spent not more than $97? [Answer: At best 5 pencils.]
12. Solve the quadratic equation: (i) $x^2 - 8x + 15 = 0$ [Answer: 3, 5]

(ii) $x^2 - 10x - 9 = 0$ [Answer: –1, 9]

13. Kazi bought a bi-cycle with Rs. 6000 and sold it to Faruk earning profit x%. After then Faruk sold it to Mizan earning profit x%. If Mizan's cost price is greater than Kazi's cost price by Rs. 2640, what is the value of x? [Answer: x = 20.]
14. In the memorable cricket match between Bangladesh and Pakistan in the World Cup'99 that Bangladesh won, one of Bangladeshi player scored 42 runs. As a result his average run increased by 2. Before this match he played x number of matches and his total score was 198. Find the value of x. [Answer: 11]
15. If a and b be two roots of $x^2 - 3x - 10 = 0$, then find the quadratic equation whose roots are a^2 and b^2. [Answer: $x^2 - 29x + 100 = 0$]
16. Solve the following systems of linear equations by Cramer's rule:

i)
$$x + 2y + z = 0$$
$$x - 2y - 2z = 0$$
$$3x + y + 2z = 11$$
[Answer: (x, y, z) = (2, -3, 4)]

ii)
$$3x + 2y - 5z = -8$$
$$2x + 3y + 6z = 37$$
$$x - y + 6z = 7$$
[Answer: x =2, y = 3, z = 4]

iii)
$$2x + y + 2z = 10$$
$$x - y + 2z = 5$$
$$x + y + z = 6$$
[Answer: (x, y, z) = (1, 2, 3)]

iv)
$$\frac{x}{a} + \frac{y}{b} = 2$$
$$ax - by = a^2 - b^2$$
[Answer: (x, y) = (a, b)]

17. Solve the following systems of linear equations by any method:

i) $x + 2y - z = 2$
$2x + y + z = 1$
$x + 5y - 4z = 5$
[Answer: It has many solutions. (-1, 2, 1) is a particular solution.]

ii) $x + 2y - 3z = -1$
$5x + 3y - 4z = 2$
$3x - y + 2z = 7$
[Answer: No solution.]

iii) $x + 2y - 3z = 4$
$x + 3y + z = 11$
$2x + 5y - 4z = 13$
$2x + 6y + 2z = 22$
[Answer: (1, 3, 1)]

iv) $x + y + z = 4$
$2x - y + z = 3$
$x - 2y + 3z = 5$
[Answer: (1, 1, 2)]

18. The sum of the digits of a two-digit number is 12 and if the digits are reversed the number is decreased by 18. Find the number? [Answer: 75]

19. Solve the following simultaneous linear equations using determinant:
$2x - 3y = 3$
$4x - y = 11$ [Answer: x = 3, y = 1] [CMA-93]

20. Solve the system:
$2x + 3y = 5$ [NU-97]
$xy = 1$

21. Solve the following simultaneous equations:
$3x^2 - 5x - 3y = -4$ [NU-2000]
$2x + 3y = 10$

22. It takes 20 minutes and costs \$2 to make one chair, whereas it takes 30 minutes and costs \$1 to make one table. If 600 minutes and \$40 are available, how many chairs and tables can be made? [Answer: 15 chairs and 10 tables]

23. Market Equilibrium problem: Given the following demand and supply functions for two competing products,

$$q_{d1} = 82 - 3p_1 + p_2$$

$$q_{s1} = 15p_1 - 5$$

$$q_{d2} = 92 + 2p_1 - 4p_2$$

$$q_{s2} = 32p_2 - 6$$

Determine whether there are prices, which bring the supply and demand levels into equilibrium for the two products. If so, what are the equilibrium quantities?

[Answer: $p_1 = 5$, $p_2 = 3$, $q_1 = 70$, $q_2 = 90$]

24. Demand (D) and supply (S) functions of a product, $D = 17 - 4p - p^2$ and $S = 4p - 3$; where p means the price of the product per unit. Determine equilibrium price and quantity. [Answer: p = 2, D = S = 5]

25. Demand and supply equations are $2p^2 + q^2 = 11$ and $p + 2q = 7$ respectively, where p stands for price and q for quantity. Find the equilibrium price and quantity. [Answer: $(p, q) = (1, 3)$ or $(\frac{5}{9}, \frac{29}{9})$] [RU-80 A/C, CMA-93]
26. A manufacturer of keyboards has a fixed cost of $10000 and variable cost per keyboard made is $5. Selling price per unit is $10.

(a) Write the revenue and cost equations. [Answer: R = 10q, C = 5q + 10000]

(b) At what number of units will break-even occur? [Answer: 2000 units]

(c) At what sales volume (revenue) will break-even occur? [Answer: $20,000]

(d) Compute the loss if 1000 keyboards are produced and sold. [Answer: $5000]

27. A manufacturer has a fixed cost of Tk.7500 and variable cost per unit made is Tk.7. Selling price per unit is Tk.10.

(a) Write the revenue and cost equations. [Answer: R = 10q, C = 7q + 7500]

(b) At what number of units will break-even occur? [Answer: 2500 units]

(c) At what sales volume (revenue) will break-even occur? [Answer: Tk.25000]

(d) Draw the break-even chart. [RU-95 A/C]

28. A manufacturer of handbag has a fixed cost of Tk.1,20,000 and a variable cost of Tk.20 per unit made and sold. Selling price is Tk.50 per unit.

(a) Write the revenue and cost equations, using C for cost and q for number of units. [Answer: R = 50q, C = 20q + 1,20,000]

(b) Find the break-even quantity. [Answer: 4,000 units]

(c) Find the break-even sales volume. [Answer: Tk. 2,00,000]

(d) Compute profit if 10,000 handbags are made and sold. [Answer: Tk. 1,80,000]

(e) Compute loss if 1,000 handbags are made and sold. [Answer: Tk. 90,000]

29. Compute total cost and profit/loss if sales are Tk. 40000, fixed expense is Tk. 12000 and margin is 25%. [Answer: Tk. 42000 and loss = Tk. 2000]

07

Chapter

Exponential and Logarithmic Functions

Highlights:

7.1 Introduction: The main object of this chapter is to review the nature and properties of exponents, exponential functions, logarithms and logarithmic functions. The concept of exponential and logarithmic functions is very useful in various parts of mathematics. We shall look at some very important applications of these functions here and in the chapter of mathematics of finance. Until twenty years ago, students labored with extensive tables of logarithms and exponential values, but today we are fortunate to have these numerical values at our fingertips via the scientific calculator and computer.

7.2 Exponential function: If a, x $\in$R, a > 0 and a $\neq$ 1 then the function $f(x) = a^x$ is called an exponential function. Here, "a" is called 'base' and "x" is called the 'exponent'.

Example: $f(x) = 5e^x$

$f(x) = 10^x$

$f(x) = (2.5)^{x+1}$ are exponential function

But $f(x) = x^{10}$ is not an exponential function because the exponent "10" is not variable.

7.2.1 Properties of Exponents:

1. Any number to the power 0 is 1

i.e., $a^0 = 1$

2. Always add exponents when multiplying two powers of the same base

i.e., $a^x . a^y = a^{x+y}$

3. When dividing a^x by a^y , Subtract exponents

i.e., $\dfrac{a^x}{a^y} = a^{x-y}$

4. The quantity a^x to the power y is equal to a^{xy}

i.e., $\left(a^x\right)^y = a^{xy}$

5. The base ab to the power x is equal to a^x times a^y

i.e., $(ab)^x = a^x . b^y$

6. The base a/b to the power x is equal to a^x over b^x

i.e., $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

7. A base to the power –x is equivalent to one over that base to the power x

i.e., $a^{-x} = \dfrac{1}{a^x}$

Example: Apply the law of exponents to simplify the following expression and write the result with positive exponents.

$$\frac{3x(x^{-2})y^5}{4x^4y^2}$$

Solution: Given that,

$$\frac{3x(x^{-2})y^5}{4x^4y^2} = \frac{3x^{1-2}y^5}{4x^4y^2}$$

$$= \frac{3x^{-1}y^5}{4x^4y^2}$$

$$= \frac{3}{4}x^{-1-4}.y^{5-2}$$

$$= \frac{3}{4}x^{-5}.y^3$$

$= \dfrac{3y^3}{4x^5}$ (Answer)

Example: Show that $\dfrac{1}{1+x^{a-b}+x^{a-c}} + \dfrac{1}{1+x^{b-c}+x^{b-a}} + \dfrac{1}{1+x^{c-a}+x^{c-b}} = 1$

Solution: L.H.S = $\dfrac{1}{1+x^{a-b}+x^{a-c}} + \dfrac{1}{1+x^{b-c}+x^{b-a}} + \dfrac{1}{1+x^{c-a}+x^{c-b}}$

$$= \frac{1}{1+x^a.x^{-b}+x^a.x^{-c}} + \frac{1}{1+x^b.x^{-c}+x^b.x^{-a}} + \frac{1}{1+x^c.x^{-a}+x^c.x^{-b}}$$

$$= \frac{x^{-a}.1}{x^{-a}(1+x^{a}.x^{-b}+x^{a}.x^{-c})} + \frac{x^{-b}.1}{x^{-b}(1+x^{b}.x^{-c}+x^{b}.x^{-a})} + \frac{x^{-c}.1}{x^{-c}(1+x^{c}.x^{-a}+x^{c}.x^{-b})}$$

[Multiplying both the numerator & the denominator of first, second and third terms by x^{-a}, x^{-b} and x^{-c} respectively.]

$$= \frac{x^{-a}}{x^{-a}+x^{-b}+x^{-c}} + \frac{x^{-b}}{x^{-b}+x^{-c}+x^{-a}} + \frac{x^{-c}}{x^{-c}+x^{-a}+x^{-b}}$$

$$= \frac{(x^{-a}+x^{-b}+x^{-c})}{(x^{-a}+x^{-b}+x^{-c})}$$

= 1 = R.H.S (Proved)

8. **<u>Graph of exponential function</u>:** The graph of an exponential function is easy to sketch. Consider the function f(x) = 2^x. To sketch the graph we determine the following table of ordered pairs, which satisfy the function.

x	0	1	2	3	-1	-2	-3
f(x) = 2^x	1	2	4	8	0.5	0.25	0.125

Using the above ordered pairs we get the following graph:

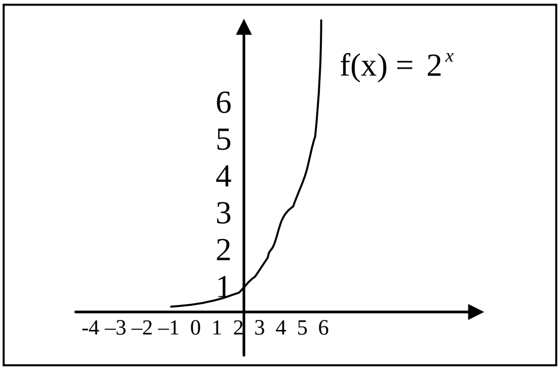


Figure 7.1

From the graph we see that as x becomes large positively, 2^x increases rapidly; and as x takes on values more and more negative, 2^x seems to decrease to zero. That is, the graph of exponential function 2^x is asymptotic to the negative x-axis.

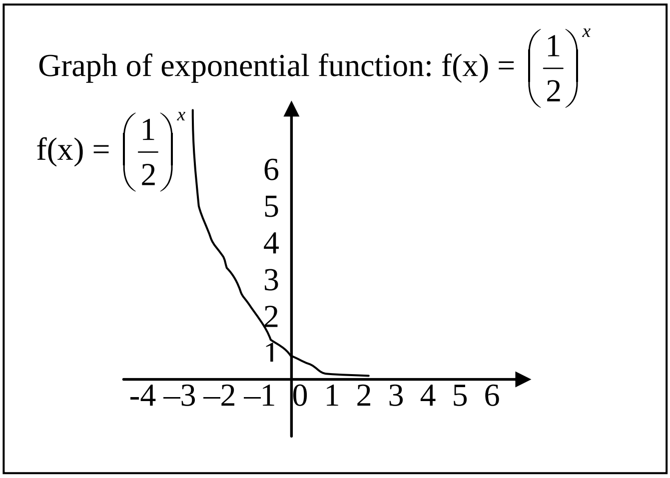


Figure 7.2

From the above two graph we can see that exponential functions comes in two forms; those with $a > 1$ increase to the right and those with $0 < a < 1$ decrease to the right. All exponential functions of the form a^x

1. pass through the point (0, 1);
2. are positive for all values of x; and
3. tend to infinity in one direction and zero in the other.

7.2.3 <u>Applications of exponential functions</u>:

<u>Example 1</u>: (Compound Interest) The equation

$$F = P(1+i)^n$$

can be used to determine the amount F that an investment of P taka will grow to if it receives interest of i percent per compounding period for n compounding periods, assuming reinvestment of any accrued interest. F is referred to as the compound amount and P as the principal. If F is considered to be a function of n, the above equation can be viewed as having the form of exponential function. That is,

$$F = f(n)$$

Or, $F = Pa^n$; where $a = 1+i$

Assume that P = Tk. 1000 and i = 0.08 per period, so a = 1.08. Then we find

$$F = f(n) = (1000)(1.08)^n$$

If we want to know to what sum Tk. 1000 will grow after 25 periods, we must evaluate $(1.08)^{25}$. And we find the compound amount

$$F = f(25) = \text{Tk. } 6848.50$$

<u>Example 2</u>: (Advertising Response) A large recording company sells tapes and CDs by direct mail only. Advertising is done through network television. Much experience with response to an advertising approach has allowed analysts to determine the expected response to an advertising program. Specifically, the response function for classical music CDs and tapes is $R = f(t) = 1 - e^{-0.05t}$, where R is the percentage of customers in the target market actually purchasing the CD or tape and t is the number of times an advertisement is run on national TV.

a) What percentage of the target market is expected to buy a classical music offering if advertisements are run one time on TV? 5 times? 10 times? 15 times? 20 times?
b) Sketch the response function R = f(t).

Solution: a) Given that the expected response to the advertising program,

$R = f(t) = 1 - e^{-0.05t}$; where t is the number of times to play the advertisement.

If the advertisement plays 1 time then the expected percentage of response,

$R = f(1) = 1 - e^{(-0.05)(1)} = 1 - 0.9512294 = 0.0488 = 0.0488 \times 100\% = 4.88\%$

If the advertisement plays 5 time then the expected percentage of response,

$R = f(5) = 1 - e^{(-0.05)(5)} = 1 - 0.7788007 = 0.2212 = 0.2212 \times 100\% = 22.12\%$

If the advertisement plays 10 time then the expected percentage of response,

$R = f(10) = 1 - e^{(-0.05)(10)} = 1 - 0.6065306 = 0.3935 = 0.3935 \times 100\% = 39.35\%$

If the advertisement plays 15 time then the expected percentage of response,

$R = f(15) = 1 - e^{(-0.05)(15)} = 1 - 0.4723665 = 0.5276 = 0.5276 \times 100\% = 52.76\%$

If the advertisement plays 20 time then the expected percentage of response,

$R = f(20) = 1 - e^{(-0.05)(20)} = 1 - 0.3678794 = 0.6321 = 0.6321 \times 100\% = 63.21\%$

b) The graph is as follows

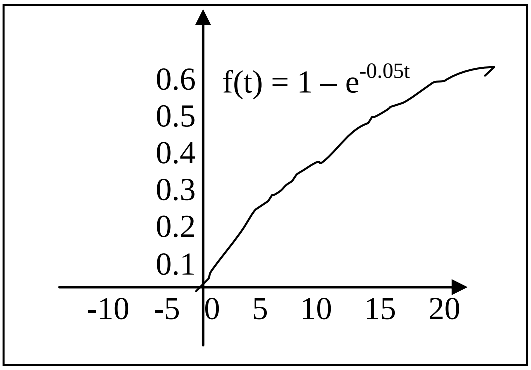


Figure 7.3

7.3 Surds: A surd is defined as the irrational root of a rational number of the type $\sqrt[n]{a}$, where it is not possible to extract exactly the nth root of "a". In other words, a real number $\sqrt[n]{a}$ is called a surd, if and only if

i) it is an irrational number, and
ii) it is a root of a rational number.

In the surd $\sqrt[n]{a}$, the index "n" is called the order of the surd and "a" the radicand. A surd can always be expressed with fractional indices.

i.e., $\sqrt[n]{a} = a^{\frac{1}{n}}$

And $\sqrt[10]{5} = 5^{\frac{1}{10}}$ etc.

Illustration: $\sqrt{2}$, $\sqrt{3}$ and $\sqrt[3]{7}$ are surds, since $\sqrt{2}$, $\sqrt{3}$ and $\sqrt[3]{7}$ are the irrational roots of the rational numbers 2, 3 and 7 respectively.

But $\sqrt[4]{16}$ is not a surd, because $\sqrt[4]{16} = \left(2^4\right)^{\frac{1}{4}} = 2$ is not an irrational number.

7.3.1 Formulae of surd: 1. $\sqrt[n]{a}.\sqrt[n]{b} = \sqrt[n]{ab}$; $p\sqrt[n]{a} \times q\sqrt[n]{b} = pq\sqrt[n]{ab}$

2. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$; $\dfrac{p\sqrt[n]{a}}{q\sqrt[n]{b}} = \dfrac{p}{q}\sqrt[n]{\dfrac{a}{b}}$

3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$

4. $\sqrt[n]{a^m} = \sqrt[pn]{a^{pm}}$

5. $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

6. $\sqrt[n]{a} = a^{\frac{1}{n}}$; $\sqrt{a} = a^{\frac{1}{2}}$

7. $p\sqrt[n]{a} \pm q\sqrt[n]{a} = (p \pm q)\sqrt[n]{a}$

7.3.2 Rationalization of surd: If multiplication of two surds makes a rational number then this multiplication is known as rationalization of surds. One of these surds is called the rationalizing factor of other. $\sqrt{5}$ is the rationalizing factor of $\sqrt{5}$ because $\sqrt{5} \times \sqrt{5} = 5$, rational number.

Example: Rationalize the denominator of the irrational fraction: $\dfrac{7\sqrt{3} - 5\sqrt{2}}{\sqrt{48} + \sqrt{18}}$

Solution: Given that $\dfrac{7\sqrt{3} - 5\sqrt{2}}{\sqrt{48} + \sqrt{18}} = \dfrac{(7\sqrt{3} - 5\sqrt{2})(\sqrt{48} - \sqrt{18})}{(\sqrt{48} + \sqrt{18})(\sqrt{48} - \sqrt{18})}$

[Multiplying numerator and denominator by $(\sqrt{48} - \sqrt{18})$]

$$= \frac{(7\sqrt{3} - 5\sqrt{2})(4\sqrt{3} - 3\sqrt{2})}{48 - 18}$$

$$= \frac{84 - 21\sqrt{6} - 20\sqrt{6} + 30}{30}$$

$$= \frac{114 - 41\sqrt{6}}{30}$$ [Answer]

Example: Show that $\left\{\frac{9^{n+\frac{1}{4}}.\sqrt{3.3^n}}{3\sqrt{3^{-n}}}\right\}^{\frac{1}{n}} = 27$ [AUB-2002]

Solution: L.H.S $= \left\{\frac{9^{n+\frac{1}{4}}.\sqrt{3.3^n}}{3\sqrt{3^{-n}}}\right\}^{\frac{1}{n}}$

$$= \left\{\frac{(3^2)^{n+\frac{1}{4}}.(3^{n+1})^{\frac{1}{2}}}{3.(3^{-n})^{\frac{1}{2}}}\right\}^{\frac{1}{n}}$$

$$= \left(\frac{3^{2n+\frac{1}{2}}.3^{\frac{n+1}{2}}}{3^{1-\frac{n}{2}}}\right)^{\frac{1}{n}}$$

$$= \left(\frac{3^{2n+\frac{1}{2}+\frac{n+1}{2}}}{3^{\frac{2-n}{2}}}\right)^{\frac{1}{n}}$$

$$= \left(3^{2n+\frac{1}{2}+\frac{n+1}{2}-\frac{2-n}{2}}\right)^{\frac{1}{n}}$$

$$= \left(3^{\frac{4n+1+n+1-2+n}{2}}\right)^{\frac{1}{n}}$$

$$= \left(3^{\frac{6n}{2}}\right)^{\frac{1}{n}}$$

$$= 3^{\frac{6n}{2}.\frac{1}{n}}$$

$$= 3^3$$

$= 27 = $ R.H.S

So, L.H.S = R.H.S (Proved)

7.4 Logarithmic function: If $a^x = n$; $a > 0$ and $a \neq 1$ then "x" is said to be the logarithm of the number n to the base "a". Symbolically it can be expressed as follows:

$Log_a n = x$

We read it as the logarithm of n to the base a is x

Note: 1) The logarithm of a number to the base "e" (e = 2.718282 approximately) is called "Natural logarithm" or "The Napierian logarithm" and is denoted by ln N.

That is, $\ln N = \log_e N$

2. The logarithm of a number to the base 10 is called "Common logarithm" or "The Briggsian logarithm" and is denoted by log N.

That is, $\log N = \log_{10} N$

When no base is mentioned, it is understood to be 10, that is, the common logarithm. Logarithmic functions are inverses of exponential functions. Ln x is inverse of e^x and $\log_{10} x$ is inverse of 10^x. That is,

$\text{Antilog}_{10}\, x = 10^x$ so, $\text{antilog}(\log_{10} x) = x = 10^{\log_{10} x}$

$\text{Antiln}\, x = e^x$ so, $\text{antiln}(\ln x) = x = e^{\ln x}$

7.4.1 <u>Laws of logarithmic operations</u>:

1. The logarithm of 1 to any base is 0,

   i.e., $\log_a 1 = 0$ because of $a^0 = 1$

2. The logarithm of any quantity to the same base is unity,

i.e., $\log_a a = 1$ because of $a^1 = a$

3. The logarithm of the product of two numbers is equal to the sum of the logarithms of that numbers to the same base,

i.e., $\log_a (mn) = \log_a m + \log_a n$

Proof: Let $\log_a m = x$ so that $a^x = m$ --- (i)

and $\log_a n = y$ so that $a^y = n$ --- (ii)

Multiplying (i) and (ii) we get

$a^x \times a^y = m \times n$

=> $a^{x+y} = mn$

Then by the definition of logarithm, we have

$\log_a (mn) = x + y$

=> $\log_a (mn) = \log_a m + \log_a n$ (Proved)

Remark: This formula can be extended in a similar way to the product of any number of quantities, that is,

$$\log_a (mnpq...) = \log_a m + \log_a n + \log_a p + \log_a q + ...$$

We should remember that $\log_a (m + n) \neq \log_a m + \log_a n$

4. The logarithm of the quotient of two numbers is equal to the difference of their logarithms to the same base,

i.e., $\log_a \frac{m}{n} = \log_a m - \log_a n$

5. The logarithm of the number raised to a power is equal to the index of the power multiplied by the logarithm of the number to the same base,

   i.e., $\log_a m^n = n \log_a m$

Proof: Let $\log_a m = x$ so that $a^x = m$ --- (i)

Raising the power "n" on both sides of equation (i), we get

$$(a^x)^n = m^n$$

=> $a^{xn} = m^n$

Then by the definition of logarithm, we get

$$\log_a m^n = nx$$

So, $\log_a m^n = n \log_a m$ (Proved)

6. The logarithm of any number "b" to the base "a" is equal to the reciprocal to the logarithm of "a" to the base "b"

i.e., $\log_a b = \dfrac{1}{\log_b a}$

Or, $\log_a b \times \log_b a = 1$

Proof: Let $\log_a b = x$ so that $a^x = b$ --- (i)

and $\log_b a = y$ so that $b^y = a$ => $b = a^{\frac{1}{y}}$ --- (ii)

From equation (i) we have

$a^x = b$

Or, $a^x = a^{\frac{1}{y}}$ [Using equation (ii)]

Or, $x = \dfrac{1}{y}$

i.e., $\log_a b = \dfrac{1}{\log_b a}$

So, $\log_a b \times \log_b a = 1$ (Proved)

7.4.2 <u>Relation between natural and common logarithms</u>: The natural logarithm of a number is equal to the quotient of the common logarithm of that number and the common logarithm of e

that is, $\ln m = \dfrac{\log m}{\log e}$ Or, $\log_e m = \dfrac{\log_{10} m}{\log_{10} e}$

<u>Proof</u>: Let $\log_e m = x$ so that $e^x = m$ --- (1)

$\log_{10} m = y$ so that $10^y = m$ --- (2)

and $\log_{10} e = z$ so that $10^z = e$ => $10 = e^{\frac{1}{z}}$ --- (3)

From equations (1) and (2), we get

$e^x = 10^y$

$$\Rightarrow \quad e^x = \left(e^{\frac{1}{z}}\right)^y \qquad \text{[Using equation (3)]}$$

$$\Rightarrow \quad e^x = e^{\frac{y}{z}}$$

$$\Rightarrow \quad x = \frac{y}{z}$$

i.e., $\log_e m = \dfrac{\log_{10} m}{\log_{10} e}$

Or, $\ln m = \dfrac{\log m}{\log e}$ (Proved)

7.4.3 <u>Graph of logarithmic function</u>: The graph of a logarithmic function is easy to sketch. Consider the function f(x) = ln x. To sketch the graph we determine the following table of ordered pairs, which satisfy the function.

x	0	1	2	3	4	5	6
f(x) = ln x	- ∞	0	0.69	1.99	1.39	1.61	1.79

Using the above ordered pairs we get the following graph:

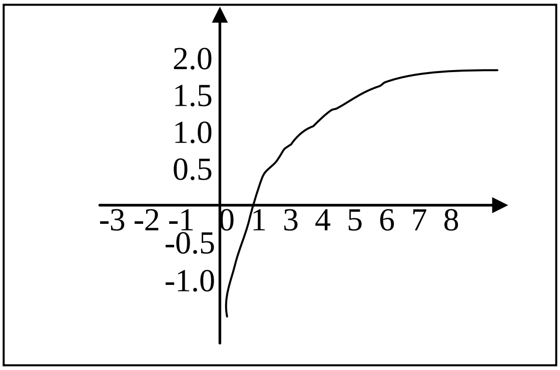


Figure 7.4

From the graph we see that the curve rises slowly, but always rises, to the right of x = 1. Recalling that x can not be zero or negative, we conclude that to the left of x = 1 the curve falls indefinitely as x gets closer to zero, but never touches the y-axis.

7.5 <u>Some worked out Examples</u>:

<u>Example (1)</u>: Find the value of $\log_6 6\sqrt{6}$.

Solution: Let x = $\log_6 6\sqrt{6}$

Then by the definition of logarithm, we get

$6^x = 6\sqrt{6}$

Or, $6^x = 6.6^{\frac{1}{2}} = 6^{1+\frac{1}{2}} = 6^{\frac{3}{2}}$

So, $x = \frac{3}{2}$ (Answer)

Example (2): Find the value of x: $\log_x 25 = 2$.

Solution: Given that, $\log_x 25 = 2$

By the definition of logarithm, we get

$x^2 = 25$

Or, $x^2 = 5^2$

So, x = 5 (Answer)

Example (3): Using logarithm find the value of $1000(1.06)^{110.5}$

Solution: Let $x = 1000\,(1.06)^{110.5}$

Taking common logarithm on both sides, we have

$\log_{10} x = \log_{10}\{1000\,(1.06)^{110.5}\}$

$= \log_{10} 1000 + \log_{10}\,(1.06)^{110.5}$

$= \log_{10}\,(10)^3 + 110.5\,\log_{10} 1.06$

$= 3\,\log_{10} 10 + 110.5\,\log_{10} 1.06$

$= 3 \times 1 + 110.5 \times 0.0253$

$= 3 + 2.79565$

$= 5.79565$

Taking anti logarithm, we have,

anti log($\log_{10} x$) = anti log 5.79565

$x = 10^{5.79565}$

$= 624669.07$ (Answer)

Example (4): Assume that, for some base, log x = 0.5, log y = 1.5 and log z = 3. Compute the value of $\log\left(\frac{xy}{z^{\frac{1}{3}}}\right)$. [RU-91]

Solution: Given that, log x = 0.5, log y = 1.5 and log z = 3

Now, $\log\left(\frac{xy}{z^{\frac{1}{3}}}\right) = \log(xy) - \log z^{\frac{1}{3}}$

$= \log x + \log y - \frac{1}{3}\log z$

$= 0.5 + 1.5 - \frac{1}{3} \times 3$

$= 2.0 - 1$

$= 1$ (Answer)

Example (5): Solve for x: $3^x = 8$

Solution: Given that, $3^x = 8$

Taking common logarithm on both sides, we have

$\log 3^x = \log 8$

Or, $x \log 3 = \log 8$

Or, $x = \frac{\log 8}{\log 3}$

Or, $x = \frac{0.903090}{0.477121}$

So, $x = 1.893$ (approximately) (Answer)

Example (6): Solve for x: $9^x = 2$

Solution: Given that, $9^x = 2$

Taking common logarithm of both sides, we have

$\log 9^x = \log 2$

Or, $x \log 9 = \log 2$

Or, $x = \frac{\log 2}{\log 9}$

Or, $x = \frac{0.3010}{0.9542}$ [Using calculator]

So, $x = 0.31546$ (approximately)

Example (7): Find the value of i when $10000(1 + i)^{10} = 30000$ [AUB-03,]

Solution: Given that, $10000(1 + i)^{10} = 30000$

Or, $(1+ i)^{10} = \frac{30000}{10000}$

Or, $(1+ i)^{10} = 3$

Or, $\ln(1+ i)^{10} = \ln 3$ [Taking natural logarithm of both sides]

Or, $10 \ln(1+ i) = 1.0986$ [Using calculator]

Or, $\ln(1+ i) = 0.10986$

Or, $\text{antiln}\{\ln(1+ i)\} = \text{antiln } 0.10986$

Or, $1+ i = 1.11612$ [Using calculator]

Or, $i = 1.11612 - 1$
So, $i = 0.11612$ (Answer)

Example (8): Solve the equation for x: ln (1 + x) = – 0.5
Solution: Given that, ln (1 + x) = – 0.5
Or, Antiln ln (1+ x) = Antiln (–0.5) [Taking Antiln of both sides]
Or, 1 + x = 0.60653
Or, x = 0.60653 – 1
So, x = – 0.39347 (Answer)

Example (9): If $\log_{10} 2 = 0.3010$, then find the value of $\log_8 25$ [AUB-02]
Solution: Given that, $\log_{10} 2 = 0.3010$

Now, $\log_8 25 = \dfrac{\log_{10} 25}{\log_{10} 8}$ [We know, $\log_e m = \dfrac{\log_{10} m}{\log_{10} e}$]

$$= \frac{\log_{10}\left(\frac{100}{4}\right)}{\log_{10} 2^3}$$

$$= \frac{\log_{10} 100 - \log_{10} 4}{3\log_{10} 2}$$

$$= \frac{\log_{10}(10)^2 - \log_{10}(2)^2}{3\log_{10} 2}$$

$$= \frac{2\log_{10} 10 - 2\log_{10} 2}{3\log_{10} 2}$$

$$= \frac{2\times 1 - 2\times 0.3010}{3\times 0.3010}$$

= 1.548 (Answer)

Example (10): Show that $\log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80} = 1$
[CMA-94, NU-94]

Proof: L.H.S $= \log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80}$

$$= \log 2 + \log \left(\frac{16}{15}\right)^{16} + \log \left(\frac{25}{24}\right)^{12} + \log \left(\frac{81}{80}\right)^{7}$$

$$= \log \left\{2\times\left(\frac{2^4}{3.5}\right)^{16}\times\left(\frac{5^2}{2^3.3}\right)^{12}\times\left(\frac{3^4}{2^4.5}\right)^{7}\right\}$$

$$= \log \left(2\times\frac{2^{64}}{3^{16}.5^{16}}\times\frac{5^{24}}{2^{36}.3^{12}}\times\frac{3^{28}}{2^{28}.5^{7}}\right)$$

$$= \log \left(2^{1+64-36-28}\times 3^{28-12-16}\times 5^{24-16-7}\right)$$

$$= \log \left(2^1\times 3^0\times 5^1\right)$$

$$= \log \left(2\times 1\times 5\right)$$

$$= \log 10$$

$= 1$ [We know that $\log_{10} 10 = 1$]

= R.H.S (Proved)

<u>Example (11)</u>: Find the value of $\frac{1}{6}.\frac{\sqrt{3\log 1728}}{1+\frac{1}{2}\log 0.36+\frac{1}{3}\log 8}$ [CMA-93]

Solution: $\frac{1}{6}.\frac{\sqrt{3\log 1728}}{1+\frac{1}{2}\log 0.36+\frac{1}{3}\log 8} = \frac{1}{6}.\frac{\sqrt{3\times 3.2375437}}{1+\frac{1}{2}\times(-0.4436975)+\frac{1}{3}\times 0.9030899}$

[Using calculator]

$$= \frac{1}{6}.\frac{\sqrt{9.7126311}}{1-0.2218487+0.3010299}$$

$$= \frac{1}{6}.\frac{3.1165094}{1.0791812}$$

$$= \frac{3.1165094}{6.4750872}$$

$= 0.4813077$ (Answer)

<u>Example (12)</u>: Show that $\frac{1}{\log_a bc+1}+\frac{1}{\log_b ca+1}+\frac{1}{\log_c ab+1}=1$ [NU-95]

Solution: L.H.S $= \frac{1}{\log_a bc+1}+\frac{1}{\log_b ca+1}+\frac{1}{\log_c ab+1}$

$$= \frac{1}{\log_a bc+\log_a a}+\frac{1}{\log_b ca+\log_b b}+\frac{1}{\log_c ab+\log_c c}$$

$$= \frac{1}{\log_a abc}+\frac{1}{\log_b abc}+\frac{1}{\log_c abc}$$

$= \log_{abc} a + \log_{abc} b + \log_{abc} c$

$= \log_{abc}(abc)$

= 1 = R.H.S (Proved)

Example (13): Find the value of $\log_2\left[\log_2\left\{\log_3\left(\log_3 27^3\right)\right\}\right]$ [AUB-02 MBA]

Solution: $\log_2\left[\log_2\left\{\log_3\left(\log_3 27^3\right)\right\}\right] = \log_2\left[\log_2\left\{\log_3\left(\log_3 3^9\right)\right\}\right]$

$= \log_2\left[\log_2\left\{\log_3\left(9\log_3 3\right)\right\}\right]$

$= \log_2\left[\log_2\left\{\log_3\left(9\times 1\right)\right\}\right]$

$= \log_2\left[\log_2\left\{\log_3 9\right\}\right]$

$= \log_2\left[\log_2\left\{\log_3 3^2\right\}\right]$

$= \log_2\left[\log_2\left\{2\log_3 3\right\}\right]$

$= \log_2\left[\log_2 2\right]$

$= \log_2 1$

= 0 (Answer)

7.6 <u>Exercise</u>:

1. Define exponential function. And discuss the properties of exponents.
2. What do you mean by surd?
3. Define natural and common logarithms. What is difference between natural and common logarithms?
4. State and prove relation between natural and common logarithms.
5. (Credit Card Collections) A major bank offers a credit card, which can be used internationally. Data gathered over time indicate that the collection percentage for credit issued in any month is an exponential function of the time since the credit was issued. Specifically, the function approximating this relationship is

$C = f(t) = 0.92(1- e^{-0.1t}); t \geq 0$

where C equals the percentage of accounts receivable (in taka) collected t months after the credit is granted.

(a) What percentage is expected to be collected after 1 month? [Answer:8.75%]
(b) What percentage is expected after 3 months? [Answer: 23.84%]
(c) What value does C approach as t increases without limit ($t \to \infty$)? [Answer:92%]

6. Find the value of (i) $(2^3)^2$ (ii) $(121)^{0.5}$ (iii) $16^{-\frac{3}{4}}$ [Answer: (i) 64 (ii) 11 (iii) 1/8]
7. Find the value of $\left[1-\left\{1-\left(1-x^3\right)^{-1}\right\}^{-1}\right]^{\frac{1}{3}}$ [Answer: $\frac{1}{x}$]

8. Show that $\left(x^{\frac{q+r}{r-p}}\right)^{\frac{1}{p-q}} \times \left(x^{\frac{r+p}{p-q}}\right)^{\frac{1}{q-r}} \times \left(x^{\frac{p+q}{q-r}}\right)^{\frac{1}{r-p}} = 1$

9. Show that $\frac{16(32)^m - 2^{3m-2}(4)^{m+1}}{15(2)^{m-1}(16)^m} - \frac{5(5)^{m-1}}{\sqrt{5^{2m}}} = 1$ [AUB-02]

10. Find the value of $\log_5\left(\sqrt[3]{5}\right)\left(\sqrt{5}\right)$ [Answer: $\frac{5}{6}$]

11. Find the value of x: (i) $\log_x 81 = 4$ (ii) $\log_{2\sqrt{5}} 400 = x$ [Answer: (i) 3, (ii) 4]

12. Using the definition of logarithm find the value of x: $\log_x(4x+3) - \log_x 4 = 2$

[Answer: $\frac{3}{2}, -\frac{1}{2}$] [NU-2000]

13. Solve for x: (i) $10^x = 8$. (ii) $2^x.3^{2x+1} = 7^{4x+3}$ [Answer: (i) 0.90309 (ii) – 0.9685]

14. Solve for x: $\log_x 3 + \log_x 9 + \log_x 729 = 9$ [Answer: 3] [AUB-01]

15. Assume that for some base, log a = 0.3, log b = 2.8642 and log c = 1.7642. Find the value of $\log\left(\frac{a^3 b^{\frac{1}{2}}}{c}\right)$ [Answer: 0.5676] [NU-01]

16. Compute the value of x from $10^{\log 5} = x$ [Answer: 5]

17. Using logarithm find the value of $789.45^{\frac{1}{8}}$ [Answer: 2.3023]

18. Find the value of $\frac{\sqrt[5]{10} \times \sqrt[3]{10}}{\sqrt[3]{10^{-2}} \times \sqrt[3]{10^{-3}}}$ [Answer: 158.49] [RU-84]

[Hints: After simplification, we get $10^{\frac{33}{15}}$. Let x = $10^{\frac{33}{15}}$ and then use log and antilog respectively to find the value of x.]

19. Simplify: $7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80}$ [Answer: $\log 2$] [NU-99]

20. Find the value of $\log_2\sqrt{6} + \log_2\sqrt{2/3}$ [Answer: 1]

21. Show that $\log_b \log_b \log_b \left(b^{b^{b^a}}\right)$ = a.

22. Show that $\log_2\sqrt{\frac{3}{2}} + \log_2\sqrt{\frac{5}{3}} - \log_2\sqrt{5} = -\frac{1}{2}$

23. Show that (i) $\frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)} = 1$ [Hints: $\frac{1}{\log_a(abc)} = \log_{abe} a$]

(ii) $\frac{1}{\log_{pq}(pqr)}+\frac{1}{\log_{qr}(pqr)}+\frac{1}{\log_{rp}(pqr)}=2$

(iii) $\frac{1}{\log_{6}24}+\frac{1}{\log_{8}24}+\frac{1}{\log_{12}24}=2$ [CMA-95]

24. Using the definition of logarithm find the value of x from $\log_2\{\log_3(\log_2 x)\}=1$.
[Answer: 512] [AUB-02]

25. If $\frac{\log a}{x-y}=\frac{\log b}{y-z}=\frac{\log c}{z-x}$, prove that

(i) abc = 1

(ii) $a^{x+y}.b^{y+z}.c^{z+x}=1$ [Hints: let $\frac{\log a}{x-y}=\frac{\log b}{y-z}=\frac{\log c}{z-x}$ = M, So, $\frac{\log a}{x-y}=M$
Or, log a = M(x – y) Or, (x + y)log a = M(x – y)(x + y) Or, log $a^{(x+y)}$ = M(x^2 – y^2).
Similarly, log $a^{(y+z)}$ = M(y^2 – z^2) and log $a^{(z+x)}$ = M(z^2 – x^2). Now add.]

08
Chapter

Mathematics of Finance

Highlights:

8.1 Introduction: This chapter is concerned with interest rates and their effects on the value of money. We shall discuss the nature of interest and its computational processes. Interest rates have widespread influence over decisions made by businesses and by us in our personal lives. Corporations pay millions of dollars in interest each year for the use of money they have borrowed. We earn money on sums we have invested in savings accounts, certificates of deposit, and money market funds. We also pay for the use of money, which we have borrowed for school loans, credit card purchases or mortgages. The interest concept also has applications that are not related to money such as population growth.

8.2 Simple interest and the future value: Interest rates are generally quoted in percentage form and, for use in calculations, must be converted to the equivalent decimal value by dividing the percentage by 100; that is, by moving the decimal point in the percentage two places to the left. For example,

$$i = 5\frac{1}{4}\% = 5.25\% = 0.0525.$$

Unless otherwise stated, a quoted rate is a rate per year. Thus, Tk.1 at 5 percent means that interest of Tk. 0.05 will be earned in a year, and Tk. 100 at this rate provides.

$$\text{Tk. } 100(0.05) = \text{Tk. } 5$$

of interest in one year. Interest on Tk. 100 at 5 percent for 9 months is interest for 9/12 year; that is,

$$\text{Interest} = \text{Tk.}100 \quad (0.05) \quad (9/12) \quad = \text{Tk. } 3.75.$$

$$\text{Interest} = (\text{Principal})\ (\text{Rate of interest})\ (\text{Time in year})$$

The last line introduces the following definitions of simple interest, which apply in simple interest calculation:

8.2.1 <u>Definition of simple interest</u>: In this case, we do not calculate interest of interest; here we calculate only interest of capital or principal. The simple interest formula is as follow:

$$\mathbf{I = Pin}$$

And $\mathbf{F = P + I = P + Pin = P\,(1 + in)}$

Where I = Total interest

P = Capital, the sum of money on which interest is being earned

i = Rate of interest, interest of per unit capital for a year

n = Number of years

F = Future value, the sum of capital and total interest

When time is given in days, there are two ways of computing the interest: the exact method and the ordinary method (often called the Banker's Rule). If the exact method is used, then the time is

$$n = \frac{Number\ of\ days}{365}$$

but if the ordinary method is used, then the time is

$$n = \frac{Number\ of\ days}{360}$$

Bank, for convenience, often count a year as twelve 30-day months, 360 days for a year.

<u>Example</u>: Compute the interest on Tk. 480 at $6\frac{1}{4}$ % for 9 months.

Solution: Here,

Capital, P = 480 Tk.

Rate of interest, $i = 6\frac{1}{4}\ \% = \frac{25}{4}\ \% = \frac{25}{4x100} = 0.0625$

Number of years, $n = \frac{9}{12}$ year

We know, simple interest, I = Pin

$$= 480 \times 0.0625 \times \frac{9}{12} \text{ taka}$$

$$= 22.50 \text{ taka} \qquad \text{(Answer)}$$

Example: Find the interest rate if Tk.1000 earns Tk.45 interest in 6 months.
Solution: Here,

Capital, P = Tk. 1000
Interest, I = Tk. 45
Number of years, $n = \frac{6}{12}$
$= 0.5$
Rate of interest, i = ?

We know,

$$I = Pin$$
$$\text{Or, } i = \frac{I}{Pn}$$
$$= \frac{45}{1000 X 0.5}$$
$$= 0.09$$
$$= 0.09 \times 100\% \qquad \left[\text{We know, } 100\% = \frac{100}{100} = 1\right]$$
$$= 9\%$$

Hence, the required interest rate is 9%.

Example: Find the exact and ordinary interest on Rs. 1460 for 72 days at 10 percent interest. [AUB-02]
Solution: In both case:

Capital, P = Rs. 1460
Interest rate, i = 10% = 0.1

For exact interest,

Number of years, $n = \frac{72}{365}$

For ordinary interest,

Number of years, $n = \frac{72}{360}$

So, exact interest, I = Pin

$$= \text{Rs. } 1460 \times 0.1 \times \frac{72}{365}$$
$$= \text{Rs. } 28.80 \quad \text{(Answer)}$$

And ordinary interest, I = Pin

$$= \text{Rs } 1460 \times 0.1 \times \frac{72}{360}$$
$$= \text{Rs } 29.20 \quad \text{(Answer)}$$

<u>Example</u>: Find the future value of $ 5000 at 10 percent for 9 months.
Solution: Here,

Capital = $ 5000

Interest rate, $i = 10\% = \frac{10}{100} = 0.1$

Number of years, $n = \frac{9}{12} = \frac{3}{4}$

We know that,

Future value, $F = P(1+in)$

$$= \$\,5000\,(1 + 0.1 \times \frac{3}{4})$$
$$= \$\,5000\,(1+ 0.075)$$
$$= \$\,5375 \qquad \text{(Answer)}$$

<u>Example</u>: Lovlu has placed Tk. 500 in a savings account that pays 8% simple interest. How long will it be, in months, until the investment amounts to Tk. 530?
Solution: Here,

Capital, P = Tk. 500

Future value, F = Tk. 530

Interest rate, $i = 8\% = \frac{8}{100} = 0.08$

Number of years, n =?

We know that,

$F = P(1 + in)$

Or, $1 + in = \frac{F}{P}$

Or, $in = \frac{F}{P} - 1$

Or, $n = \frac{F-P}{Pi}$

$$= \frac{530-500}{500\times 0.08}$$
$$= \frac{30}{500\times 0.08}$$
$$= \frac{3}{4}$$

So, the required time = $\frac{3}{4}$ of a year = $\frac{3}{4} \times 12$ months = 9 months (Answer)

8.3 <u>The yield on the common stock of a company</u>: The yield on the common stock of a company is a percent obtained by dividing the amount (called the dividend) that a shareholder receives per share of stock held by the price of a share of the stock. A stock market report showing

Shain Pu 3.50 87 ½

Means that at the time of the quotation, a share of Shain Pukur sold for Tk. 87.50 and the annual dividend was estimated to be Tk. 3.50 per share.

<u>Example</u>: Compute the yield for Shain Pukur from the market report Shain Pu 3.50 87 ½.
Solution: Given that,

The annual dividend = Tk. 3.50
The value of a share = Tk. 87 ½ = Tk. 87.50

So, the yield for general electric = $\frac{3.50}{87.50} \times 100\%$

= 4% (Answer)

8.4 <u>Bank discount</u>: In many loans, the interest charge is computed not on the amount the borrower receives, but on the amount that is repaid later. A charge for a loan computed in this manner is called the bank discount, and the amount borrower receives is called the proceeds of a loan and is denoted by 'P'. The future amount to be paid back is 'F', now called the maturity value of the loan. The bank discount rate is denoted by 'd' and the loan time is denoted by 'n', which is in years.
The formula is to calculate the proceeds: P = F (1- dn)

<u>Example</u>: (a) A borrower signs a deed promising to pay a bank $ 5000 in ten months from now. How much will the borrower receive if the discount rate is 8.4%? (b) How much would the borrower have to repay in order to receive $ 5000 now?
Solution: (a) Here,
The maturity value, F = $ 5000

The bank discount rate, d = 8.4% = 0.084

The loan time, n = 10 months = $\frac{10}{12}$ of a year.

We know that,

The proceeds, P = F (1- dn)

$= 5000\,(1 - 0.084 \times \frac{10}{12})$ dollar

= 5000(1- 0.07) dollar
= 4650 dollar

So, the borrower will receive $ 4650 (Answer)

(b) Here, The proceed value, P = $ 5000

The bank discount rate, d =8.4% = 0.084

The loan time, n = 10 months = $\frac{10}{12}$ of a year

The maturity value, F =?

We know that,

$$P = F(1 - dn)$$

$$\text{Or,} \quad F = \frac{P}{1-dn}$$

$$= \frac{5000}{1-0.084\times\frac{10}{12}} \text{ dollar}$$

$$= \frac{5000}{1-0.07} \text{ dollar}$$

$$= \frac{5000}{0.93} \text{ dollar}$$

$$= 5376.34 \text{ dollar}$$

So, the borrower will have to repay $ 5376.34 (Answer)

8.5 <u>Compound interest and the future value</u>: To see how compound interest works and develop a formula for computing the future value, suppose Tk. 4000 in invested at 10% interest compounded each year. The amount at the end of the first year would be

$$F_1 = \text{Tk. } [4000 + 4000\,(0.10)\,(1)]$$
$$= \text{Tk. } (4000+400)$$
$$= \text{Tk. } 4400$$

This Tk. 4400 becomes the principal at the beginning of the second year, and the amount at the end of the second year is

$$F_2 = \text{Tk. } [4400 + 4400\,(0.10)\,1]$$
$$= \text{Tk. } (4400+440)$$
$$= \text{Tk. } 4840$$

Thus, in the second year, interest is earned not only on Tk. 4000 invested, but also on Tk.400 of interest earned in the first year. This common practice of computing interest on interest is called compounding interest.

To formulate a formula for computing the future value, we will use 'i' as the interest rate per period. **Period** means the time interval of two consecutive compounding; it may be a year, a month, a semiannual etc.

Assuming, that the compounding period is 1 year, a capital of Tk. P will amount to

$$F_1 = P(1 + i)$$

at the end of the first year. At the beginning of the second year, P(1+ i) becomes the new principal, which is multiplied by (1+ i) to find the future value at the end of the second year.
Thus,

$$F_2 = P(1 + i)(1 + i) = P(1 + i)^2$$

after two years. At the beginning of the third year, the new principal is $P(1 + i)^2$, and to obtain the future value at the end of the third year, this must be multiplied by (1 + i). Thus,

$$F_3 = P(1 + i)^2(1 + i) = P(1+ i)^3$$

after three years. Similarly, the future value at the end of 20 years would be

$$F_{20} = P(1 + i)^{20}$$

and, in general, at the end of n years, the future value will be

$$F_n = P(1 + i)^n \, .$$

Thus in this case of compound interest, we calculate the interest of interest and the capital time to time. The compound interest formula is given by

$$F = P(1 + i)^n$$

Where, F = Future value, the sum of the capital and total interest.
P = Capital or principal or present value
i = Interest per unit per period
n = Total number of periods

<u>Example</u>: Find the compound future value of Tk. 1000 at 7 % interest compounded annually for 10 years.
Solution: Here,

Capital, P = Tk. 1000
Interest rate, i = 7% = 7/100 = 0.07
Total number of periods, n = 10
We know that,

$$\begin{aligned} \text{Future value, } F &= P(1 + i)^n \\ &= \text{Tk. } 1000\,(1 + 0.07)^{10} \\ &= \text{Tk. } 1000\,(1.07)^{10} \\ &= \text{Tk. } 1000\,(1.96715) \\ &= \text{Tk. } 1967.15 \qquad \text{(Answer)} \end{aligned}$$

Example: If $500 is invested at 6 percent compounded annually, what will be the future value 30 years later?

Solution: Here, Capital, P = $ 500

Interest rate, i = 6% = 6/100 = 0.06

Total number of periods, n = 30

Future value, F =?

We know that, $F = P(1 + i)^n$

$= \$\ 500\ (1 + 0.06)^{30}$

$= \$\ 500\ (1.06)^{30}$

$= \$\ 500\ (5.7435)$

$= \$\ 2871.75$

So, the future value = $ 2871.75 (Answer)

Example: If Tk. 800 is invested at 6% compounded semiannually, what will be the amount in 5 years?

Solution: Here, Capital, P = 800 taka

Interest rate per period, $i = \frac{6\%}{2} = 3\% = 0.03$

Total number of periods, n = (5 years)(2 periods per year)

= 10 periods

Future value, F =?

We know that,

$$F = P(1 + i)^n$$
$$= 800\ (1 + 0.03)^{10} \text{ taka}$$
$$= 800\ (1.03)^{10} \text{ taka}$$
$$= 800\ (1.3439) \text{ taka}$$
$$= 1075.13 \text{ taka}$$

So, after 5 years, total amount will be 1075.13 taka. (Answer)

Example: Find how many years it will take at 9% compounded annually for $1000 to grow to $ 2000. [AUB-01]

Solution: Here, Capital, P = $ 1000

Future value, F = $ 2000

Interest rate, i = 9% = 0.09

Number of periods, n =?

We know that,

$$F = P(1 + i)^n$$

Or, $2000 = 1000\ (1 + 0.09)^n$

Or, $(1.09)^n = \frac{2000}{1000}$

Or, $(1.09)^n = 2$

Or, $n \, Ln (1.09) = Ln \, 2$ [Taking natural logarithm on both sides.]

Or, $n = \frac{Ln2}{Ln1.09}$

Or, $n = \frac{0.6931471}{0.0861776}$

Or, $n = 8.0432317$

So, $n = 8.04$ (Approximately)

Hence, the required time = 8.04 years.

Example: A man built up a scholarship fund to give prize of Tk.500.00 every year. If the fund provides 10% interest compounded semiannually, what is the amount of the fund? [DU-78]

Solution: Since the fund gives prize of Tk.500 every year, the amount of the interest of a year must be equal to Tk.500.

Given that, Interest rate, $i = \frac{10\%}{2} = 5\% = 0.05$

Number of periods (in a year) = 1×2

Let the amount of the fund, P = x taka

So, the future value, F = (x + 500) taka

We know that,

$F = P(1 + i)^n$

Or, $x + 500 = x(1 + 0.05)^2$

Or, $x + 500 = 1.1025x$

Or, $0.1025x = 500$

Or, $x = 500/0.1025$

So, $x = 4878.05$

Therefore, the amount of the fund = Tk.4878.05 (Answer)

Example: Find how many months it will take at 10% compounded quarterly of a year for Tk. 5000 to grow to Tk. 20000. [DU-78]

Solution: Here, Capital, P = $ 5000

Future value, F = Tk.20000

Interest rate, $i = \frac{10\%}{4} = 2.5\% = 0.025$

Number of periods (Quarter year), n =?

We know that,

$F = P(1 + i)^n$

Or, $20000 = 5000 (1 + 0.025)^n$

Or, $(1.025)^n = \frac{20000}{5000}$

Or, $(1.025)^n = 4$

Or, $n \operatorname{Ln}(1.025) = \operatorname{Ln} 4$ [Taking natural logarithm on both sides.]

Or, $n = \dfrac{Ln4}{Ln1.025}$

Or, $n = \dfrac{1.386294}{0.024693}$

Or, $n = 56.1411736$

So, $n = 56.14$ (Approximately)

Hence, the required time = $\dfrac{56.14 \times 12}{4}$ months.

= 168.42 months. (Answer)

Example: Find the rate of interest that compounded annually, will result in tripling a sum of money in 10 years. [AUB-03, DU-77]

Solution: Here, let the capital = P taka

So, the future value, F = 3P taka

Total number of period, n = 10

The rate of interest, i =?

We know that,

$F = P(1 + i)^n$

Or, $3P = P(1 + i)^{10}$

Or, $(1 + i)^{10} = \dfrac{3P}{P}$

Or, $(1 + i)^{10} = 3$

Or, $\operatorname{Ln}(1 + i)^{10} = \operatorname{Ln}3$

Or, $10 \operatorname{Ln}(1 + i) = \operatorname{Ln} 3$ [Taking natural logarithm on both sides.]

Or, $\operatorname{Ln}(1 + i) = \dfrac{1.0986123}{10}$

Or, $\operatorname{Ln}(1 + i) = 0.10986123$

Or, $1 + i = \text{Anti ln } 0.10986123$

Or, $i = 1.1161231-1$

Or, $i = 0.1161231$

Or, $i = 0.1161231 \times 100\%$

So, $i = 11.61\%$

Hence, the required rate of interest = 11.61%. (Answer)

Example: What is the present value of $ 2500 payable 4 years from now at 10% compounded quarterly of a year?

Solution: Here, the future value, F = $ 2500

Interest rate per period, $i = \frac{10\%}{4}$
$= 2.5\%$
$= 0.025$
Number of periods, $n = 4 \times 4 = 16$
Present value (Capital), P =?

We know that, $F = P(1 + i)^n$

Or, $P = \frac{F}{(1+i)^n}$

Or, $P = \frac{2500}{(1+0.025)^{16}}$

Or, $P = \frac{2500}{1.4845056}$

So, P = 1684.0624

Thus, the present value = $ 1684.0624.

<u>Example</u>: To buy a car Mr. Amin pays Tk.500000 in cash and promises to pay Tk.300000 (including interest) in 3 years later. Find the present value of the car if he pays 12% annual interest compounded semiannually. [NU-96]

Solution: The present value of the car = Tk.500000 + present value of Tk.300000

Here, F = Tk.300000, $i = \frac{12\%}{2} = 6\% = 0.06$, $n = 3 \times 2 = 6$, P = ?

We know that, $F = P(1 + i)^n$

Or, $300000 = P(1 + 0.06)^6$

Or, $300000 = 1.4185191P$

Or, $P = \frac{300000}{1.4185191}$

Or, P = 211488.16

Therefore, the present value of the car = Tk.(500000 + 211488.16)
= Tk. 711488.16 (Answer)

8.5.1 <u>Effective interest rate</u>: Because of lack of comparability, it is hard to judge whether interest quoted at 18% compounded semiannually result in more or less interest than would be the case if the rate were 17% compounded monthly. To make the comparison possible, we change both to their equivalent annual rates; these equivalents are called effective rates. To find out effective rate formula let us consider annual interest rate, j compounded m times in a year, Tk.1 grows to

$F = (1)(1+i)^m$ [Interest rate per period, $i = \frac{j}{m}$]

at the end of a year. At the effective rate, r_e, Tk. 1 grows to

$$F = 1 + r_e$$

in a year. So,

$1 + r_e = (1)(1+i)^m$

Hence, the effective rate, $\mathbf{r_e = (1+i)^m - 1}$, where m = number of periods in a year and i = interest rate per period.

Example: Find the effective rate of 16% compounded quarterly of a year.

Solution: Here, interest rate per period, $i = \frac{16\%}{4} = 4\% = 0.04$

Number of period in a year, m = 4

We know that,

$$\begin{aligned}\text{Effective rate, } r_e &= (1+i)^m - 1\\ &= (1 + 0.04)^4 - 1\\ &= (1.04)^4 - 1\\ &= 0.1699\\ &= 0.1699 \times 100\%\\ &= 16.99\% \qquad \text{(Answer)}\end{aligned}$$

8.5.2 Future value with continuous compounding: In this case we calculate future value at every time (moment) for compound interest. To formulate a formula, let us consider compound interest formula:

$$F = P(1 + i)^n$$

Where, F = Future value, the sum of the capital and total interest.

P = Capital or principal

$i = \frac{j}{m}$ = Interest per unit per period (j = Interest rate per year, m = Periods in a year)

n = mt = Total number of periods (t = Time in years, m = Periods in a year)

For continuous compounding a period is very very small, so m is very very large that means m tends to infinity.

So, we can write the formula as follows:

$$F = P\left(1 + \frac{j}{m}\right)^{mt} \quad [\text{Let, } x = \frac{m}{j}, \text{ so } m = xj \text{ and when } m \to \infty, x \to \infty]$$

Or, $$F = P\left(1 + \frac{1}{x}\right)^{xjt} \quad [x \to \infty]$$

Or, $$F = P\left\{\left(1 + \frac{1}{x}\right)^x\right\}^{jt} \quad [x \to \infty]$$

So, $$F = Pe^{jt} \quad [\text{In the chapter of limit, we knew, } Lt_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e]$$

Hence, the formula for the future value with continuous compounding is

$F = Pe^{jt}$.

Where, F = Future Value

P = The present value

j = Interest rate per year

t = Time in years.

Example: Find the future value of $ 500 at 8 percent compounded continuously for 9 years and 3 months.

Solution: Here, The present value, P= $ 500

Interest rate per year, j = 8% = 0.08

Number of years, $t = (9+\frac{3}{12})$ years

$= 9.25.$

We know that, Future value, $F = Pe^{jt}$

$= \$ 500\, e^{(0.08)(9.25)}$

$= \$ 500\, e^{0.74}$

= $ 1047.97. (Answer)

8.6 Ordinary annuity: An ordinary annuity is a series of equal periodic payments in which each payment is made at the end of the period. To find out a formula, we shall use the following symbols:

n = Number of periods

i = Interest rate per period

R = Payment per period

F = Future value of the annuity

Since, ordinary annuity is a series of equal periodic payments, the first payment of Tk. R accumulates interest for n-1 periods, the second payment of Tk. R for n-2 periods, etc. The next-to-last payment R accumulates one period of interest, and the last payment R accumulates no interest. So using the future value formula for compound interest, we set the future value of the annuity:

$F = R(1+i)^{n-1} + R(1+i)^{n-2} + \text{------} + R(1+i)^1 + R$

$= R1 + R(1+i)^1 + - - - + R(1+i)^{n-2} + R(1+i)^{n-1}$ [This is a geometric progression of first term R and common ratio 1+i > 1.]

$= R\left[\frac{(1+i)^n - 1}{(1+i) - 1}\right]$

$= R\left[\frac{(1+i)^n - 1}{i}\right]$

That is, $F = R\left[\frac{(1+i)^n - 1}{i}\right]$

Example: If $100 is deposited in an account at the end of every quarter for the next 10 years, how much will be in the account at the time of the final deposit if interest is 8% compounded quarterly?

Solution: Here, Payment per period, R = $100

Number of periods, n = (10 years) (4 quarters per year)

= 40 periods

Interest rate per period, I = $\frac{8\%}{4}$ = 2% = 0.02

The future value, F =?

We know that, F = R [$\frac{(1+i)^n - 1}{i}$]

= R [$\frac{(1+0.02)^{40} - 1}{0.02}$] dollar

= 100 (60.401983) dollar

= 6040.20 dollar (Answer)

Example: A company issues $ 1 million of bonds and sets up a sinking fund at 8 percent compounded quarterly to accumulate $ 1 million 15 years hence to redeem the bonds. Find the quarterly payment to the sinking fund. [AUB-02, CMA-96]

Solution: Here, The future value, F = $ 1 million = $ 10,00,000

The rate of interest, i = $\frac{8\%}{4}$ = 2% = 0.02

Number of periods, n = (15 years) (4 quarters per year)= 60 periods

Periodic payment, R =?

We Know that,

F = R [$\frac{(1+i)^n - 1}{i}$]

10,00,000 = R [$\frac{(1+0.02)^{60} - 1}{0.02}$]

10,00,000 = R (114.0515)

R = $\frac{10,00,000}{114.0515}$

R = 8767.97.

So, the payment of every period = $ 8767.97 (Answer)

8.6.1 <u>Present value of ordinary annuity</u>: Present value annuity calculations arise when we wish to determine what lump sum must be deposited in an account now if this some and the interest it earns are to provides equal payment for a stated number of periods, with the last payment making the account balance zero.

Form the compound interest formula we found that

Present value, $P = F(1 + i)^{-n}$ [F is future value.]

So the present value of any amount due n periods from now is found by multiplying the amount by the compound discount factor,

$$(1 + i)^{-n}$$

If we multiply the future amount of an annuity of R per period, R $\left[\frac{(1+i)^n - 1}{i}\right]$, by the compound discount factor we have the present value of an annuity of R per period. Hence

$$P = R\left[\frac{(1+i)^n - 1}{i}\right](1 + i)^{-n}$$

$$\text{So, } P = R\left[\frac{1-(1+i)^{-n}}{i}\right];$$

where, P = Present value

R = Payment of every period

i = Interest rate for a period

n = Total numbers of periods.

The process of repaying a loan by installment payments is referred to as **amortizing a loan**. To find the amortization payment we use the above formula. In the formula R is referred to as amortization payment or installment payment per period.

<u>Example</u>: a) A sum of money invested now at 10 percent compounded semiannually is to provide payments of \$ 1500 every 6 months for 8 years, the first payment due 6 months from now. How much should be invested? b) How much interest will the investment earn?

Solution: Here, number of periods, n = (8 years) (2 periods per year)

= 16 periods

Interest rate, $i = \frac{10\%}{2} = 5\% = 0.05$

Payment for every period, R = \$ 1500.

Present Value, P =?

We know that, $P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$

$$= \$\,1500\left[\frac{1-(1+0.05)^{-16}}{0.05}\right]$$

= \$ 1500 (10.83777)

= \$ 16256.65

So, \$ 16256.65 should be invested.

b) The amount of interest = (16 × 1500 – 16256.65) dollar

= 7743.35 Dollar.

<u>Example</u>: To buy a computer Nahar borrowed Tk.50000.00 at 10% interest per year, compounded quarterly. She will amortize the debt by equal payments each quarter over 15 years. a) Find the quarterly payment. b) How mach interest will be paid?

Solution: a) Here, number of periods, n = (15 years) (4 periods per year)

= 60 periods

Interest rate, $i = \frac{10\%}{4} = 2.5\% = 0.025$

Present Value, P = Tk.50000.00

Payment for every period, R = ?

We know that, $P = R[\frac{1-(1+i)^{-n}}{i}]$

Or, $50000 = R[\frac{1-(1+0.025)^{-60}}{0.025}]$

Or, $50000 = R(30.908656)$

Or, $R = \frac{50000}{30.908656}$

So, $R = 1617.67$

Therefore, the quarterly payment = Tk.1617.67

b) Payment for 60 quarters will be Tk.1617.67 $\times$ 60 = Tk.97060.20

So, interest paid will be (Tk.97060.20 – Tk.50000) = Tk.47060.20

8.7 <u>Exercises</u>:

1. Discuss simple and compound interest formulae.
2. What is difference between simple and compound interest?
3. What do you mean by period in compound interest? Discuss that the interest will increase as the duration of a period decreases.
4. What do you understand by amortization of loan?
5. Compute the interest on Tk.5480 at $9\frac{1}{4}$ % for 9 months. (Answer: Tk.380.18)
6. Find the interest rate if Tk.5250 earns Tk.55 interest in 6 months. (Answer: 2.1%)
7. Find the exact and ordinary interest on $ 2190 for 75 days at 12 percent interest. (Answer: $54.00, $54.75)
8. Find the future value if Tk.1000 is borrowed at 10 percent for 5 years. (Answer: Tk.1250)
9. Find the future value if Tk.20000 is invested at 6 percent for 3 months. (Answer: Tk.20300)
10. Compute the future value of Tk.480 at $6\frac{1}{4}$ % interest for 1 year and 6 months. (Answer: Tk.525)

11. Liton has invested $5350 in a savings account that pays 12% simple interest. How long will it be, in years, until the investment amount to Tk.10165?

(Answer: 7.5 years)

12. Lalu received Rs.50 for a diamond at a pawnshop and a month later paid Rs.53.50 to get the diamond back. Find the present interest rate. (Answer: 84%)
13. (a) A borrower signs a deed promising to pay a bank Tk.15000 in 5 years from now. How much will the borrower receive if the bank discount rate is 10%? (b) How much would the borrower have to repay in order to receive Tk.15000 now? (Answer: (a) Tk.7500, (b) Tk.30000)
14. Find the Future value of Tk.500 at 8% compounded quarterly for 10 Years. (Answer: Tk.1104.02)
15. Find the future value of Rs.10000 at 12% compounded monthly for 3 years and 4 months. (Answer: Rs.14888.64)
16. Compute the future value of Tk.5000 at 9 percent compounded monthly for 10 years. (Answer: Tk.12256.79)
17. How many years it will take for $5630 to grow to $12657.45 where interest rate is 10% and compounded yearly? (Answer: 8.5 years)
18. Find how many months it will take at 11% compounded quarterly of a year for Tk.550 to grow to Tk.946.24. (Answer: 60 months)
19. Find the rate of interest that compounded yearly for $1200 to grow to $1636.40 in 5 years. (Answer: 6.4%)
20. Find the rate of interest that compounded semiannually for $1550 to grow to $3144.32 in 10 years. (Answer: 7.2%)
21. A man built up a scholarship fund to give prize of $2562.50 every year. If the fund provides 10% interest compounded semiannually, what is the amount of the fund? (Answer: $25000.00)
22. Find the effective rate of 24 percent compounded monthly. (Answer: 26.824%)
23. How much will a deposit of 5000 taka grow to in 20 years at 6% interest compounded continuously? (Answer: 16600.59 taka)
24. Sums of Tk.1000 are deposited in an account at the end of each 6 months period for 9 years. Find the amount in the account after the last deposit has been made if interest is earned at the rate of 10% compounded semiannually. (Answer: Tk.28132.39)
25. How much should be deposited in a sinking fund at the end of each quarter for 8 years to accumulate Tk.15000 if the fund earns 10 percent compounded quarterly? (Answer: Tk.311.52)
26. a) A sum of money invested now at 10 percent compounded quarterly is to provide payments of $ 1000 every 3 months for 12 years, the first payment due 3 months from now. How much should be invested? b) How much interest will the investment earn? (Answer: (a) $27773.15, (b) $20226.85)
27. A real estate developer borrows Tk.100000 at 12% compounded monthly. The debt is to be discharged by monthly payments for the next 6 years. (a) Find the

monthly payment. (b) How much interest will be paid? (Answer: (a) Tk.1955.02, (b) Tk.40761.44)

28. (a) Determine the annual payment necessary to repay a Tk.350000 loan if interest is computed at 9% per year, compounded annually. Assume the period of the loan is 6 years. (b) How much interest will be paid over the 6-year period? (Answer: (a) Tk.78021.92, (b) Tk.118131.52)
29. (a) Determine the quarterly payment necessary to repay a Tk.25000 loan if interest is computed at the rate of 14% per year, compounded quarterly. Assume the loan is to be repaid in 10 years. (b) How much interest will be paid over the 10-year period? (Answer: (a) Tk.1170.68, (b) Tk.21827.20)

09

Chapter

Limit and Continuity

Highlights:

9.1 Introduction: Limit and continuity are the core concept in the development of calculus. In the calculus there is often an interest in the limiting value of a function as the independent variable approaches some specific real number. There are different procedures for finding the limit of a function. One procedure is simply to substitute the value x = a in the function. Another one procedure is to substitute values of the independent variable into the function while observing the behavior of f(x) as the value of x comes closer and closer to a from both sides. In an informal sense, a function is said to be continuous if it can be sketched without lifting our pen or pencil from the paper (that is, it has no jumps, no breaks and no gaps). A function, which is not continuous, is termed discontinuous.

9.2 Limit: If corresponding to a positive number $\in$, however small, we are able to find a number δ such that $|f(x)-l|<\in$ for all values of x satisfying $|x-a|<\delta$ then we say that f (x) → l as x → a; and write this symbolically as

$\lim_{x\to a} f(x)=l,$ this is the limiting value as x→a.

Note: It should be remembered that the function may not actually reach the limit l but it may get closes and closer to l as x approaches a so that $|f(x)-l|$ is less than any given value.

Another definition of limit: The limit of $f(x)$ as x approaches a is *l*,

$$\lim_{x\to a} f(x) = l,$$

if and only if $f(x)$ approaches '*l*' as x approaches 'a' along any sequence of values.

<u>Example</u>: Let us have a function

$f(x) = x^2 - 2$

The function approaches the limit 7 as x approaches 3, we can express it is

$\lim_{x\to 3}(x^2 - 2) = 7$. This can be shown below first with x approaching closer and closer to 3 from the lower side:

When $x = 2.99$, $f(x) = 6.9401$

” $x = 2.999$, $f(x) = 6.994001$

” $x = 2.9999$, $f(x) = 6.99940001$

” $x = 2.99999$, $f(x) = 6.9999400001$

So, when x approaching closer and closer to 3 from the lower side, then f(x) tends to 7.
Now when x approaches 3 from the higher side, we have

When $x = 3.01$, $f(x) = 7.0601$

” $x = 3.001$, $f(x) = 7.006001$

” $x = 3.0001$, $f(x) = 7.00060001$

” $x = 3.00001$, $f(x) = 7.0000600001$

Thus, when x approaching closer and closer to 3 from the higher side, then f(x) tends to 7.
It is evident from the above that as x is taken closer and closer to 3 from any side, $f(x)$ moves closer and closer to 7.

9.3 <u>Difference between $\lim_{x\to a} f(x)$ and $f(a)$</u>: The statement $\lim_{x\to a} f(x)$ is a statement about the value of f(x) when x has any value arbitrarily near to a, except a. In this case, we do not care to know what happens when x is put equal to a. But f(a) stands for the value of f(x) when x is exactly equal to a, obtained either by the definition of the function at a, or else by substitution of a for x in the expression f(x), when it exists.

9.4 <u>Methods of evaluating limit of a function</u>: The following are some theorems on the limits, which are often used for evaluating the limits of a function.

If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} Q(x) = m$, then

1) $\lim_{x\to a}\left[f(x) \pm Q(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} Q(x) = l \pm m,$

2) $\lim_{x\to a}\left[f(x).Q(x)\right] = \lim_{x\to a} f(x).\ \lim_{x\to a} Q(x) = lm,$

3) $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x) = k.l$

4) $\lim\limits_{x\to a} \dfrac{f(x)}{Q(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} Q(x)} = \dfrac{l}{m}$

5) $\lim\limits_{x\to a} \log f(x) = \log \{\lim\limits_{x\to a} f(x)\} = \log l.$

6) $\lim\limits_{x\to a} \{f(x)\}^n = \{\lim\limits_{x\to a} f(x)\}^n$

The proofs of these formulae are beyond the scope of this book.

9.5 <u>Some important limits</u>: 1) $\lim\limits_{x\to a}(c) = c;\quad c$ is constant.

2) $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x-a} = na^{n-1}$

3) $\lim\limits_{n\to 0}(1+n)^{1/n} = \lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$

4) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{x\to 0} \dfrac{\tan x}{x} = \lim\limits_{x\to 0}$ cos x =1

The proofs of these limits are beyond the scope of this book.

9.6 <u>Left hand side and right hand side limits</u>:

L.H.S $\lim\limits_{x\to a^-} f(x) = \lim\limits_{h\to 0} f(a-h) =$ limit of $f(x)$;

when x approaches "a" from the L.H.S

R.H.S $\lim\limits_{x\to a^+} f(x) = \lim\limits_{h\to 0} f(a+h) =$ Limit of $f(x)$;

when x approaches "a" from the R.H.S

Note: If $\lim\limits_{x\to a^-} f(x) = l = \lim\limits_{x\to a^+} f(x)$, then we say limit of f(x) exists when x→a and is *l*, that is $\lim\limits_{x\to a} f(x) = l$; otherwise, we say the limit does not exist.

<u>Example</u>: If $f(x) = \begin{cases} x, & \text{if } x > 0 \\ -1, & \text{if } x \le 0 \end{cases}$

Find $\lim\limits_{x\to 0} f(x)$.

Solution: L.H.S $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}(-1) = -1$ [For x ≤ 0, f(x) = -1]

And R.H.S $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}(x) = 0$ [For x > 0, f(x) = x]

Since, L.H.S limit = -1 ≠ 0 = R.H.S limit, the limit

$\lim\limits_{x\to 0} f(x)$ does not exist.

The graph of the function is as follows:

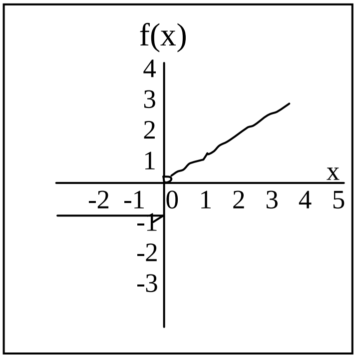


Figure 9.1

<u>Example</u>: Find f(5) and $\lim\limits_{x\to 5} f(x)$ where f(x) = $3x^2 + 2$.

Solution: Given that, f(x) = $3x^2 + 2$

So, f(5) = $3(5)^2 + 2$

= 77. (Answer)

And $\lim\limits_{x\to 5} f(x) = \lim\limits_{x\to 5}(3x^2 + 2)$

$= \lim\limits_{x\to 5}(3x^2) + \lim\limits_{x\to 5}(2)$

[Since, $\lim\limits_{x\to a}\left[f(x) \pm Q(x)\right] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} Q(x)$]

$= 3\lim\limits_{x\to 5}(x^2) + 2$

$= 3 \times 25 + 2$

= 77. (Answer)

<u>Example</u>: Evaluate $\lim\limits_{x\to 1} f(x)$ *where* $f(x) = 5x^3 + 3x + 1.$

Solution: Given that, $f(x) = 5x^3 + 3x + 1.$

So, $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}(5x^3 + 3x + 1)$

$= 5 \times 1 + 3 \times 1 + 1$

= 9 (Answer)

<u>Example</u>: Find the values of f(3) and $\lim\limits_{x\to 3} f(x)$ where f(x) = $\dfrac{(2x+1)(x-3)}{(5x-2)(x-3)}$

Solution: Given that, f(x) = $\dfrac{(2x+1)(x-3)}{(5x-2)(x-3)}$

So, f(3) = $\dfrac{(2\times 3+1)(3-3)}{(5\times 3-2)(3-3)} = \dfrac{7\times 0}{13\times 0} = \dfrac{0}{0}$, which is indeterminate form.

Therefore, f(3) does not exist.

Now, $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \frac{(2x+1)(x-3)}{(5x-2)(x-3)}$

$$= \lim_{x\to 3} \frac{(2x+1)}{(5x-2)}$$

$$= \frac{\lim\limits_{x\to 3}(2x+1)}{\lim\limits_{x\to 3}(5x-2)} \qquad \text{[Since, } \lim_{x\to a}\frac{f(x)}{Q(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} Q(x)}\text{]}$$

$$= \frac{7}{13} \qquad \text{(Answer)}$$

Example: Prove that $\lim\limits_{x\to 0}\frac{\sqrt{a+x^2}-\sqrt{a-x^2}}{x^2} = \frac{1}{\sqrt{a}}$ [AUB-02]

Solution: If we put x = 0 in given function, we get $\frac{0}{0}$, which is indeterminate form. In such cases rationalizing the numerator, we have

$$\lim_{x\to 0}\frac{\sqrt{a+x^2}-\sqrt{a-x^2}}{x^2} = \lim_{x\to 0}\left[\frac{\left(\sqrt{a+x^2}-\sqrt{a-x^2}\right)\left(\sqrt{a+x^2}+\sqrt{a-x^2}\right)}{x^2\left(\sqrt{a+x^2}+\sqrt{a-x^2}\right)}\right]$$

$$= \lim_{x\to 0}\frac{\left(\sqrt{a+x^2}\right)^2-\left(\sqrt{a-x^2}\right)^2}{x^2\left(\sqrt{a+x^2}+\sqrt{a-x^2}\right)}$$

$$= \lim_{x\to 0}\frac{\not{a}+x^2-\not{a}+x^2}{x^2\left(\sqrt{a+x^2}+\sqrt{a-x^2}\right)}$$

$$= \lim_{x\to 0}\frac{2\not{x}^2}{\not{x}^2\left(\sqrt{a+x^2}+\sqrt{a-x^2}\right)}$$

$$= \lim_{x\to 0}\frac{2}{\sqrt{a+x^2}+\sqrt{a-x^2}}$$

$$= \frac{2}{\sqrt{a}+\sqrt{a}} = \frac{\not{2}}{\not{2}\sqrt{a}} = \frac{1}{\sqrt{a}} \qquad \text{(Proved)}$$

Example: Evaluate $\lim\limits_{x\to 3}\frac{x^2+2x-15}{x^2-9}$

Solution: Replacing x by 3 in the expression, we get $\frac{0}{0}$, which is indeterminate form, (x – 3) must therefore be a factor of the numerator as well as of the denominator. Factorizing, we get

$$\lim_{x \to 3} \frac{x^2 + 2x - 15}{x^2 - 9} = \lim_{x \to 3} \frac{x^2 + 5x - 3x - 15}{x^2 - 3^2}$$

$$= \lim_{x \to 3} \frac{x(x+5) - 3(x+5)}{(x+3)(x-3)}$$

$$= \lim_{x \to 3} \frac{(x+5)(x-3)}{(x+3)(x-3)}$$

$$= \lim_{x \to 3} \frac{(x+5)}{(x+3)} \quad \text{[For all } x \neq 3\text{]}$$

$$= \frac{8}{6}$$

$$= \frac{4}{3} \quad \text{(Answer)}$$

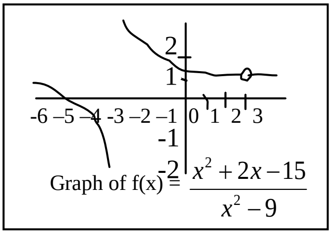


Figure 9.2

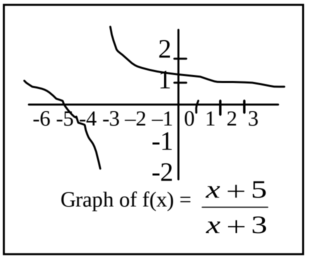


Figure 9.3

Example: Find the limiting value of $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$ [NU-99 Mgt.]

Solution: Given that $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$

$$= \lim_{h \to 0} (1+h)^{\frac{a}{h}} \qquad \left[\text{Let } \frac{a}{x} = h, \text{ so } h \to 0 \text{ as } x \to \infty\right]$$

$$= \lim_{h \to 0} \left\{(1+h)^{\frac{1}{h}}\right\}^a$$

$$= \left\{\lim_{h \to 0} (1+h)^{1/h}\right\}^a \quad \left[\text{Since } \lim_{x \to a} \{f(x)\}^n = \left\{\lim_{x \to a} f(x)\right\}^n\right]$$

$$= e^a \qquad \left[\text{We know, } \lim_{n \to 0} (1+n)^{1/n} = e\right]$$

9.7 <u>Continuity</u>: A function $f(x)$ is said to be continuous at a point 'a' if the following three conditions are met

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

If a function is not continuous at a point 'a', we say it is discontinuous at 'a' or has a discontinuity at 'a'.

Example: Discuss the continuity of $f(x) = \dfrac{x+2}{x^2-5x+6}$

Solution: First, we factor the denominator

$$f(x) = \frac{x+2}{x^2-3x-2x+6}$$
$$= \frac{x+2}{x(x-3)-2(x-3)}$$
$$= \frac{x+2}{(x-3)(x-2)}$$

Condition 1: $f(x)$ is defined for all x except x = 2 and x = 3.

Condition 2: By definition of limit, $\lim\limits_{x\to a} f(x)$ exists for all x except x = 2 and x = 3.

Condition 3: Again by definition of limit, we find that $\lim\limits_{x\to a} f(x) = f(a)$ for all x except x = 2 and x = 3.

Hence, $f(x)$ is continuous at all x except x = 2 and x = 3, where it has discontinuities.

Example: Show that the function $f(x)$ as defined below, is discontinuous at $x = \dfrac{1}{2}$.

$$f(x) = \begin{cases} x & , for \quad 0 \le x < 1/2 \\ 1 & , for \quad x = 1/2 \\ 1-x & , for \quad 1/2 < x < 1 \end{cases}$$

[AUB-03, DU-88]

Solution: We are given that

$f(x)$ =1 when $x = \dfrac{1}{2}$, which means that $f(\dfrac{1}{2}) = 1$

$\therefore$ the first condition is satisfied.

Now let us find $\lim\limits_{x\to \frac{1}{2}^-} f(x)$ and $\lim\limits_{x\to \frac{1}{2}^+} f(x)$

$\therefore$ L.H.S $\lim\limits_{x\to \frac{1}{2}^-} f(x) = \lim\limits_{x\to \frac{1}{2}^-} (x) = \dfrac{1}{2}$

And R.H.S $\lim\limits_{x\to \frac{1}{2}^+} f(x) = \lim\limits_{x\to \frac{1}{2}^+} (1-x) = \dfrac{1}{2}$

Since, L.H.S limit = $\dfrac{1}{2}$ = R.H.S limit,

$\lim\limits_{x\to \frac{1}{2}} f(x) = \dfrac{1}{2}$, exist

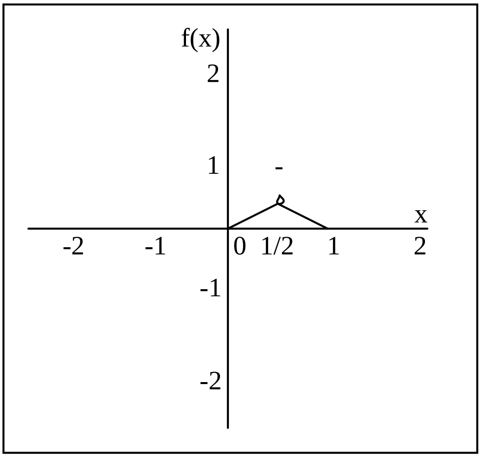


Figure 9.4

$\therefore$ the second condition is satisfied.

But $\lim\limits_{x\to\frac{1}{2}} f(x) = \frac{1}{2} \neq 1 = f(\frac{1}{2})$

So, the third condition is failed.

Therefore, the given function $f(x)$ is discontinuous at $x = \frac{1}{2}$. (Proved)

9.8 <u>Some solved problems</u>:

<u>Problem (1)</u>: If $f(x) = 10x^4 + 5x^3 + x^2 + 9$, then find the value of $f(3)$.

Solution: Given that $f(x) = 10x^4 + 5x^3 + x^2 + 9$

So, $f(3) = 10(3)^4 + 5(3)^3 + (3)^2 + 9$

$= 10 \times 81 + 5 \times 27 + 9 + 9$

$= 963$ (Answer)

<u>Problem (2)</u>: Find g(a) – g(x – a) if g(x) = x^2 +10 [AUB-01]

Solution: Given that g(x) = x^2 + 10

$\therefore g(a) = a^2 + 10$

$\therefore g(x - a) = (x - a)^2 + 10$

Now $g(a) - g(x - a) = a^2 + 10 - \{(x-a)^2 + 10\}$

$= a^2 + 10 - \{x^2 - 2ax + a^2 + 10\}$

$= a^2 + 10 - x^2 + 2ax - a^2 - 10\}$

$= - x^2 + 2ax$

$= x(2a - x)$ (Answer)

<u>Problem (3)</u>: Evaluate the limit: $\lim\limits_{x\to 3} \frac{x^2 - 9}{x - 3}$ [NU-98]

Solution: Given that, $\lim\limits_{x\to 3} \frac{x^2 - 9}{x - 3} = \lim\limits_{x\to 3} \frac{x^2 - 3^2}{x - 3}$

$= 2.3^{(2-1)}$ [Since $\lim\limits_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$]

$= 2.3$

$= 6$ (Answer)

<u>Problem (4)</u>: Evaluate the limit: $\lim\limits_{x\to 0} \frac{\sqrt{1+2x} - \sqrt{1-3x}}{x}$ [RU-90]

Solution: If we put x = 0 in given function, we get $\frac{0}{0}$, which is indeterminate form. In such cases rationalizing the numerator, we have

$$\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1-3x}}{x} = \lim_{x\to 0}\frac{\left(\sqrt{1+2x}-\sqrt{1-3x}\right)\left(\sqrt{1+2x}+\sqrt{1-3x}\right)}{x\left(\sqrt{1+3x}+\sqrt{1-3x}\right)}$$

$$= \lim_{x\to 0}\frac{\left(\sqrt{1+2x}\right)^2-\left(\sqrt{1-3x}\right)^2}{x\left(\sqrt{1+2x}+\sqrt{1-3x}\right)}$$

$$= \lim_{x\to 0}\frac{1+2x-1+3x}{x\left(\sqrt{1+2x}+\sqrt{1-3x}\right)}$$

$$= \lim_{x\to 0}\frac{5x}{x\left(\sqrt{1+2x}+\sqrt{1-3x}\right)}$$

$$= \lim_{x\to 0}\frac{5}{\sqrt{1+2x}+\sqrt{1-3x}}$$

$$= \frac{5}{1+1}$$

$$= \frac{5}{2} \qquad \text{(Answer)}$$

<u>Problem (5)</u>: Show that $\lim_{x\to 1} f(x)$ exists and is equal to $f(1)$,

where $f(x)=\begin{cases} x+1 & \text{for } x\le 1 \\ 3-x^2 & \text{for } x>1 \end{cases}$ [NU-96]

Solution: Given that f(x) = x + 1 for $x \le 1$

So, f(1) = 1 + 1
= 2

To find $\lim_{x\to 1} f(x)$, we have to find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$

$\therefore$ L.H.S $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+1) = \lim_{h\to 0}\{(1-h)+1\} = 2$

And R.H.S $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(3-x^2) = \lim_{h\to 0}\{3-(1+h)^2\} = 2$

Since, L.H.S limit = 2 = R.H.S limit,

$\lim_{x\to 1} f(x) = 2$, exist

Therefore, $\lim_{x\to 1} f(x) = f(1) = 2$ (Shown)

Note: This function satisfies all three conditions of continuity. So it is continuous at x = 1.

<u>Problem (6)</u>: Discuss the continuity of $f(x)=\dfrac{x+2}{x^2-3x+2}$

Solution: Given that, $f(x)=\dfrac{x+2}{x^2-3x+2}$

$$= \frac{x+2}{x^2-2x-x+2}$$

$$= \frac{x+2}{x(x-2)-1(x-2)}$$

$$= \frac{x+2}{(x-2)(x-1)}$$

Condition 1: $f(x)$ is defined for all x except x = 1 and x = 2.

Condition 2: By definition of limit, $\lim_{x\to a} f(x)$ exists for all x except x = 1 and x = 2.

Condition 3: Again by definition of limit, we find that $\lim_{x\to a} f(x) = f(a)$ for all x except x = 1 and x = 2.

Hence, $f(x)$ is continuous at all x except x = 1 and x = 2, where it has discontinuities.

9.9 Exercise:

1. Define limit and continuity.
2. What is difference between limit and continuity?
3. When does the limit of a function exist?
4. If $f(x) = 10x^4 + 5x^3 + x^2 + 9$, then find the value of $f(0)$. [Answer: 9]
5. If $f(x) = x^4 + 2x^2 + 5$, then find f(0), f(-2) and f(3). [Answer: 5, 29 and 104]
6. Evaluate $\lim_{x\to 2} f(x)$ *where* $f(x) = 3x^3 + 2x^2 + 6.$ [Answer: 38]
7. Find f(5) and $\lim_{x\to 5} f(x)$ where f(x) = $3x^2$ + 10x + 5 [Answer: 130, 130]
8. Show that f(2) and $\lim_{x\to 2} f(x)$ are equal where f(x) = $3x^2$ + 2x – 1.
9. For the following exercises, find the indicated limit.

(i) $\lim_{x\to 0}(3x^2 - 5x + 3)$; (ii) $\lim_{x\to -4}\frac{x^2+8x-14}{2x+7}$; (iii) $\lim_{x\to 9}\frac{x^2-81}{x-9}$;

[Answer: 3] [Answer: 14] [Answer: 18]

(iv) $\lim_{x\to 3}\frac{x^2+3x}{x+3}$; (v) $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\sqrt{1+x^2}}{x}$; (vi) $\lim_{x\to -1}\frac{x^{10}-1}{x+1}$; (vii) $\lim_{x\to\infty}\frac{2x}{x+3}$

[Answer: 3] [Answer: 0] [Answer: -10] [Answer: 2]

(viii) $\lim_{x\to\infty}\frac{x+1}{x^2+1}$ [Hints: $\frac{\frac{1}{x^2}(x+1)}{\frac{1}{x^2}(x^2+1)} = \frac{(\frac{1}{x}+\frac{1}{x^2})}{1+\frac{1}{x^2}} = \frac{h+h^2}{1+h^2}$; h→0 as x →∞]

[Answer: 0]

10. A function f(x) is defined as follows:

$$f(x) = \begin{cases} x & when \;\; x > 0 \\ 0 & when \;\; x = 0 \\ -x & when \;\; x < 0 \end{cases}$$ [Answer: 0]

Find the value of $\lim_{x \to 0} f(x)$.

11. Find $\lim_{x \to 5} f(x)$ where f(x) = $\begin{cases} 2x & for \;\; x < 5 \\ 20 - 2x & for \;\; x \ge 5 \end{cases}$

[Answer: 10]

12. Discuss the continuity of $\dfrac{x+5}{x^2+7x+12}$.

13. A function f(x) is defined as follows:

$$f(x) = \begin{cases} \frac{1}{2} - x, & when \;\; 0 < x < \frac{1}{2} \\ 0, & when \;\; x = \frac{1}{2} \\ \frac{3}{2} - x, & when \;\; \frac{1}{2} < x < 1 \end{cases}$$

Show that f(x) is continuous at x = ½ .

14. Show that the function defined below is discontinuous at x = 1

$$f(x) = \begin{cases} x+1, & for \; x > 1 \\ x, & for \; x \le 1 \end{cases}$$

10

Chapter

Differentiation and its applications

Highlights:

10.1 Introduction: Differential calculus is the most important part of mathematics. The word differentiation means the rate of change in one variable with reference to an infinitesimal variation in the other variables. There is then a dependent variable which gets an impulse for changing in the dependent variables. Differential calculus is concerned with the average rate of changes, whereas Integral calculus, by it nature, considered the total rate of changes in variables. It has a large use in business problems. For example with a given cost function it would be possible to find average change of cost, i.e., marginal cost with reference to a small change in product or in other related factors and also be found out the minimum value of the function. In this chapter we discuss nature of differentiation, how to find the differential coefficient of various functions and the use in business problems.

10.2 Differential coefficient (or Derivation): Let y = f(x) be a finite and single valued function defined in any interval of x and assume x to have any particular value in the interval. Let Δx (or h) be the increment of x, and let Δy (or k) = $f(x+\Delta x)-f(x)$ be the corresponding increment of y. If the ratio Δy/Δx of these increments tends to a definite finite limit as Δx tends to zero, then this limit is called the differential coefficient (or derivative) of $f(x)$ $(or\ y)$ for the particular value of x and is denoted by $f_1(x)$ or $f^{/}(x)$ or $\frac{d}{dx}\{f(x)\}$ or $D\{f(x)\}$ or $\frac{dy}{dx}$ (here, $\frac{d}{dx}$ is called differential operator with respect to x).

Thus, symbolically, the differential coefficient of y [= $f(x)$] with respect to x [for any particular value of x] is

$$f^{/}(x) \text{ or } \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ provided this limit exists.}$$

Note 1: The process of finding the differential coefficients is called differentiation, and we are said to differentiate $f(x)$ with respect to x.

Note 2: The right-hand limit $\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$ for any particular value of x, when it exists, is called the right-hand derivative of f(x) at that point and is denoted by $Rf^{/}(x)$. Similarly, the left-hand limit $\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$, when it exists is called the left-hand derivative of f(x) at x, denoted by $Lf^{/}(x)$. When these two derivatives both exist and are equal, it is then only that the derivative of f(x) exists at x. When, however, the left-hand and right-hand derivatives of f(x) at x are unequal, or one or both are non-existent then f(x) is said to have no proper derivative at x.

Note 3: If $f^{/}(a)$ is finite, the function f(x) must be continuous at x = a.

<u>Example</u>: If $f(x) = x^2 + 3x$ then find $f(x + \Delta x) – f(x)$. What is the change of f(x) if x changes from 5 to 5.5?

Solution: Given that $f(x) = x^2 + 3x$

Then $f(x + \Delta x) – f(x) = \{(x + \Delta x)^2 + 3(x + \Delta x)\} – (x^2 + 3x)$

$= x^2 + 2x(\Delta x) + (\Delta x)^2 + 3x + 3(\Delta x) – x^2 – 3x$

$= \Delta x(2x + \Delta x + 3)$ (Answer)

Here, $\Delta x = 5.5 – 5 = 0.5$

So, the change of f(x) is $f(5 + 0.5) – f(5) = 0.5\{2(5) + 0.5 + 3\}$

$= 0.5(13.5)$

$= 6.75$ (Answer)

<u>Example</u>: Find the differential coefficient of $f(x) = x^2$ using first principle.

Solution: Given that $f(x) = x^2$.

$\therefore$ $f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$.

We know that $f^{/}(x) \text{ or } \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h)}{h}$$

$$= \lim_{h \to 0} (2x + h)$$

$$= 2x$$

Therefore, $\frac{d}{dx}(x^2) = 2x$ (Answer)

Example: A function f(x) is defined as follows:

$$f(x) = \begin{cases} x & when \;\; x > 0 \\ 0 & when \;\; x = 0 \\ -x & when \;\; x < 0 \end{cases}$$

Show that $f'(0)$ does not exist.

Solution: We know that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\therefore \; f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

Now, $Rf'(0) = \lim_{h \to 0^+} \frac{f(h)}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$

And $Lf'(0) = \lim_{h \to 0^-} \frac{f(h)}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1$

Since the right-hand derivative is not equal to the left-hand derivative, the derivative at x = 0 does not exist.

10.3 Fundamental theorem on differentiation:

1. $\frac{d}{dx}(c) = 0$; $c = Constant \;\; with \;\; respect \;\; to \;\; x$
2. $\frac{d}{dx}\{c \; f(x)\} = c\frac{d}{dx}\{f(x)\} = c \, f'(x)$
3. $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x) = f'(x) \pm g'(x)$
4. $\frac{d}{dx}\{ f(x) \times g(x)\} = f(x)\frac{d}{dx} g(x) + g(x)\frac{d}{dx} f(x) = f(x) \, g'(x) + g(x) f'(x)$
5. $\frac{d}{dx}\{f_1(x).f_2(x)...f_n(x)\} = f_1'(x)\{f_2(x).f_3(x)...f_n(x)\} + f_2'(x)\{f_1(x).f_3(x)...f_n(x)\}$
   $+ \ldots + f_n'(x)\{f_1(x).f_2(x)...f_{n-1}(x)\}$
6. $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx} f(x) - f(x)\frac{d}{dx} g(x)}{\{g(x)\}^2} = \frac{g(x)f'(x) - f(x)g'(x)}{\{g(x)\}^2}$; $g(x) \neq o$

7. If y = f(u), u = f(x) then $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

That is, if y = f(g(x)) then $\frac{dy}{dx} = \frac{d}{dg(x)} f(g(x)) . \frac{d}{dx} g(x)$

8. $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, that is, $\frac{dy}{dx} \times \frac{dx}{dy} = 1$

9. If y = f(t) and x = g(t) then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

10. If $y = \{f(x)\}^{g(x)}$ *than* $\frac{dy}{dx} = \{f(x)\}^{g(x)} \frac{d}{dx}\{g(x).\log_e f(x)\}$

The proofs of these formulae are beyond the scope of this book.

10.4 <u>Meaning of the derivatives and differentials</u>:

- $\frac{dy}{dx} = \tan\theta$, *when* θ is the angle which the tangent at any point to the curve y = $f(x)$ makes with the positive direction of the x -axis , that means , $\frac{dy}{dx}$ is the slope of the tangent of the curve y = $f(x)$ at the point (x, $f(x)$).
- $\frac{dy}{dx}$ =Rate of change of dependent variable y with respect to independent variable x
- $dy = f^{/}(x)dx$, if y = $f(x)$.

10.5 <u>Some standard derivatives</u>:

1) $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$;

2) $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \frac{d}{dx} f(x)$;

3) $\frac{d}{dx}(e^x) = e^x$;

4) $\frac{d}{dx}(e^{mx}) = me^{mx}$;

5) $\frac{d}{dx}(a^x) = a^x \log_e a$;

6) $\frac{d}{dx}(a^{mx}) = m\, a^{mx} \log_e a$;

7) $\frac{d}{dx}(\log_e x) = \frac{d}{dx}(\ln x) = \frac{1}{x}$;

8) $\frac{d}{dx}((\log_a x) = \frac{1}{x}\log_a e$

Note: We may use these standard derivatives as formula where necessary.

Example: Find $f^{/}(x)$ if (a) $f(x) = x^{10}$; (b) $f(x) = 3x^{\frac{2}{3}}$ and (c) $f(x) = \frac{1}{x^3}$

Solution: (a) Given that $f(x) = x^{10}$

Differentiating with respect to x, we get

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(x^{10})$$

Or, $f^{/}(x) = 10x^{10-1}$ [We know $\frac{d}{dx}(x^n) = nx^{n-1}$]

So, $f^{/}(x) = 10x^9$ (Answer)

(b) Given that $f(x) = 3x^{\frac{2}{3}}$

Differentiating with respect to x, we get,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(3x^{\frac{2}{3}})$$

Or, $f^{/}(x) = 3\frac{d}{dx}(x^{\frac{2}{3}})$ [We know $\frac{d}{dx}\{c\ f(x)\} = c\frac{d}{dx}\{f(x)\}$]

Or, $f^{/}(x) = 3.\frac{2}{3}x^{\frac{2}{3}-1}$ [We know $\frac{d}{dx}(x^n) = nx^{n-1}$]

$= 2x^{-\frac{1}{3}}$ (Answer)

(c) Given that $f(x) = \frac{1}{x^3}$

$= x^{-3}$

Differentiating with respect to x, we get

$$\frac{d}{dx}f(x) = \frac{d}{dx}(x^{-3})$$

So, $f^{/}(x) = -3x^{-3-1} = -3x^{-4} = -3\frac{1}{x^4}$ (Answer)

Example: Find $h^{/}(x)$ if $h(x) = (3x^2 + 4)^{100}$

Solution: Given that $h(x) = (3x^2 + 4)^{100}$

Differentiating it we respect to x, we get

$$h^{/}(x) = \frac{d}{dx}\{(3x^2 + 4)^{100}\}$$

$=100(3x^2 + 4)^{100-1}\frac{d}{dx}(3x^2 + 4)$ [$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\frac{d}{dx}f(x)$]

$= 100(3x^2 + 4)^{99}(6x + 0)$

$= 100(3x^2 + 4)^{99}.6x$

$= 600x(3x^2 + 4)^{99}$ (Answer)

<u>Example</u>: Find the differential coefficient of $f(x) = 4x^4 + (2x+1)^3 - ¼\, e^{4x}$.

Solution: Given that, $f(x) = 4x^4 + (2x+1)^3 - ¼\, e^{4x}$

Differentiating with respect to x, we get

$$f'(x) = \frac{d}{dx}\{4x^4 + (2x+1)^3 - ¼\, e^{4x}\}$$

$$= \frac{d}{dx}(4x^4) + \frac{d}{dx}\{(2x+1)^3\} - \frac{d}{dx}(¼\, e^{4x}) \quad \left[\text{Since } \frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)\right]$$

$$= 4\frac{d}{dx}(x^4) + \frac{d}{du}(u^3)\frac{d}{dx}(2x+1) - \frac{1}{4}\frac{d}{dx}(e^{4x}) \quad [\text{Let } u = 2x+1]$$

$$= 4.4x^3 + 3u^2.2 - \frac{1}{4}.4e^{4x}$$

$$= 16x^3 + 6(2x+1)^2 - e^{4x} \quad \text{(Answer)}$$

<u>Example</u>: Find the differential coefficient of $f(x) = (\ln x)^2$.

Solution: Given that $f(x) = (\ln x)^2$.

Differentiating with respect to x, we get

$$\therefore \quad f'(x) = \frac{d}{dx}\{(\ln x)^2\} \quad [\text{Let } u = \ln x]$$

$$= \frac{d}{du}(u^2).\frac{d}{dx}(\ln x)$$

$$= 2u.\frac{1}{x}$$

$$= 2.\ln x.\frac{1}{x} \quad [\text{Putting the value of u}]$$

$$= \frac{2\ln x}{x} \quad \text{(Answer)}$$

<u>Example</u>: Find $f'(x)$ if $f(x) = (2x)^{(5x+2)}$.

Solution: Given that, $f(x) = (2x)^{(5x+2)}$.

$$\therefore \quad f'(x) = \frac{d}{dx}[(5x+2)\ln(2x)] \quad [\text{Using fundamental formula 10}]$$

$$= (5x+2)\frac{d}{dx}\{\ln(2x)\} + \ln(2x)\frac{d}{dx}(5x+2)$$

$$= (5x+2).\frac{1}{2x}.\frac{d}{dx}(2x) + \ln(2x).5$$

$$= (5x + 2).\frac{1}{2x}.2 + 5\ln(2x)$$

$$= \frac{5x+2}{x} + 5\ln(2x) \quad \text{(Answer)}$$

Example: Find the differential coefficient $\frac{dy}{dx}$ of the following implicit function:

$$x^2 + y - 2x = 0$$

Solution: Given that $x^2 + y - 2x = 0$

Differentiating with respect to x, we get

$$\frac{d}{dx}(x^2 + y - 2x) = \frac{d}{dx}(0)$$

Or, $\frac{d}{dx}(x^2) + \frac{dy}{dx} - \frac{d}{dx}(2x) = 0$

Or, $2x + \frac{dy}{dx} - 2 = 0$

So, $\frac{dy}{dx} = 2 - 2x$ (Answer)

Example: Find the differential coefficient $\frac{dy}{dx}$ of the following implicit function:

$x^2 - y^2 + 3x = 5y$ [NU-99 A/C]

Solution: Given that $x^2 - y^2 + 3x = 5y$

Differentiating with respect to x, we get

$$\frac{d}{dx}(x^2 - y^2 + 3x) = \frac{d}{dx}(5y)$$

Or, $\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) + \frac{d}{dx}(3x) = 5\frac{dy}{dx}$

Or, $2x - \frac{d}{dy}(y^2).\frac{dy}{dx} + 3 = 5\frac{dy}{dx}$

Or, $2x - 2y\frac{dy}{dx} + 3 = 5\frac{dy}{dx}$

Or, $-2y\frac{dy}{dx} - 5\frac{dy}{dx} = -2x - 3$

Or, $-(2y + 5)\frac{dy}{dx} = -(2x + 3)$

So, $\frac{dy}{dx} = \frac{2x+3}{2y+5}$ (Answer)

<u>Example</u>: Find the slope of $f(x) = x^3 - \frac{1}{2}x^2 + x + 1,$ at x = – 1. [AUB-02]

Solution: Let us consider, y = f(x)

$\therefore \quad y = x^3 - \frac{1}{2}x^2 + x + 1$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - \frac{1}{2}x^2 + x + 1,)$$

$$= \frac{d}{dx}(x^3) - \frac{d}{dx}(\frac{1}{2}x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(1)$$

$$= 3x^{3-1} - \frac{1}{2}.\frac{d}{dx}(x^2) + 1.x^{1-1} + 0$$

$$= 3x^2 - \frac{1}{2}.2\,x^{2-1} + 1.x^o + 0$$

$$= 3x^2 - x + 1.1$$

$$= 3x^2 - x + 1$$

So slope of the given curve at x = –1 is

$$\left[\frac{dy}{dx}\right]_{x=-1} = 3(-1)^2 - (-1) + 1$$

$$= 3 + 1 + 1$$

$$= 5 \qquad \text{(Answer)}$$

10.6 <u>Successive differentiation</u>: We have seen that the first derivative (or first differential coefficient) of a function $y = f(x)$ of x is in general a function of x. This new function may have a derivative, which is called the second derivative (or second differential coefficient) of $f(x)$ and is denoted by $f^{//}(x)$ or $\frac{d^2y}{dx^2}$. Similarly, the derivative of the second derivative is called the third derivative of $f(x)$ and is denoted by $f^{///}(x)$ or $\frac{d^3y}{dx^3}$, and so on for the nth derivative $f^n(x)$ or $\frac{d^ny}{dx^n}$.

Thus, if y = x^5, $\frac{dy}{dx} = 5x^4$ is the first derivative of y with respect to x.

Again, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(5x^4) = 20x^3$ is the second derivative of y with respect to x.

Again, $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(20x^3) = 60x^2$ is the third derivative of y with respect to x.

<u>Example</u>: Find the third derivative of f(x) = $5x^7 + 3x^3 - 2x^2 + 10$.
Solution: Given that f(x) = $5x^7 + 3x^3 - 2x^2 + 10$.

$$\therefore \quad f'(x) = \frac{d}{dx}(5x^7 + 3x^3 - 2x^2 + 10) = 35x^6 + 9x^2 - 4x$$

$$\text{Or,} \quad f''(x) = \frac{d}{dx}(35x^6 + 9x^2 - 4x) = 210x^5 + 18x - 4$$

So, third derivative $f'''(x) = \frac{d}{dx}(210x^5 + 18x - 4) = 1050x^4 + 18$ (Answer)

10.7 <u>Maxima, minima and point of inflection</u>: A function f(x) is said to be maximum at x = a if f(a) is greater than every other values assumed by f(x) in the immediate neighbourhood of x = a.

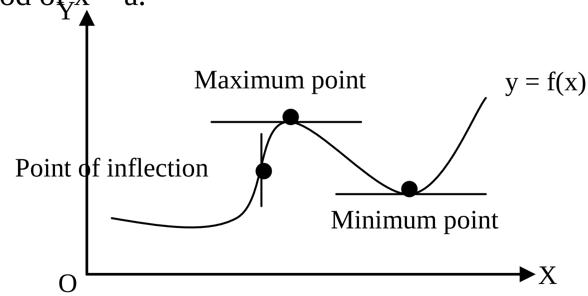


Figure 10.1

A function f(x) is said to be minimum at x = a if f(a) is less than every other values assumed by f(x) in the immediate neighbourhood of x = a. And a function f(x) has a point of inflection at x = a if f(a) is neither less nor greater than every other values assumed by f(x) in the immediate neighbourhood of x = a.
The gradient of the curve measured by $\frac{dy}{dx}$ = 0, at all turning points. At the maximum point the sign of $\frac{dy}{dx}$ changes from positive to negative as x increases and at the minimum point the sign of $\frac{dy}{dx}$ changes from negative to positive as x increases. At the point of inflection $\frac{dy}{dx}$ does not change it sign as x increases or decreases.

10.8 <u>Determination of maximum and minimum</u>:
(A) If c be a point in the interval in which the function $f(x)$ is defined, and if $f'(c) = 0$ and $f''(c) \neq 0$, then $f(c)$ is (i) a maximum if $f''(c)$ is negative and (ii) a minimum if $f''(c)$ is positive.
(B) If $f'(c) = f''(c) = \ \dots \ = f^{n-1}(c) = 0 \ \ and \ \ f^{n}(c) \neq 0$, than

(i) If n be even, $f(c)$ is a maximum or a minimum according as $f^{//}(c)$ is negative or positive.
(ii) If n be add, $f(c)$ is neither a maximum nor a minimum.

Example: Find for what values of x, the following expression is maximum and minimum respectively: $15x^4 + 8x^3 – 18x^2$
Find also the maximum and minimum values of the expression. [AUB-03]
Solution: Let us consider,

$$f(x) = 15x^4 + 8x^3 - 18x^2$$

$$\therefore f^{/}(x) = \frac{d}{dx}(15x^4 + 8x^3 - 18x^2) = 60x^3 + 24x^2 - 36x$$

And $f^{//}(x) = \frac{d}{dx}(60x^3 + 24x^2 - 36x) = 180x^2 + 48x - 36$

Now, when $f(x)$ is a maximum or a minimum, $f^{/}(x) = 0$
Therefore, we should have, $60x^3 + 24x^2 – 36x = 0$
Or, $12x(5x^2 + 2x – 3) = 0$
Or, $x(5x^2 + 2x – 3) = 0$
Or, $x(5x^2 + 5x – 3x – 3) = 0$
Or, $x\{5x(x +1) – 3(x +1)\} = 0$
Or, $x(x +1)(5x – 3) = 0$
$\therefore$ $x = 0$, $x = –1$ and $x = \frac{3}{5}$
Now, when $x = 0$, $f^{//}(0) = –36$, *which* is negative
when $x = –1$, $f^{//}(–1) = 96$, *which* is positive
And when $x = \frac{3}{5}$, $f^{//}\left(\frac{3}{5}\right) = \frac{288}{5}$, *which* is positive.

Hence, the given expression is maximum at $x = 0$ and minimum at $x = –1$ and $x = \frac{3}{5}$

Therefore, $f(0) = 0$
$f(–1) = –11$
And $f\left(\frac{3}{5}\right) = – 2.808$

Thus, the maximum value of the expression is 1 and the minimum values are –10 and –1.808.

10.9 Calculus of multivariate functions:

10.9.1 Definition of the function of two independent variables: If three variables x, y, z are so related that for every pair of values of x and y within the defined domain, z has a single definite value, z is said to be a function of the two independent variables x and y, and is denoted by $z = f(x, y)$.

$z = f(x, y) = x^2 + 2hxy + y^2$ is a function of two independent variables x and y.

10.9.2 Partial derivatives: The result of differentiating $z = f(x, y)$, with respect to x, treating y as a constant, is called the partial derivatives of z with respect to x, and is denoted by one of the symbols $\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, z_x, f_x(x, y)$ *or briefly*, f_x, etc.

Analytically, $\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$

Similarly, $\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$.

And $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y} \ (f_y \neq 0)$

Illustration: Let $z = x^2 + xy + y^2$; then

$\frac{\partial z}{\partial x} = 2x + y$; $\frac{\partial z}{\partial y} = x + 2y$. So, $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$.

Example: Find the differential coefficient $\frac{dy}{dx}$ of the following implicit function:

$$f(x, y) = x^2 + y - 2x$$

Solution: Given that $f(x, y) = x^2 + y - 2x$

Here, $f_x = 2x - 2$; $f_y = 1$

So, $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x-2}{1} = 2 - 2x$ (Answer)

10.9.3 Successive partial derivatives: Since each of the first order partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ is in general, a function of x and y, each may possess partial derivatives with respect to these two independent variables, and these are called the second order partial derivatives of z. The usual notations for these second order partial derivatives are

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right),\quad \text{i.e.,}\quad \frac{\partial^2 z}{\partial x^2}\quad \text{or}\quad f_{xx}\,.$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right),\quad \text{i.e.,}\quad \frac{\partial^2 z}{\partial x\partial y}\quad \text{or}\quad f_{xy}\,.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right),\quad \text{i.e.,}\quad \frac{\partial^2 z}{\partial y\partial x}\quad \text{or}\quad f_{yx}\,.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right),\quad \text{i.e.,}\quad \frac{\partial^2 z}{\partial y^2}\quad \text{or}\quad f_{yy}\,.$$

And always $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x}$.

Illustration: Let $z = x^2 + xy + y^2$; then $\frac{\partial z}{\partial x} = 2x + y$; $\frac{\partial z}{\partial y} = x + 2y$.

So, $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(2x + y) = 2$; $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}(x + 2y) = 1$; $\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}(2x + y) = 1$ and $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(x + 2y) = 2$.

10.9.4 <u>Determination of maximum and minimum of multivariate function</u>: If (a, b) be a point in the domain in which the function f(x, y) is defined, and if $f_x(a, b) = 0$, $f_y(a, b) = 0$ and $f_{xx}(a, b).f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0$, then f(a, b) is a maximum or a minimum according as $f_{xx}(a, b) <$ or > 0 (and consequently $f_{yy}(a, b) <$ or > 0).
But if $f_{xx}(a, b).f_{yy}(a, b) - [f_{xy}(a, b)]^2 < 0$, f(a, b) is neither a maximum nor a minimum and if $f_{xx}(a, b).f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$, further analysis is necessary.
In other word we can say that f(a, b) is a maximum (or a minimum) if the determinant formed by the Hessain, $H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y} \\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$ is positive and the leading diagonal elements are positive (or negative).

Example: Examine for extreme values of the function: $f(x, y) = x^2 + y^2 + (x + y + 1)^2$.
Solution: Given that $f(x, y) = x^2 + y^2 + (x + y + 1)^2$.
$\therefore \quad f_x = 2x + 2(x + y + 1) = 4x + 2y + 2$
$\therefore \quad f_y = 2y + 2(x + y + 1) = 2x + 4y + 2$
$\therefore \quad f_{xx} = 4,\ f_{yy} = 4$ and $f_{xy} = 2$.
The equations $f_x = 0$ and $f_y = 0$ are equivalent to

$$2x + y + 1 = 0$$
and $$x + 2y + 1 = 0$$
These give $\frac{x}{1-2} = \frac{y}{1-2} = \frac{1}{4-1}$

Or, $\frac{x}{-1} = \frac{y}{-1} = \frac{1}{3}$

Or, $x = -\frac{1}{3}$ and $y = -\frac{1}{3}$

The function may have an extreme value at $(-\frac{1}{3}, -\frac{1}{3})$

Now, $f_{xx}.f_{yy} - (f_{xy})^2 = 4.4 - 2^2 = 12 > 0$,
Also $f_{xx} > 0$.

Therefore, f(x, y) is maximum at $(-\frac{1}{3}, -\frac{1}{3})$ and the maximum value is $\frac{1}{3}$.

10.9.5 <u>Constrained optimization with Lagrangian multipliers</u>: Differential calculus is also used to maximize or minimize a function subject to constraints. Consider a function f(x, y) subject to a constraint h(x, y) = k, a new function F can be formed as follows:

$$F(x, y, \lambda) = f(x, y) - \lambda[h(x, y) - k]$$

Here, F(x, y, λ) is called Lagrangian function.
Since the constraint is always set equal to 0, the value of the new function f(x, y, λ) is same as the original objective function f(x, y). Now solving simultaneously:

$$F_x(x, y, \lambda) = 0, F_y(x, y, \lambda) = 0 \text{ and } F_\lambda(x, y, \lambda) = 0$$

we shall get some critical point. At these critical points the function will be maximum or minimum according the maximum or minimum criteria of multivariate function.

Example: Determine the maximum value of the objective function $f(x, y) = 4x - x^3 + 2y$ subject to x + y = 1.
Solution: Here, the Lagrangian function: $F(x, y, \lambda) = 4x - x^3 + 2y - \lambda(x + y - 1)$

∴ $F_x = 4 - 3x^2 - \lambda$
$F_y = 2 - \lambda$
and $F_\lambda = -(x + y - 1)$

The equations $F_x = 0$, $F_y = 0$ and $F_\lambda = 0$ are equivalent to
$4 - 3x^2 - \lambda = 0$ - - - (1)
$2 - \lambda = 0$ - - - (2)
and $x + y - 1 = 0$ - - - (3)

From equation (2), we get $\lambda = 2$
Substituting $\lambda = 2$ in equation (1), we get $x = 0.8165$
Now from equation (3), we get $y = 0.1835$

So, (0.8165, 0.1835, 2) is the critical point of F(x, y, λ) as well as (0.8165, 0.1835) is the critical point of f(x, y).
Now, at the point $f_{xx} = -4.899$, $f_{xy} = 0$, $f_{yx} = 0$ and $f_{yy} = 0$.
So, $f_{xx}.f_{yy} - (f_{xy})^2 = -4.899 \times 0 - 0 = 0$, also $f_{xx} < 0$.
Hence, (0.8165, 0.1835) maximizes the objective function and the maximum value is 3.0887 (Answer)

10.10 <u>Business Applications of Differential Calculus</u>:

Key Concepts: Profit: Profit is defined as the excess of total revenue over total cost. That is, Profit = Total revenue – Total cost. If *c(x)* is the total cost and *r(x)* is the total revenue, then profit, *p(x) = r(x) – c(x).*

Marginal cost: Marginal cost is the rate of change of total cost with respect to units of production. If $c(x)$ is the total cost of producing x units of a products, $c^{/}(x)$ is the point of marginal cost.

Marginal production: Marginal production is the incremental production i.e., the additional production added to the total production. If *tp(x)* is the total production, $\frac{d}{dx}[tp(x)]$ is the marginal production.

Marginal revenue: Marginal revenue is the rate of change in revenue with respect to total output. If *tr(x)* is the total revenue, $\frac{d}{dx}[tr(x)]$ is the marginal revenue.

<u>Example</u>: If the total cost of producing p units of pen is $c(p) = 0.0015\,p^3 - 0.9p^2 + 200p + 60000$; compute the marginal cost at outputs of (a) 100 units, (b) 200 units, (c) 300 units.

Solution: Given that, cost: $c(p) = 0.0015p^3 - 0.9p^2 + 200p + 60000$
Differentiation with respect to p, we have,

$$c^{/}(x) = \frac{d}{dx}(0.0015p^3 - 0.9p^2 + 200p + 60000)$$
$$= 0.0015 \times 3p^2 - 0.9 \times 2p + 200 + 0$$
$$= 0.0045p^2 - 1.8p + 200, \text{ this the marginal cost,}$$

(a) $c^{/}(100) = 0.0045(100)^2 - 1.8(100) + 200$
$= 65$ taka per unit pen
(b) $c^{/}(200) = 0.0045(200)^2 - 1.8(200) + 200$
$= 20$ taka per unit pen
(c) $c^{/}(300) = 0.0045(300)^2 - 1.8(300) + 200$
$= 65$ taka per unit pen

<u>Example</u>: Total cost of producing x units is $\frac{5}{4}x^2 + 175x + 125$ and the price at which each unit can be sold for $250 - \frac{5}{2}x$. What should be the output for a maximum profit? Calculate the maximum profit.

Solution: Given that, cost: $c(x) = \frac{5}{4}x^2 + 175x + 125$

And revenue: $tr(x) = (250 - \frac{5}{2}x)\times x = 250x - \frac{5}{2}x^2$

So, profit: $p(x) = 250x - \frac{5}{2}x^2 - (\frac{5}{4}x^2 + 175x + 125)$

$= -125 + 75x - \frac{15}{4}x^2$

Now, $\frac{d}{dx}[p(x)] = \frac{d}{dx}(-125 + 75x - \frac{15}{4}x^2)$

$= 75 - \frac{30}{4}x$

The necessary and sufficient conditions for maximization are that the first derivative of a profit function is equal to zero and the second derivative is negative.

So, $75 - \frac{30}{4}x = 0$

Or, $\frac{30}{4}x = 75$

Or, $x = 10$

And $\frac{d^2}{dx^2}[p(x)] = -\frac{30}{4}$, which is negative.

Therefore, the profit is maximum at the output, x = 10 (Answer)

Putting x = 10 in the profit function, we get

The maximum profit $= -125 + 75\times 10 - \frac{15}{4}\times(10)^2 = 250$

Hence, the maximum profit = Tk. 250 (Answer)

10.11 <u>Some worked out examples</u>:

<u>Example (1)</u>: Find the differential coefficient of $f(x) = \sqrt{x}$ using first principle.

Solution: Given that $f(x) = \sqrt{x}$

$\therefore$ $f(x + h) = \sqrt{x+h}$

We know that $f^{/}(x)$ or $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Therefore, $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ (Answer)

<u>Example (2)</u>: Find $f^{/}(x)$ *for* $f(x) = \frac{x^{15}.x^{\frac{3}{2}}}{x^{20}}$.

Solution: Given that, $f(x) = \frac{x^{15}.x^{\frac{3}{2}}}{x^{20}}$

$$= \frac{x^{15+\frac{3}{2}}}{x^{20}}$$

$$= x^{15+\frac{3}{2}-20}$$

$$= x^{\frac{-7}{2}}$$

Differentiating with respect to x, we get

$$f^{/}(x) = \frac{d}{dx}(x^{-\frac{7}{2}})$$

$$= -\frac{7}{2.}x^{-\frac{7}{2}-1}$$

$$= -\frac{7}{2.}x^{-\frac{9}{2}} \qquad \text{(Answer)}$$

Example (3): Find the differential coefficient with respect to x of the following implicit function: $x^2 + 2hxy + y^2 = 0$; h is a constant. [NU-99 Mgt.]

Solution: Given that $x^2 + 2hxy + y^2 = 0$

Differentiating with respect to x, we get

$$\frac{d}{dx}(x^2 + 2hxy + y^2) = \frac{d}{dx}(0)$$

Or, $\frac{d}{dx}(x^2) + \frac{d}{dx}(2hxy) + \frac{d}{dx}(y^2) = 0$

Or, $2x + 2h\frac{d}{dx}(xy) + 2y\frac{dy}{dx} = 0$

Or, $2x + 2h\{x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\} + 2y\frac{dy}{dx} = 0$

Or, $2x + 2h\{x\frac{dy}{dx} + y.1\} + 2y\frac{dy}{dx} = 0$

Or, $2x + 2hx\frac{dy}{dx} + 2hy + 2y\frac{dy}{dx} = 0$

Or, $(2hx + 2y)\frac{dy}{dx} = -2x - 2hy$

Or, $\frac{dy}{dx} = \frac{-2(x+hy)}{2(hx+y)}$

So, $\frac{dy}{dx} = -\frac{(x+hy)}{(hx+y)}$ (Answer)

Example (4): Find $f^{/}(x)$ if $f(x) = (2x^3 - 3x^2 - 10)^{\frac{4}{3}} - \frac{45}{(5x^2+4)^{\frac{1}{2}}} + e^{9x}$ [AUB-02]

Solution: Given that, $f(x) = (2x^3 - 3x^2 - 10)^{\frac{4}{3}} - \frac{45}{(5x^2+4)^{\frac{1}{2}}} + e^{9x}$

Differentiating with respect to x, we get

$$f'(x) = \frac{d}{dx}\left\{\left(2x^3 - 3x^2 - 10\right)^{\frac{4}{3}} - \frac{45}{\left(5x^2+4\right)^{\frac{1}{2}}} + e^{9x}\right\}$$

$$= \frac{d}{dx}\left(2x^3 - 3x^2 - 10\right)^{\frac{4}{3}} - \frac{d}{dx}\left\{45\left(5x^2+4\right)^{-\frac{1}{2}}\right\} + \frac{d}{dx}(e^{9x})$$

$$= \frac{4}{3}\left(2x^3 - 3x^2 - 10\right)^{\frac{4}{3}-1}\frac{d}{dx}\left(2x^3 - 3x^2 - 10\right) - 45\left(-\frac{1}{2}\right)\left(5x^2+4\right)^{-\frac{1}{2}-1}\frac{d}{dx}\left(5x^2+4\right)$$
$$+ 9e^{9x}$$

$$= \frac{4}{3}6x(x-1)(2x^3 - 3x^2 - 10)^{\frac{1}{3}} + \frac{45\times 10x}{2(5x^2+4)^{\frac{3}{2}}} + 9e^{9x}$$

$$= 8x(x-1)(2x^3 - 3x^2 - 10)^{\frac{1}{3}} + \frac{225x}{(5x^2+4)^{\frac{3}{2}}} + 9e^{9x} \qquad \text{(Answer)}$$

Example (5): Find the differential coefficient of $y = e^{ax^2+bx+c} + (\ln x)^5$ [RU-96 A/C]

Solution: Given that $y = e^{ax^2+bx+c} + (\ln x)^5$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[e^{ax^2+bx+c} + (\ln x)^5]$$

Or, $\frac{dy}{dx} = \frac{d}{dx}(e^{ax^2+bx+c}) + \frac{d}{dx}[(\ln x)^5]$

Or, $\frac{dy}{dx} = e^{ax^2+bx+c}\cdot\frac{d}{dx}(ax^2 + bx + c) + 5(\ln x)^4\frac{d}{dx}(\ln x)$

Or, $\frac{dy}{dx} = e^{ax^2+bx+c}.(2ax + b) + 5(\ln x)^4.\frac{1}{x}$

So, $\frac{dy}{dx} = (2ax + b)e^{ax^2+bx+c} + \frac{5}{x}(\ln x)^4$ (Answer)

Example (6): Find the differential coefficient of f(x), where

$$f(x) = (2x+1)(4x-5)^{\frac{1}{2}} - \frac{3x+2}{(6x^2+5)^{\frac{1}{3}}} + e^{2x}\log_e 2x \qquad \text{[AUB-03]}$$

Solution: Given that,

$$f(x) = (2x+1)(4x-5)^{\frac{1}{2}} - \frac{3x+2}{(6x^2+5)^{\frac{1}{3}}} + e^{2x}\log_e 2x$$

Differentiation with respect to x, we get

$$f^{/}(x)=\frac{d}{dx}\left\{(2x+1)(4x-5)^{\frac{1}{2}}-\frac{3x+2}{(6x^2+5)^{\frac{1}{3}}}+e^{2x}\log_e 2x\right\}$$

$$=\frac{d}{dx}\left\{(2x+1)(4x-5)^{\frac{1}{2}}\right\}-\frac{d}{dx}\left\{\frac{3x+2}{(6x^2+5)^{\frac{1}{3}}}\right\}+\frac{d}{dx}\left(e^{2x}\log_e 2x\right)$$

$$=(2x+1)\frac{d}{dx}(4x-5)^{\frac{1}{2}}+(4x-5)^{\frac{1}{2}}\frac{d}{dx}(2x+1)$$

$$-\frac{(6x^2+5)^{\frac{1}{3}}\frac{d}{dx}(3x+2)-(3x+2)\frac{d}{dx}(6x^2+5)^{\frac{1}{3}}}{\left\{(6x^2+5)^{\frac{1}{3}}\right\}^2}+e^{2x}\frac{d}{dx}(\log_e 2x)+\log_e 2x\frac{d}{dx}(e^{2x})$$

$$=$$

$$(2x+1).\frac{1}{2}(4x-5)^{\frac{1}{2}-1}.4+(4x-5)^{\frac{1}{2}}.2-\frac{(6x^2+5)^{\frac{1}{3}}.3-(3x+2).\frac{1}{3}(6x^2+5)^{\frac{1}{3}-1}.12x}{(6x^2+5)^{\frac{2}{3}}}$$

$$+e^{2x}.\frac{1}{2x}.2+\log_e 2x.2e^{2x}$$

$$=2(2x+1)(4x-5)^{-\frac{1}{2}}+2(4x-5)^{\frac{1}{2}}-\frac{3(6x^2+5)^{\frac{1}{3}}-4x(3x+2)(6x^2+5)^{-\frac{2}{3}}}{(6x^2+5)^{\frac{2}{3}}}$$

$$+\frac{1}{x}e^{2x}+2e^{2x}\log_e 2x$$

$$=\frac{2(2x+1)}{(4x-5)^{\frac{1}{2}}}+2(4x-5)^{\frac{1}{2}}-\frac{3(6x^2+5)^{\frac{1}{3}}-4x(3x+2)(6x^2+5)^{-\frac{2}{3}}}{(6x^2+5)^{\frac{2}{3}}}+$$

$$e^{2x}\left(\frac{1}{x}+\log_e 2x\right)$$

Example (7): Find $\frac{dy}{dx}$ of $y = x^{x^{x}}$. [RU-91 Mgt., RU-90 A/C]

Solution: Given that $y = x^{x^{x}}$.

Taking natural logarithm on both sides, we get

$$\log y = x^x \log x$$

Again taking natural logarithm on both sides, we get

$$\log(\log y) = x\log x + \log(\log x)$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}\ (\log(\log y)\,) = \frac{d}{dx}(\,x\log x + \log(\log x)\,)$$

Or, $\dfrac{1}{\log y}.\dfrac{d}{dx}(\log y) = x\dfrac{d}{dx}(\log x) + \log x.\dfrac{d}{dx}(x) + \dfrac{1}{\log x}\dfrac{d}{dx}(\log x)$

Or, $\dfrac{1}{\log y}.\dfrac{1}{y}\dfrac{dy}{dx} = x.\dfrac{1}{x} + (\log x).1 + \dfrac{1}{\log x}.\dfrac{1}{x}$

Or, $\dfrac{dy}{dx} = y\log y\left(1 + \log x + \dfrac{1}{x\log x}\right)$

Or, $\dfrac{dy}{dx} = x^{x^{x}}.\,x^x \log x\left(1 + \log x + \dfrac{1}{x\log x}\right)$

Or, $\dfrac{dy}{dx} = x^{x^{x}}.[x^x \log x(1 + \log x) + x^{x-1}]$

<u>Example (8)</u>: Find $f^{/}(x)$ if $f(x) = (12x^2 + 6)^{(5x+2)}$ [AUB-01]

Solution: Given that, $f(x) = (12x^2 + 6)^{(5x+2)}$

Taking natural logarithm of bath sides, we get

$$\log_e f(x) = \log_e (12x^2 + 6)^{(5x+2)}$$
$$= (5x + 2)\log_e (12x^2 + 6)$$

Now differentiating with respect to x, we have

$$\frac{d}{dx}\{\log_e f(x)\} = \frac{d}{dx}\{(5x+2)\log_e(12x^2+6)\}$$

Or, $\dfrac{1}{f(x)}\dfrac{d}{dx}[f(x)] = (5x+2)\dfrac{d}{dx}\{\log_e(12x^2+6)\} + \log_e(12x^2+6)\dfrac{d}{dx}(5x+2)$

Or, $\dfrac{d}{dx}[f(x)] = f(x)[(5x+2).\dfrac{1}{12x^2+6}.\dfrac{d}{dx}(12x^2+6) + \log_e(12x^2+6).5]$

Or, $f^{/}(x) = (12x^2+6)^{(5x+2)}[\dfrac{5x+2}{6(2x^2+1)}(24x) + 5\log_e(12x^2+6)$

$\therefore\ f^{/}(x) = (12x^2+6)^{(5x+2)}[\dfrac{4x(5x+2)}{2x^2+1} + 5\log_e(12x^2+6)]$ (Answer)

<u>Example (9)</u>: Find the maximum and minimum value of $f(x) = x^3 - 4x^2 + 4x - 10$.

[RU-91 A/C]

Solution: Given that, $f(x) = x^3 - 4x^2 + 4x - 10$.

$\therefore \quad f'(x) = \frac{d}{dx}(x^3 - 4x^2 + 4x - 10) = 3x^2 - 8x + 4$

And $f''(x) = \frac{d}{dx}(3x^2 - 8x + 4) = 6x - 8$

Now, when $f(x)$ is a maximum or a minimum, $f'(x) = 0$

Therefore, we should have, $3x^2 - 8x + 4 = 0$

Or, $3x^2 - 6x - 2x + 4 = 0$

Or, $3x(x - 2) - 2(x - 2) = 0$

Or, $(x - 2)(3x - 2) = 0$

$\therefore$ $x = 2$ and $x = \frac{2}{3}$

Now, when x = 2, $f''(2) = 4$, *which* is positive.

And when $x = \frac{2}{3}$ $f''\left(\frac{2}{3}\right) = -4$, *which* is negative.

Hence, the given function is maximum at $x = \frac{2}{3}$ and minimum at x = 2

Therefore, $f(2) = -10$

And $f\left(\frac{2}{3}\right) = -\frac{238}{27} = -8.815$

Thus, the maximum value of the function is – 8.815 and the minimum values is –10

<u>Example (10)</u>: Average cost of each radio for producing x radios is $x^2 - 10x + 30$ (in hundred taka). Find the marginal cost when x = 6. [RU-87 A/C & RU-95 BBA]

Solution: Given that, average cost = $x^2 - 10x + 30$

So, the total cost: $c(x) = (x^2 - 10x + 30)x = x^3 - 10x^2 + 30x$

We know that, marginal cost = $\frac{d}{dx}[c(x)]$

$$= \frac{d}{dx}(x^3 - 10x^2 + 30x)$$

$$= 3x^2 - 20x + 30$$

When x = 6, marginal cost = $3(6)^2 - 20(6) + 30$ hundred taka = Tk.1800 (Answer)

<u>Example (11)</u>: A study has shown that the cost of producing pencils of a manufacturing concern is given by $c(x) = 30 + 1.5x + 0.0008x^2$. What is the marginal cost at x = 1000 units? If the pencils are sold for Tk.5 each for what values of x does marginal cost equal to marginal revenue? [AUB-99]

Solution: Given that cost: $c(x) = 30 + 1.5x + 0.0008x^2$.

We know that, marginal cost = $\frac{d}{dx}[c(x)]$

$$= \frac{d}{dx}(30 + 1.5x + 0.0008x^2)$$

$$= 1.5 + 0.0016x$$

Putting the value x = 1000, we get

Marginal cost = $1.5 + 0.0016\times 1000 = 3.1$ (Answer)

We know that, Total revenue [tr(x)] = Selling × price output.

So, tr(x) = 5x

We know that, Marginal revenue = $\frac{d}{dx}[tr(x)]$

$$= \frac{d}{dx}(5x)$$

$$= 5$$

When marginal cost = marginal revenue, then

$1.5 + 0.0016x = 5$

Or, $0.0016x = 3.5$

Or, $x = 2187.5$

Therefore, marginal cost will equal to marginal revenue when x = 2187.5 (Answer)

Example (12): Production function of a firm is $f(x, y) = 20x + 10y - 2x^2$, where x means number of labours and y means number of units raw material. If the cost of per unit labour is Tk. 4, the cost of per unit raw material is Tk. 5 and total spent is Tk. 24, find the maximum production of the firm.

Solution: Here, the objective function is $f(x, y) = 20x + 10y - 2x^2$,

And the constraint is $4x + 5y = 24$.

So, the Lagrangian function: $F(x, y, \lambda) = 20x + 10y - 2x^2 - \lambda(4x + 5y - 24)$

$\therefore$ $F_x = 20 - 4x - 4\lambda$

$F_y = 10 - 5\lambda$

and $F_\lambda = -(4x + 5y - 24)$

The equations $F_x = 0$, $F_y = 0$ and $F_\lambda = 0$ are equivalent to

$20 - 4x - 4\lambda = 0$ - - - (1)

$10 - 5\lambda = 0$ - - - (2)

and $4x + 5y - 24 = 0$ - - - (3)

From equation (2), we get $\lambda = 2$. Then from equation (1), we get $x = 3$. And using $x = 3$ in equation (3), we have $y = 2.4$.

We know that Hessain, $H_F = \begin{pmatrix} F_{xx} & F_{xy} & F_{x\lambda} \\ F_{yx} & F_{yy} & F_{y\lambda} \\ F_{\lambda x} & F_{\lambda y} & F_{\lambda\lambda} \end{pmatrix}$

Here, $F_{xx} = -4$, $F_{xy} = 0$, $F_{x\lambda} = -4$, $F_{yx} = 0$, $F_{yy} = 0$, $F_{y\lambda} = -5$, $F_{\lambda x} = -4$, $F_{\lambda y} = -5$ and $F_{\lambda\lambda} = 0$.

So, the determinant formed by the Hessain is $\begin{vmatrix} -4 & 0 & -4 \\ 0 & 0 & -5 \\ -4 & -5 & 0 \end{vmatrix} = 100 > 0.$

Therefore, the production will be maximum when x = 3, y = 2.4 and the value of the maximum product = 20(3) + 10(2.4) – $2(3)^2$ = 66 units. (Answer)

10.12 <u>Exercise</u>:

1. Define differential coefficient and calculus of multivariate functions.
2. Why is differentiation necessary for business education?
3. If $f(x) = x^2 + 8x$ then find $f(x + \Delta x) - f(x)$. What the change of f(x) if x changes from 6 to 5.5? [Answer: $\Delta x(2x + \Delta x + 8)$, – 9.75]
4. Using first principle find the differential coefficient of (i) $f(x) = x^3$ (ii) $f(x) = \sqrt{ax}$ [Answer: (i) $3x^2$ (ii) $\sqrt{ax}/2x$]
5. A function f(x) is defined as follows:

$$f(x) = \begin{cases} 5x & when \ \ x > 0 \\ 0 & when \ \ x = 0 \\ -3x & when \ \ x < 0 \end{cases}$$

Show that $f'(0)$ does not exist.

6. Find the differential coefficient of the following function:

(i) $y = x^3 - 20x^2 + 3x + 4$ [Answer: $3x^2 - 40x + 3$]
(ii) $y = ax^4 + bx^2 + c$ [Answer: $4ax^3 + 2bx$]
(iii) $y = \left(x^2 + \frac{1}{x}\right)^2$ [Answer: $4x^3 - 2x^{-3} + 2$]

7. Find the first derivatives with respect to x of the following functions:

(i) $f(x) = (7x + 3)(4 - 3x)$ [Answer: $19 - 42x$]
(ii) $f(x) = x^2/(3x + 2)^{½}$ [Answer: $x(9x + 8)/2(3x + 2)^{3/2}$]
(iii) $f(x) = x^3(6x - 1)^{2/3}$ [Answer: $x^2(22x - 3)/(6x - 1)^{1/3}$]
(iv) $f(x) = 5x/(3 - 4x)$ [Answer: $15/(3 - 4x)^2$]
(v) $f(x) = (1 + x)^{2x}$ [Answer: $2(1 + x)^{2x}\{x/(1 + x) + \log(1 + x)\}$]
(vi) $f(x) = x^{\log x}$ [Answer: $2x^{\log x - 1}.\log x$]
(vii) $f(x) = (2x^2 - 5x - 8)^2$ [Answer: $2(2x^2 - 5x - 8)(4x - 5)$]
(viii) $f(x) = (1 - x)(1 - 2x)(1 - 3x)(1 - 4x)$ [Answer: $96x^3 - 150x^2 + 7x - 10$]
(ix) $f(x) = \frac{3}{\sqrt{9x}} + \frac{1}{2x^2} + \sqrt{\frac{2}{5}}$ [Answer: $-\frac{1}{2\sqrt{x^3}} - \frac{1}{x^3}$]

8. Find the $\frac{dy}{dx}$ of the following functions:

(i) $2x^2 + 3x + y^2 = 0$ [Answer: $-(4x + 3)/2y$]
(ii) $5x^3 - 3x^2 + 3y^2 - 6 = 7$ [Answer: $2(2x + 1)/(2y - 5)$]
(iii) $x^3 + 3bxy + y^3 = b^4$ [Answer: $-(bx + y^2)/(x^2 + by)$]
(iv) $ax^2 + 2mxy + by^2 = k$ [Answer: $-(ax + my)/(mx + by)$]
(v) $\log(xy) = x^2 + y^2$ [Answer: $y(2x^2 - 1)/x(1 - 2y^2)$]
(vi) $e^{xy} - 4xy = 2$ [Answer: $-y/x$]

9. Find the partial derivatives f_x, f_y and then find $\frac{dy}{dx}$ of the following functions:

(i) $f(x, y) = 2x^2 + 3x + y^2$ [Answer: $4x + 3$, $2y$, $-(4x + 3)/2y$]
(ii) $f(x, y) = ax^2 + 2mxy + by^2$ [Answer: $2ax+2my$, $2mx+2by$, $-(ax+my)/(mx+by)$]
(ii) $f(x, y) = -3x^5 + 4y^3 + 6y$ [Answer: $-15x^4$, $12y^2 + 6$, $15x^4/(12y^2 + 6)$]

10. If $z = e^{x^2 y}$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

11. Find the differential coefficient of $y = \log(\sqrt{x-1} - \sqrt{x+1})$ [Answer: $\frac{-1}{2\sqrt{x^2 - 1}}$]

12. Find the differential coefficient of $y = e^{ax^2+bx+c} + \ln x$ [NU-96 A/C]

[Answer: $(2ax + b)e^{ax^2+bx+c} + 1/x$]

13. Differentiate the function $f(x) = \log\left\{e^x \left(\frac{x-1}{x+1}\right)^{\frac{3}{2}}\right\}$ [Answer: $\frac{x^2 + 2}{x^2 - 1}$]

14. Find the differential coefficient of $y = e^{x^2+2x}$ [Answer: $(2x + 2)\, e^{x^2+2x}$]

15. Differentiate x^5 with respect to x^2. [Answer: $\frac{5}{2}x^3$]

16. Differentiate $\ln x$ with respect to x^2. [Answer: $\frac{1}{2x^2}$]

17. If $y = 2x + \frac{4}{x}$, then prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$.

18. If $y = \left(x + \sqrt{1+x^2}\right)^m$, then show that $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2 y = 0$.

19. Find for what values of x, the following expression is maximum and minimum respectively: $15x^4 + 8x^3 - 18x^2 + 1$. Find also the maximum and minimum values of the expression. [Answer: Max. at $x = 0$, min. at $x = -1$, $\frac{3}{5}$ and maximum value is 1 and minimum values are -10, -1.808]

20. Examine $f(x) = x^3 - 9x^2 + 24x - 12$ for maximum or minimum values. [Answer: Maximum at x = 2, max. value f(2) = 8 and minimum at x = 4, min. value f(4) = 4]
21. Show that $f(x) = x^5 - 5x^4 + 5x^3 - 1$ is a maximum when x = 1, a minimum when x = 3 and neither when x = 0.
22. The profit function of a company can be represented by $p(x) = x - 0.00001x^2$, where x is unit sold. Find the optimal sales volume and the amount of profit to be expected at that volume. [Answer: 50000 & 25000]
23. Total cost of producing x units is $x^3 - 10x^2 + 17x + 66$ and the price at which each unit can be sold for Tk. 5. What should be the output for a maximum profit? Calculate the maximum profit. [Answer: 6 units & Tk. 6]
24. If cost: c(x) = 50x + 30000 and profit: p(x) = 100 – 0.01x, find the functions of marginal cost and marginal revenue. Also find the value of x when marginal cost is equal to marginal revenue. [Answer: marginal cost eq^n = 50, marginal revenue eq^n = 100 – 0.02x & x = 2500]
25. The cost function and the revenue function of a company are $c(x) = 100 + 0.015x^2$ and r(x) = 3x, where x is the number of units of product, respectively. Find the number of units of product that will maximize the profit. What is the maximum profit? [Answer: 100 units & 50]
26. Show that the function $f(x, y) = x^2 + y^2 - 4x + 6y$ is minimum at (2, –3).
27. The yearly profit of A company depends upon the number of workers (x) and the number of units of advertising (y), according to the function

$$p(x, y) = 412x + 806y - x^2 - 4y^2 - xy - 50000$$

(i) Determine the number of workers and the number of units of advertising that results in maximum the profit. [Answer: x = 166, y = 80]

(ii) Determine the maximum profit. [Answer: 16595]

28. The total cost of making x gallons of oil is C(x) dollars, where

$$C(x) = 50 + 1.5x + 0.02x^2.$$

a) Write the expression for the marginal cost of the xth gallon.
b) Find the marginal cost of the fifth gallon.
c) Find the marginal cost of the 40^{th} gallon.

[Answer: a) 1.5 + 0.04x, b) \$1.7, c) \$3.1]

29. Use the Lagrangian function to optimize the function

$$f(x, y) = 3x^2 + 5xy - 6y^2 + 26x + 12y; \quad \text{subject to } 3x + y = 170$$

[Answer: $x = \frac{145}{3}$, y = 25, $\lambda = -\frac{139}{3}$ and Optimum value = $-\frac{139}{3}$]

30. Determine the maximum value of the objective function

$$f(x, y) = 10x + 4y - 2x^2 - y^2 \quad \text{subject to } 2x + y = 5.$$

[Answer: Critical point $(\frac{11}{6}, \frac{4}{3})$ and Maximum value = $\frac{91}{6}$]

11

Chapter

Integration and its applications

Highlights:

11.1 Introduction: Integral calculus is also the most important part of mathematics. There are two types of integration. One is indefinite integration and the other is definite integration. Indefinite integration deals with the inverse operation of differentiation, i.e., anti-derivative. Definite integration is the limit of a special type of addition process of infinitesimal parts of a region. So, it may be expressed as the area enclosed by a set of curves. Integral calculus has a great use in business problems. For example with a given marginal cost function it would be possible to find cost function. In this chapter we discuss nature of integration, how to find the integral value of some given functions and the use in business problems.

11.2 Definition of integration: If $F(x)$ be any differentiable function of x such that

$$\frac{d}{dx}\left[F(x)\right] = f(x)$$

then $F(x)$ is called an anti-derivative or an indefinite integral or simply an integral of $f(x)$. Symbolically, we write this as follows:

$$F(x) = \int f(x)dx$$

and is read as '$F(x)$ is the integral of $f(x)$ with respect to x'. Here, the function $f(x)$ is known as integrand.

The process of finding the integral of a given function is called integration and the given function is the integrand.

<u>Note</u>: 1. Integration is the sum of a certain infinite series. The symbol $\int$ used for integral is a distorted form of the letter S, the first letter of the word 'Sum'.

2. The symbol $\int \; dx$ is the integral operator with respect to x.

3. It is clear from the definition that $\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$.

11.3 <u>Indefinite integral</u>: Let us consider x^3, $x^3 + 2$ and $x^3 + c$, c is a constant. Since integration is anti-derivative, we have

$$\frac{d}{dx}(x^3) = 3x^2, \qquad \int 3x^2 \; dx = x^3;$$

$$\frac{d}{dx}(x^3 + 2) = 3x^2, \qquad \int 3x^2 \; dx = x^3 + 2;$$

And $$\frac{d}{dx}(x^3 + c) = 3x^2, \qquad \int 3x^2 \; dx = x^3 + c.$$

So, $\int 3x^2 \; dx$ does not give a definite value. For this such type of integration is called indefinite integral. The general value of $\int 3x^2 \; dx$ is $x^3 + c$, where c is called integral constant. So, every indefinite integral of *F(x)* can be obtained from *f(x) + c*, by taking a suitable value of c. After this we shall use 'c' as integral constant in this chapter.

11.4 <u>Fundamental theorem on integration</u>:

1. $\int kf(x)dx = k\int f(x)dx$; where k is constant.
2. $\int [f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)]dx = \int f_1(x)dx \pm \int f_2(x)dx \pm \cdots \pm \int f_n(x)dx$
3. $\int (uv)dx = u\int vdx - \int \left[\frac{d}{dx}(u)\int vdx\right]dx$; where u and v are functions of x. This is known as integration by parts formula.

11.5 <u>Some standard integrals</u>:

1. $\frac{d}{dx}(x) = 1 \qquad \therefore \qquad \int dx = x + c$
2. $\frac{d}{dx}(x^n) = nx^{n-1} \qquad \therefore \qquad \int x^n dx = \frac{x^{n+1}}{n+1} + c \quad [n \neq -1]$
3. $\frac{d}{dx}(\log_e x) = \frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \therefore \qquad \int x^{-1}dx = \int \frac{1}{x}dx = \ln x + c = \log_e x + c$

4. $\frac{d}{dx}(e^x) = e^x$ $\quad\therefore\quad$ $\int e^x \, dx = e^x + c$

5. $\frac{d}{dx}(e^{mx}) = me^{mx}$ $\quad\therefore\quad$ $\int e^{mx} \, dx = \frac{e^{mx}}{m} + c$

6. $\frac{d}{dx}(a^x) = a^x \log_e a$ $\quad\therefore\quad$ $\int a^x \, dx = \frac{a^x}{\log_e a} + c$

7. $\frac{d}{dx}(a^{mx}) = m\, a^{mx} \log_e a$ $\quad\therefore\quad$ $\int a^{mx} \, dx = \frac{a^{mx}}{m \log_e a} + c$

Note: We may use these standard derivatives as formula where necessary.

<u>Example</u>: Evaluate $\int 10x^5 dx$

Solution: $\int 10x^5 dx = 10\frac{x^{5+1}}{5+1} + c = \frac{10x^6}{6} + c = \frac{5}{3}x^6 + c$ (Answer)

<u>Example</u>: Evaluate $\int (x^5 + 3x^2 - 5x)dx$

Solution: $\int (x^5 + 3x^2 - 5x)dx = \int x^5 dx + \int 3x^2 dx - \int 5x dx$

$= \int x^5 dx + 3\int x^2 dx - 5\int x dx$

$= \frac{x^{5+1}}{5+1} + 3.\frac{x^{2+1}}{2+1} - 5\frac{x^{1+1}}{1+1} + c$

$= \frac{1}{6}x^6 + x^3 - \frac{5}{2}x^2 + c$ (Answer)

<u>Example</u>: Evaluate $\int(1+ x)(1 - 3x)dx$

Solution: $\int(1+ x)(1 - 3x)dx = \int(1 - 3x + x - 3x^2)dx$

$= \int(1 - 2x - 3x^2)dx$

$= \int 1dx - 2\int xdx - 3\int x^2dx$

$= x - 2.\frac{x^2}{2} - 3.\frac{x^3}{3} + c$

$= x - x^2 - x^3 + c$ (Answer)

<u>Example</u>: Evaluate $\int(3x^{-1} + 4x^2 - 3x + 8)dx$

Solution: $\int(3x^{-1} + 4x^2 - 3x + 8)dx = 3\int x^{-1}dx + 4\int x^2 dx - 3\int xdx + \int 8dx$

$= 3\ln x + 4.\frac{x^3}{3} - 3.\frac{x^2}{2} + 8x + c$

$= 3\ln x + \frac{4}{3}x^3 - \frac{3}{2}x^2 + 8x + c$ (Answer)

Example: Evaluate $\int(2\sqrt{x} + 5x^{-1} + 3x^2)dx$

Solution: $\int(2\sqrt{x} + 5x^{-1} + 3x^2)dx = 2\int\sqrt{x}\,dx + 5\int x^{-1}\,dx + 3\int x^2\,dx$

$$= 2\int x^{\frac{1}{2}}\,dx + 5\int x^{-1}\,dx + 3\int x^2\,dx$$

$$= 2.\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 5\ln x + 3.\frac{x^3}{3} + c$$

$$= 2.\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 5\ln x + x^3 + c$$

$$= \frac{4}{3}x^{\frac{3}{2}} + 5\ln x + x^3 + c \quad \text{(Answer)}$$

Example: Integrate $\int(3x^2 + e^x + a^x)dx$

Solution: $\int(3x^2 + e^x + a^x)dx = 3\int x^2\,dx + \int e^x\,dx + \int a^x\,dx$

$$= 3.\frac{x^3}{3} + e^x + \frac{a^x}{\log_e a} + c$$

$$= x^3 + e^x + \frac{a^x}{\log_e a} + c \quad \text{(Answer)}$$

Example: Integrate $\int(3x^5 + e^{(n+1)x} + a^{5x})dx$

Solution: $\int(3x^5 + e^{(n+1)x} + a^{5x})dx = \int x^5\,dx + \int e^{(n+1)x}\,dx + \int a^{5x}\,dx$

$$= \frac{x^6}{6} + \frac{e^{(n+1)x}}{n+1} + \frac{a^{5x}}{5\log_e a} + c \quad \text{(Answer)}$$

Example: Integrate by parts: $\int xe^x\,dx$ [AUB-02]

Solution: $\int xe^x\,dx = x\int e^x\,dx - \int\left[\frac{d}{dx}(x)\int e^x dx\right]dx;$ [Let first function, u = x
Second function, v = e^x]

$$= xe^x - \int 1.e^x dx$$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + c$$

$$= e^x(x-1) + c \quad \text{(Answer)}$$

Example: Evaluate $\int \log_e x \, dx$ [RU-88, 92]

Solution: $\int \log_e x \, dx = \int (\log_e x).1 \, dx$

$$= (\log_e x) \int 1dx - \int [\frac{d}{dx}(\log_e x) \int 1dx]dx$$

$$= (\log_e x)\, x - \int \frac{1}{x}.x \, dx$$

$$= x\log_e x - \int 1dx$$

$$= x\log_e x - x + c$$

$$= x(\log_e x - 1) + c \quad \text{(Answer)}$$

Example: Integrate by parts: $\int x^2 \ln x \; dx$ [AUB-01, NU-94]

Solution: $\int x^2 \ln x \; dx = \int (\ln x).x^2 \; dx$; [Let first function, u = lnx
Second function, $v = x^2$]

$$= \ln x \int x^2 \, dx - \int \left[\frac{d}{dx}(\ln x) \int x^2 dx \right] dx$$

$$= \ln x . \frac{x^3}{3} - \int \frac{1}{x} . \frac{x^3}{3} dx$$

$$= \frac{1}{3} \ln x - \frac{1}{3} \int x^2 dx$$

$$= \frac{1}{3} \ln x - \frac{1}{3} . \frac{x^3}{3} + c$$

$$= \frac{1}{9}(3 \ln x - x^3) + c\,] \quad \text{(Answer)}$$

11.6 Integration by substitution: In this method we first change the variable of integration to a new variable by taking a proper relationship between two variables so that the given problem reduces to a known problem. After completing the integration the new variable is to be changed by the old one. Experience is the best guide to choose the suitable substitution.

Example: Evaluate $\int (2x + 3)^5 dx$

Solution: Let $u = 2x + 3$ $\Rightarrow \frac{du}{dx} = \frac{d}{dx}(2x + 3)$

$$\Rightarrow \frac{du}{dx} = 2$$

$$\Rightarrow \frac{du}{2} = dx$$

So, $\int(2x + 3)^5dx = \int u^5 \frac{du}{2}$

$= \frac{1}{2}\int u^5\, du$

$= \frac{1}{2}.\frac{u^6}{6} + c$

$= \frac{1}{12}(2x + 3)^6 + c$ [Putting value of u] (Answer)

<u>Example</u>: Evaluate $\int \frac{dx}{(5x-6)^3}$

Solution: Let $u = 5x – 6$ => $\frac{du}{dx} = 5$ => $\frac{du}{5} = dx$

So, $\int \frac{dx}{(5x-6)^3} = \int u^{-3}\frac{du}{5} = \frac{1}{5}.\frac{u^{-2}}{-2} + c = -\frac{1}{10}(5x – 6)^{-2} + c = \frac{-1}{10(5x-6)^2} + c$

<u>Example</u>: Evaluate $\int \frac{dw}{(7-2w)^3}$

Solution: Let $u = 7 – 2w$ => $\frac{du}{dw} = -2$ => $\frac{du}{-2} = dw$

So, $\int \frac{dw}{(7-2w)^3} = \int u^{-3}\ \frac{du}{-2} = -\frac{1}{2}.\frac{u^{-2}}{-2} + c = \frac{1}{4}(7 – 2w)^{-2} + c = \frac{1}{4(7-2w)^2} + c$

<u>Example</u>: Evaluate $\int x^2(2x^3 + 3)^5dx$

Solution: Let $u = 2x^3 + 3$ => => $\frac{du}{dx} = 6x^2$ => $\frac{du}{6} = x^2dx$

So, $\int x^2(2x^3 + 3)^5dx = \int(2x^3 + 3)^5.x^2dx$

$= \int u^5\ \frac{du}{6}$

$= \frac{1}{6}.\int u^5\, du$

$= \frac{1}{6}.\frac{u^6}{6} + c$

$= \frac{1}{36}(3x^3 + 3)^6 + c$ [Putting the value of u]

<u>Example</u>: Integrate $\int e^{2x-1}\,dx$

Solution: Let $u = 2x - 1 \Rightarrow \frac{du}{dx} = 2 \Rightarrow \frac{du}{2} = dx$

So, $\int e^{2x-1}\,dx = \int e^u \frac{du}{2} = \frac{1}{2}e^u + c = \frac{1}{2}e^{2x-1} + c$ (Answer)

<u>Example</u>: Evaluate $\int \frac{1}{ax+b}dx$ [AUB-03]

Solution: Let $u = ax + b \Rightarrow \frac{du}{dx} = a \Rightarrow \frac{du}{a} = dx$

So, $\int \frac{1}{ax+b}dx = \int \frac{1}{u}.\frac{du}{a} = \frac{1}{a}\log_e u + c = \frac{1}{a}\log_e(ax + b) + c$ (Answer)

Therefore, $\int \frac{1}{ax+b}dx = \frac{1}{a}\log_e(ax+b)+c$

Similarly, $\int \frac{1}{ax-b}dx = \frac{1}{a}\log_e(ax-b)+c$

We can use these as formulae

<u>Example</u>: Integrate $\int 2x\ln(a + x^2)dx$ [AUB-98]

Solution: Let $u = a + x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

So,
$$\begin{aligned}
\int 2x\ln(a + x^2)dx &= \int \ln(a + x^2)\,2x\,dx \\
&= \int \ln u\,du \\
&= \int (\ln u).1\,du \\
&= (\ln u)\int 1du - \int\left[\frac{d}{du}(\ln u)\int 1du\right]du \\
&= (\ln u)\,u - \int \left[\frac{1}{u}.u\right]du \\
&= u\ln u - \int 1du \\
&= u\ln u - u + c \\
&= u(\ln u - 1) + c \\
&= (a + x^2)[\ln(a + x^2) - 1] + c \quad \text{(Answer)}
\end{aligned}$$

11.7 <u>Integration using partial fractions</u>: When a fraction function is given, first we try to integrate it by substitution method, if it is not possible then we find the partial fractions and integrate then.

Example: Integrate $\dfrac{3x+2}{(1+x)(1+2x)}$

Solution: Let $\dfrac{3x+2}{(1+x)(1+2x)} = \dfrac{A}{(1+x)} + \dfrac{B}{(1+2x)}$

Multiplying both sides by (1 + x)(1 + 2x), we get

$$3x + 2 = A(1 + 2x) + B(1 + x)$$

Putting x = –1 and $-\dfrac{1}{2}$, we have

$$-A = -1 \quad => \quad A = 1$$

And $\dfrac{1}{2}B = \dfrac{1}{2} \quad => \quad B = 1$

So, $\dfrac{3x+2}{(1+x)(1+2x)} = \dfrac{1}{(1+x)} + \dfrac{1}{(1+2x)}$, these are the partial fractions.

Hence $\displaystyle\int \frac{3x+2}{(1+x)(1+2x)}dx = \int \frac{1}{(1+x)}dx + \int \frac{1}{(1+2x)}dx$

$$= \int \frac{1}{(x+1)}dx + \int \frac{1}{(2x+1)}dx$$

$$= \log_e(1 + x) + \frac{1}{2}\log_e(1 + 2x) + c \quad \text{(Answer)}$$

Example: Evaluate $\displaystyle\int \frac{x^3}{(x-a)(x-b)(x-c)}dx$

Solution: Let $\dfrac{x^3}{(x-a)(x-b)(x-c)} = 1 + \dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} + \dfrac{C}{(x-c)}$

Multiplying both sides by (x – a)(x – b)(x – c), we get

$$x^3 = (x - a)(x - b)(x - c) + A(x - b)(x - c) + B(x - a)(x - c) + C(x - a)(x - b)$$

Putting x = a, b and c, we have

$$A = \frac{a^3}{(a-b)(a-c)},\ B = \frac{b^3}{(b-a)(b-c)} \text{ and } C = \frac{c^3}{(c-a)(c-b)}$$

So, $\dfrac{x^3}{(x-a)(x-b)(x-c)} = 1 + \dfrac{a^3}{(a-b)(a-c)(x-a)} + \dfrac{b^3}{(b-a)(b-c)(x-b)}$

$$+ \frac{c^3}{(c-a)(c-b)(x-c)}$$

Hence, $\int \frac{x^3}{(x-a)(x-b)(x-c)}dx = \int 1dx + \int \frac{a^3}{(a-b)(a-c)(x-a)}dx + \int \frac{b^3}{(b-a)(b-c)(x-b)}dx + \int \frac{c^3}{(c-a)(c-b)(x-c)}dx$

$= x + \frac{a^3}{(a-b)(a-c)}\log_e(x-a) + \frac{b^3}{(b-a)(b-c)}\log_e(x-b) + \frac{c^3}{(c-a)(c-b)}\log_e(x-c) + k$; k is arbitrary constant.

11.8 <u>Definite integral</u>: Let *f(x)* be a continuous function on the interval [a, b] and *F(x)* is an anti-derivative for *f(x)* on [a, b] then

$$\int_a^b f(x)dx = F(b) - F(a)$$

is called the definite integral of *f(x)* from a to b. The number a and b are respectively called the lower limit and the upper limit of the definite integral. Here, the arbitrary (integral) constant, c disappears.

<u>Note</u>: $\int_a^b f(x)dx$ represents the area under the curve *f(x)* from a to b.

11.9 <u>Properties of definite integral</u>:

1. $\int_a^b f(x)dx = \int_a^b f(u)du$; i.e., Definite integral is independent of variable.
2. $\int_a^a f(x)dx = 0$;
3. $\int_a^b f(x)dx = -\int_b^a f(x)dx$;
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$; where a < c < b.
5. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$;
6. $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$; if f(x) is an even function, i.e., f(–x) = f(x).
7. $\int_{-a}^a f(x)dx = 0$; if f(x) is an odd function, i.e., f(–x) = – f(x).

All the properties can be proved simply by the definition of definite integral.

Example: Evaluate $\int_2^3 x^3 dx$

Solution: $\int_2^3 x^3 dx = \left|\frac{x^{3+1}}{3+1}\right|_2^3 = \left|\frac{x^4}{4}\right|_2^3 = \frac{3^4}{4} - \frac{2^4}{4} = \frac{81}{4} - \frac{16}{4} = \frac{65}{4}$ (Answer)

Example: Evaluate $\int_6^{10} \frac{1}{x+2} dx$

Solution: $\int_6^{10} \frac{1}{x+2} dx = \left|\log_e(x+2)\right|_6^{10}$

$= \log_e(10+2) - \log_e(6+2)$

$= \log_e 12 - \log_e 8$

$= \log_e\left(\frac{12}{8}\right)$

$= \log_e\left(\frac{3}{2}\right)$ (Answer)

Example: Evaluate $\int_2^3 e^{3x} dx$

Solution: $\int_2^3 e^{3x} dx = \left|\frac{e^{3x}}{3}\right|_2^3 = \frac{e^{3\times 3}}{3} - \frac{e^{3\times 2}}{3} = \frac{1}{3}(e^9 - e^6) = \frac{1}{3}e^6(e^3 - 1)$ (Answer)

Example: Evaluate $\int_{-1}^3 \frac{4}{2x+3} dx$

Solution: Let u = 2x + 3 => $\frac{du}{dx} = 2$ => $\frac{du}{2} = dx$

When x = – 1, u = 1 and when x = 3, u = 9

So, $\int_{-1}^3 \frac{4}{2x+3} dx = \int_1^9 \frac{4}{u}.\frac{du}{2}$

$= 2\int_1^9 \frac{1}{u} du$

$= 2\left|\ln u\right|_1^9$

$= 2(\ln 9 - \ln 1)$

$= 2(2.1972 - 0)$

$= 4.3944$ (Answer)

Example: Compute the area under $f(x) = 3x^{1/2} - 2$ over the interval $x = 4$ to $x = 16$.

Solution: The required area is computed as follows:

$$\text{Area} = \int_4^{16} (3x^{1/2} - 2)dx$$

$$= \left| 3.\frac{x^{3/2}}{3/2} - 2x \right|_4^{16}$$

$$= \left| 2x^{3/2} - 2x \right|_4^{16}$$

$$= \left[2(16)^{3/2} - 2(16)\right] - \left[2(4)^{3/2} - 2(4)\right]$$

$$= (2\times 64 - 32) - (2\times 8 - 8)$$

$$= 96 - 8$$

$$= 88 \text{ square unit} \qquad \text{(Answer)}$$

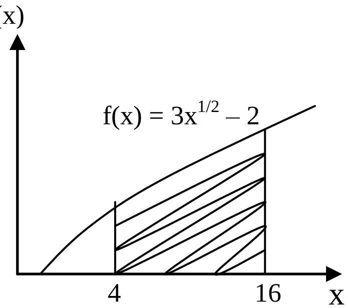


Figure 11.1

Example: Evaluate $\int_{-2}^{2} x^3 dx$

Solution: $\int_{-2}^{2} x^3 dx = \left|\frac{x^{3+1}}{3+1}\right|_{-2}^{2} = \left|\frac{x^4}{4}\right|_{-2}^{2} = \frac{2^4}{4} - \frac{(-2)^4}{4} = \frac{16}{4} - \frac{16}{4} = 0$ (Answer)

Another way: $f(x) = x^3$ is an odd function, because $f(-x) = (-x)^3 = -x^3 = -f(x)$.

So, by property (7), $\int_{-2}^{2} x^3 dx = 0$. (Answer)

11.10 Application of integration in business problems:

Integral is the anti-derivative of a function. So, the business functions which are related with differentiation are also associated with integration. For example: we know that if the total revenue function, say r(x), is given then the marginal revenue function is the first derivative of total revenue function, say $r'(x)$. Therefore, it follows that the total revenue function is the integral of the marginal revenue function.

That is, total revenue function, $r(x) = \int r'(x)\, dx$.

And the average revenue function $= \frac{r(x)}{x}$.

Example: Let the marginal revenue function and marginal cost function of a firm are $r'(x) = 16 - x^2$ and $c'(x) = 3x^2 - 2x + 8$ respectively; where x is the quantity of product. And fixed cost of the firm is Tk. 500, i.e., c(0) = 500, then find

(i) the total revenue function.
(ii) the average revenue function.
(iii) the demand function.

(iv) the maximum total revenue.
(v) the total revenue from product 1 to 3.
(vi) the total cost function.
(vii) the total profit function.

Solution: Given that, the marginal revenue function, $r'(x) = 16 - x^2$.

(i) The total revenue function: $r(x) = \int r'(x)\,dx.$

$$= \int (16 - x^2)dx$$

$$= 16x - \frac{1}{3}x^3 + c$$

We know that, total revenue = 0 if we produce no product, i.e., x = 0. So, c = 0

Therefore, the total revenue function: $r(x) = 16x - \frac{1}{3}x^3$ (Answer)

(ii) The average revenue function $= \frac{r(x)}{x} = (16x - \frac{1}{3}x^3)/x = 16 - \frac{1}{3}x^2.$ (Answer)

(ii) We know that the average revenue from a product is the demand of that product.

So, the demand function, $d(x) = 16 - \frac{1}{3}x^2.$ (Answer)

(iv) We shall find the maximum or minimum total revenue where the marginal revenue, $r'(x) = 0$. i.e., $16 - x^2 = 0 \Rightarrow x^2 = 16 \Rightarrow x = 4$ and -4

But, in Economics negative production is not possible. So, x = 4 is acceptable only.

Here, second derivative of the total revenue function, r(x) is

$$r''(x) = \frac{d}{dx}(16 - x^2) = -2x.$$

And $r''(4) = -2(4) = -8$; which is negative.

So, the maximum revenue will be get for x = 4 unit products.

Hence, the maximum total revenue $= \int_0^4 (16 - x^2)dx$

$$= \left| 16x - \frac{x^3}{3} \right|_0^4$$

$$= \left\{16(4) - \frac{(4)^3}{3}\right\} - \left\{16(0) - \frac{(0)^3}{3}\right\}$$

$$= 64 - \frac{64}{3}$$

$$= \frac{128}{3}$$ (Answer)

(v) The total revenue from product 1 to 3 $= \int_1^3 (16 - x^2)dx$

$$= \left| 16x - \frac{x^3}{3} \right|_1^3$$

$$= \left\{16(3) - \frac{(3)^3}{3}\right\} - \left\{16(1) - \frac{(1)^3}{3}\right\}$$

$$= 39 - \frac{47}{3}$$

$$= \frac{70}{3} \quad \text{(Answer)}$$

(vi) Given that, the marginal cost function: $c^{/}(x) = 3x^2 - 2x + 8$ and $c(0) = 500$.

So, the total cost function, $c(x) = \int c^{/}(x)\, dx$.

$$= \int (3x^2 - 2x + 8)\, dx$$

$$= x^3 - x^2 + 8x + c$$

But given, $c(0) = 500$. So, $c = 500$

Hence, the total cost function: $c(x) = x^3 - x^2 + 8x + 500$. (Answer)

(vii) We know that, total profit = total revenue – total cost

So, the total profit function, $p(x) = r(x) - c(x)$

$$= (16x - \frac{1}{3}x^3) - (x^3 - x^2 + 8x + 500)$$

$$= -\frac{4}{3}x^3 + x^2 + 8x - 500 \quad \text{(Answer)}$$

<u>Consumer's surplus & producer's surplus</u>: We know that demand is decreasing and supply is increasing with respect to the price. Let (x_0, p_0) is the equilibrium point at which the demand, *d(x)* and the supply, *s(x)* are equal. Then

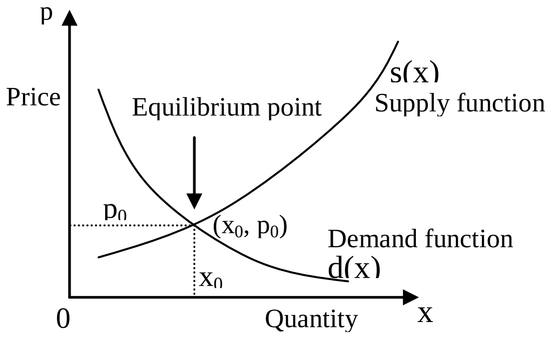


Figure 11.2

The consumer's surplus $= \int_0^{x_0} d(x)dx - x_0 \times p_0$. That is, the consumer's surplus is the difference between the cost consumers are willing to pay for a commodity and what they actually pay.

The producer's surplus $= x_0 \times p_0 - \int_0^{x_0} s(x)dx$. That is, the producer's surplus is the difference between the revenue producers actually receive and what they have been willing to receive.

Example: The demand function for a product is $p = 20 - x - x^2$. Find the consumer's surplus when the demand is 3.

Solution: When demand $x_0 = 3$, the price $p_0 = 20 - 3 - 3^2 = 8$

So, the consumer's surplus $= \int_0^{x_0} d(x)dx - x_0 \times p_0$

$$= \int_0^3 (20 - x - x^2)dx - 3 \times 8$$

$$= \left| 20x - \frac{x^2}{2} - \frac{x^3}{3} \right|_0^3 - 24$$

$$= 20(3) - \frac{3^2}{2} - \frac{3^3}{3} - 24$$

$$= 60 - \frac{9}{2} - 9 - 24$$

$$= \frac{45}{2} \qquad \text{(Answer)}$$

Rate of sales: When the rate of sales of a product is a known function of t, say $f(t)$ where t is a time measure, the total sales of this product over a time period T is

$$\int_0^T f(t)dt$$

Example: Suppose the rate of sales of a new product is given by $f(t) = 200 - 90e^{-t}$, where t is the number of days the product is on the market. Find the total sales during the first 5 days.

Solution: The total sales during the first 5 days $= \int_0^5 f(t)dt$

$$= \int_0^5 (200 - 90e^{-t})dt$$

$$= \left| 200t + 90e^{-t} \right|_0^5$$

$$= 1000 + 90e^{-5} - 90e^0$$

$$= 1000 + 90(0.0067) - 90$$

$$= 910.603 \text{ units.} \qquad \text{(Answer)}$$

<u>Amount of an annuity</u>: If an annuity consists of equal annual payments Tk. P in which an interest rate of Tk. j for one taka per annum is compounded continuously, the amount Tk. F of the annuity after n payment is

$$F = \int_0^n Pe^{jt}dt$$

<u>Example</u>: A bank pays interest at the rate of 10% per annum compounded continuously. If a person places Tk. 1000 in a saving account each year, how much will be in the account after 6 years?

Solution: Here, P = 1000, n = 6 and $j = \frac{10}{100} = 0.1$

The amount after 6 years is $F = \int_0^6 1000e^{0.1t}dt$

$$= \left| 1000 \times \frac{e^{0.1t}}{0.1} \right|_0^6$$

$$= 10000\{e^{0.1(6)} - e^{0.1(0)}\}$$

$$= 10000(1.822 - 1)$$

$$= \text{Tk. } 8220 \qquad \text{(Answer)}$$

11.11 <u>Some worked out examples</u>:

<u>Example (1)</u>: Evaluate $\int \frac{xdx}{\sqrt{1+x^2}}$ [NU-97, CU-87]

Solution: Let, $I = \int \frac{xdx}{\sqrt{1+x^2}}$

$$= \int \frac{1}{\sqrt{u}} \cdot \frac{du}{2}$$

$$= \frac{1}{2}\int u^{-\frac{1}{2}}du$$

Let, $u = 1 + x^2$

$\therefore \frac{du}{dx} = 2x$ => $\frac{du}{2} = xdx$

$$= \frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+c$$

$$= (1+x^2)^{\frac{1}{2}}+c$$

$$= \sqrt{(1+x^2)}+c \qquad \text{(Answer)}$$

Example (2): Evaluate $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}dx$ [NU-96]

Solution: Let I $= \int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}dx$

$= \int e^u .2du$

$= 2\int e^u du$

$= 2e^u + c$

$= 2e^{\sqrt{x+1}}+c$ (Answer)

Let, u = $\sqrt{x+1}$

$\therefore \frac{du}{dx} = \frac{1}{2\sqrt{x+1}}$

$\Rightarrow 2du = \frac{1}{\sqrt{x+1}}dx$

Example (3): Evaluate $\int \frac{1}{e^x+1}dx$ [RU-81]

Solution: Let, I $= \int \frac{1}{e^x+1}dx$

$= \int \frac{e^{-x}}{e^{-x}(e^x+1)}dx$ [Multiplying numerator and denominator by e^{-x}]

$= \int \frac{e^{-x}}{1+e^{-x}}dx$

$= -\int \frac{1}{u}du$

$= -\log_e u + c$

$= -\log_e(1+e^{-x})+c$ (Answer)

Let, u = 1 + e^{-x}

$\therefore \frac{du}{dx} = -e^{-x}$

$\Rightarrow -du = e^{-x}dx$

Example (4): $\int (x^2 + 1)x\,dx$ [CU-87]

Solution: Let, I $= \int (x^2 + 1)x\,dx$

$= \int u.\frac{du}{2}$

Let, u = $x^2 + 1$

$\therefore \frac{du}{dx} = 2x \Rightarrow \frac{du}{2} = xdx$

$= \frac{1}{2}\int u\, du$

$= \frac{1}{2}.\frac{u^2}{2} + c$

$= \frac{1}{4}(x^2 + 1)^2 + c$ (Answer)

Example (5): Evaluate $\int x^2e^{2x}\, dx$ [RU- '93, '81]

Solution: Let, $I = \int x^2e^{2x}\, dx$

$= x^2\int e^{2x}\, dx - \int [\frac{d}{dx}(x^2)\int e^{2x}\, dx]dx$

$= x^2.\frac{e^{2x}}{2} - \int [2x.\frac{e^{2x}}{2}]dx$

$= \frac{1}{2}x^2e^{2x} - \int xe^{2x}\, dx$

$= \frac{1}{2}x^2e^{2x} - [x\int e^{2x}\, dx - \int \{\frac{d}{dx}(x)\int e^{2x}\, dx\}dx$

$= \frac{1}{2}x^2e^{2x} - [x.\frac{e^{2x}}{2} - \int 1.\frac{e^{2x}}{2}dx]$

$= \frac{1}{2}x^2e^{2x} - [\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\, dx]$

$= \frac{1}{2}x^2e^{2x} - [\frac{1}{2}xe^{2x} - \frac{1}{2}.\frac{e^{2x}}{2}] + c$

$= \frac{1}{2}e^{2x}[x^2 - x + \frac{1}{2}] + c$ (Answer)

Example (6): Evaluate $\int \frac{3x^2+2}{x^3+2x+5}dx$ [RU-88]

Solution: Let $I = \int \frac{3x^2+2}{x^3+2x+5}dx$

$= \int \frac{1}{u}du$

$= \log_e u + c$

$= \log_e(x^3 + 2x + 5) + c$ (Answer)

Let, $u = x^3 + 2x + 5$

$\therefore \frac{du}{dx} = 3x^2 + 2$

$\Rightarrow du = (3x^2 + 2)dx$

Example (7): Evaluate $\int x^2e^x\, dx$ [RU-80]

Solution: Let, $I = \int x^2e^x\, dx$

$$= x^2\int e^x\,dx - \int\left[\frac{d}{dx}(x^2)\int e^x dx\right]dx\,;$$
$$= x^2e^x - \int 2x.e^x dx$$
$$= x^2e^x - 2\int xe^x dx$$
$$= x^2e^x - 2[x\int e^x dx - \int\{\frac{d}{dx}(x)\int e^x dx\}dx]$$
$$= x^2e^x - 2[xe^x - \int 1.e^x dx]$$
$$= x^2e^x - 2[xe^x - \int e^x dx]$$
$$= x^2e^x - 2[xe^x - e^x] + c$$
$$= e^x(x^2 - 2x + 2] + c \quad \text{(Answer)}$$

Example (8): Evaluate $\int \frac{1}{x}\log_e(\log_e x)dx$

Solution: Let $I = \int \frac{1}{x}\log_e(\log_e x)dx$

$= \int \log_e u\, du$

$= (\log_e u)\int 1 du - \int [\frac{d}{du}(\log_e u)\int 1 du]du$

$= (\log_e u)u - \int \frac{1}{u}.u\, du$

$= (\log_e u)u - \int du$

$= (\log_e u)u - u + c$

$= u(\log_e u - 1) + c$

$= (\log_e x)\{\log_e(\log_e x) - 1\} + c$ (Answer)

Let, $u = \log_e x$

$\therefore \frac{du}{dx} = \frac{1}{x}$ => $du = \frac{1}{x}dx$

Example (9): Evaluate $\int \frac{1}{x(1+\log_e x)^3}dx$ [RU-81]

Solution: Let, $I = \int \frac{1}{x(1+\log_e x)^3}dx$

$= \int \frac{1}{u^3}du$

$= \int u^{-3}du$

$= \frac{u^{-2}}{-2} + c$

Let, $u = 1 + \log_e x$

$\therefore \frac{du}{dx} = \frac{1}{x}$ => $du = \frac{1}{x}dx$

$$= -\frac{1}{2}(1+\log_e x)^{-2} + c \quad \text{(Answer)}$$

<u>Example (10)</u>: Evaluate $\int_1^3 \frac{2xdx}{1+x^2}$ [RU-80]

Solution: Let, $I = \int_1^3 \frac{2xdx}{1+x^2}$

$$= \int_2^{10} \frac{du}{u}$$

$$= \left|\ln u\right|_2^{10}$$

$$= \ln 10 - \ln 2$$

$$= \ln \frac{10}{2}$$

$$= \ln 5 \quad \text{(Answer)}$$

Let, u = 1 + x²

$\therefore \frac{du}{dx} = 2x$ => $du = 2xdx$

u = 2 when x = 1

and u = 10 when x = 3

<u>Example (11)</u>: Evaluate $\int_0^2 \frac{x}{x^2+4}dx$ [AUB-01, RU-81]

Solution: Let, $I = \int_0^2 \frac{x}{x^2+4}dx$

$$= \int_4^8 \frac{1}{u}.\frac{du}{2}$$

$$= \frac{1}{2}\int_4^8 \frac{1}{u}du$$

$$= \frac{1}{2}\left|\log_e u\right|_4^8$$

$$= \frac{1}{2}(\log_e 8 - \log_e 4)$$

$$= \frac{1}{2}\log_e\left(\frac{8}{4}\right)$$

$$= \frac{1}{2}\log_e 2 \quad \text{(Answer)}$$

Let, u = x² + 4

$\therefore \frac{du}{dx} = 2x$ => $\frac{du}{2} = xdx$

u = 4 when x = 0

and u = 8 when x = 2

<u>Example (12)</u>: Evaluate $\int_0^{\ln 2} \frac{e^x}{1+e^x}dx$ [RU-96]

Solution: Let, I = $\int_{0}^{\ln 2} \frac{e^x}{1+e^x} dx$

$= \int_{2}^{3} \frac{1}{u} du$

$= \left| \ln u \right|_2^3$

$= \ln 3 - \ln 2$

$= \ln \frac{3}{2}$

$= \ln 1.5$ (Answer)

Let, u = 1 + e^x

$\therefore \frac{du}{dx} = e^x \Rightarrow du = e^x dx$

When x = 0, u = 2 and

when x = ln 2, u =1 + e$^{\ln 2}$ =1 + 2= 3

<u>Example (13)</u>: Evaluate $\int_{1}^{2} \frac{1}{x(x+1)} dx$ [RU-82]

Solution: Let, I = $\int_{1}^{2} \frac{1}{x(x+1)} dx$

$= \int_{1}^{2} \left\{ \frac{1}{x} - \frac{1}{x+1} \right\} dx$ [By taking partial fractions]

$= \int_{1}^{2} \frac{1}{x} dx - \int_{1}^{2} \frac{1}{x+1} dx$

$= \left| \log_e x \right|_1^2 - \left| \log_e (x+1) \right|_1^2$

$= (\log_e 2 - \log_e 1) - (\log_e 3 - \log_e 2)$

$= \log_e 2 - 0 - \log_e 3 + \log_e 2$

$= 2\log_e 2 - \log_e 3$

$= \log_e 2^2 - \log_e 3$

$= \log_e \left(\frac{4}{3} \right)$ (Answer)

<u>Example (14)</u>: Evaluate $\int_{0}^{5} \frac{1}{y} dy$ [RU-86]

Solution: Let, I = $\int_{0}^{5} \frac{1}{y} dy$

$= \left| \ln y \right|_0^5$

$= \ln 5 - \ln 0$

= Not found, because the value of $\ln 0$ is not known.

Example (15): Find the area bounded by the curve f(x) = 15 – 2x – x^2 and the straight line h(x) = 9 – x. [AUB-02]

Solution: Taking f(x) = h(x), we get

$15 - 2x - x^2 = 9 - x$

Or, $x^2 + x - 6 = 0$

Or, $x^2 + 3x - 2x - 6 = 0$

Or, $x(x + 3) - 2(x + 3) = 0$

Or, $(x + 3)(x - 2) = 0$

Or, $x = -3$ and $x = 2$

So, – 3 and 2 are the x – coordinates of the points of intersection of f(x) and h(x). Therefore, the area bounded by the curve f(x) and the straight line h(x) is

$$\int_{-3}^{2} [f(x) - h(x)]dx$$

$$= \int_{-3}^{2} [(15 - 2x - x^2) - (9 - x)]dx$$

$$= \int_{-3}^{2} [(6 - x - x^2)dx$$

$$= \left| 6x - \frac{x^2}{2} - \frac{x^3}{3} \right|_{-3}^{2}$$

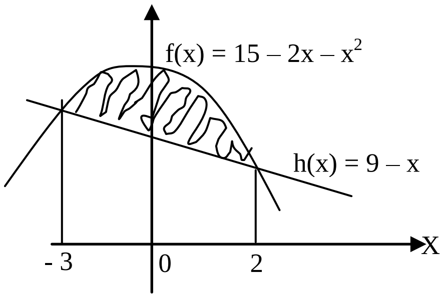


Figure 11.3

$$= \left\{6(2) - \frac{(2)^2}{2} - \frac{(2)^3}{3}\right\} - \left\{6(-3) - \frac{(-3)^2}{2} - \frac{(-3)^3}{3}\right\}$$

$$= (12 - 2 - \frac{8}{3} + 18 + \frac{9}{2} - 9)$$

$$= \frac{125}{6} \text{ square unit.} \quad \text{(Answer)}$$

Example (16): If marginal cost of a firm is $c^{/}(x) = x^2$ and c(0) = 0.1 find the total cost function. [RU-82]

Solution: Given that the marginal cost function, $c^{/}(x) = x^2$.

So, the total cost function, $c(x) = \int c^{/}(x)\, dx$

$$= \int x^2\, dx$$

$$= \frac{1}{3}x^3 + c$$

And given that c(0) = 0.1

$$\therefore \quad \frac{1}{3}(0)^3 + c = 0.1 \quad \Rightarrow \quad c = 0.1$$

Therefore, the total cost function is $c(x) = \frac{1}{3}x^3 + 0.1$ (Answer)

<u>Example (17)</u>: Let the marginal revenue function of a firm is $r^{/}(x) = 5 + \frac{6}{(x+2)^2}$; where x is the quantity of sold products. Find the total revenue function. [NU- 98 Mgt.]

Solution: Given that the marginal revenue function: $r^{/}(x) = 5 + \frac{6}{(x+2)^2}$

So, total revenue function: $r(x) = \int[5 + \frac{6}{(x+2)^2}]dx$

$$= 5x - \frac{6}{(x+2)} + c$$

Since, revenue = 0 when x = 0 unit product is sold, i.e., r(0) = 0.

So, $0 = 5(0) - \frac{6}{(0+2)} + c \quad => \quad c = 3$

Hence, the total revenue function is $r(x) = 5x - \frac{6}{(x+2)} + 3$ (Answer)

<u>Example (18)</u>: Find the consumer's surplus and producer's surplus under pure competition for the demand function $p = 61 - x^2$ and supply function $p = 11 + x^2$, where p is the price and x is quantity.

Solution: Under pure competition, market equilibrium conditions can be obtained by equating the demand and supply.

$\therefore \quad 61 - x^2 = 11 + x^2$

Or, $2x^2 = 50$

Or, $x^2 = 25$

Or, $x = 5, -5$ [$x = -5$ is not acceptable]

$\therefore \quad x_0 = 5$

So, $p_0 = 61 - 25 = 36$

The consumer's surplus $= \int_0^{x_0} d(x)dx - x_0 \times p_0$.

$$= \int_0^5 (61 - x^2)dx - 5 \times 36.$$

$$= \left| 61x - \frac{x^3}{3} \right|_0^5 - 180$$

$$= 61(5) - \frac{(5)^3}{3} - 180$$

$$= \frac{250}{3}$$ (Answer)

And the producer's surplus $= x_0 \times p_0 - \int_0^{x_0} s(x)dx$

$= 5\times 36 - \int_0^5 (11 + x^2)dx$

$= 180 - \left| 11x + \frac{x^3}{3} \right|_0^5$

$= 180 - 11(5) - \frac{(5)^3}{3}$

$= \frac{250}{3}$ (Answer)

11.12 <u>Exercise</u>:

1. Define integral and definite integral. Give examples.
2. What is difference between differentiation and integration?
3. Why does arbitrary constant c disappear in the definite integral? Explain by a example.
4. Evaluate the following integrals:

(i) $\int x^{10}dx$ [Answer: $\frac{x^{11}}{11} + c$]

(ii) $\int x^{1/2}dx$ [Answer: $\frac{2}{3}x^{3/2}$ + c]

(iii) $\int \frac{1}{x^3}$ dx [Answer: $-\frac{1}{2x^2}$ + c]

(iv) $\int 5x^4dx$ [Answer: $x^5 + c$]

(v) $\int(4x^3 + 3x^2 + 2x + 3)dx$ [Answer: $x^4 + x^3 + x^2 + 3x + c$]

(vi) $\int(x^2 - 1)^2dx$ [Answer: $\frac{1}{5}x^5 - \frac{2}{3}x^3 + x + c$]

(vii) $\int \left(x - \frac{1}{x}\right)^3$ dx [Answer: $\frac{x^4}{4} - \frac{3x^2}{2} + 3\ln x + \frac{1}{2x^2} + c$]

(viii) $\int \frac{x^4 + 1}{x^2}$dx [Answer: $\frac{x^3}{3} - \frac{1}{x} + c$]

(ix) $\int(\sqrt{x} - \frac{x}{2} + \frac{2}{\sqrt{x}})$dx [Answer: $\frac{2}{3}x^{3/2} - \frac{1}{4}x^2 + 4\sqrt{x} + c$]

(x) $\int(2^x + \frac{1}{2}e^{-x} + \frac{4}{x} - \frac{1}{\sqrt[3]{x}})$dx [Answer: $\frac{2^x}{\log_e 2} - \frac{1}{2}e^{-x} + 4\log_e x - \frac{3}{2}x^{2/3} + c$]

5. Integrate by parts:

(i) $\int \ln x \, dx$ [Answer: $x(\ln x - 1) + c$]

(ii) $\int x \ln x \, dx$ [Answer: $\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c$]

(iii) $\int x^n \log_e x \, dx$ [Answer: $\frac{x^{n+1}}{n+1}\log_e x - \frac{x^{n+1}}{(n+1)^2} + c$]

(iv) $\int xe^{-x} \, dx$ [Answer: $-e^{-x}(x + 1) + c$]

(v) $\int x^2e^{3x}dx$ [Answer: $\frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2e^{3x}}{27} + c$]

6. Evaluate the following integrals by method of substitution:

(i) $\int \frac{1}{x}\ln x dx$ [Answer: $\frac{1}{2}(\ln x)^2 + c$]

(ii) $\int (5-2x)^{-3/2} dx$ [Answer: $\frac{1}{(5-2x)^{1/2}} + c$]

(iii) $\int (ax^2 + 2bx + 2c)^5 (ax + b) dx$ [Answer: $\frac{(ax^2 + 2bx + 2c)^6}{12} + c$]

(iv) $\int \frac{dx}{1-4x}$ [Answer: $\frac{1}{4}\ln(1-4x) + c$]

(v) $\int \frac{xdx}{2x^2+3}$ [Answer: $\frac{1}{4}\ln(2x^2+3) + c$]

(vi) $\int \frac{8x^2}{(x^3+2)^3}dx$ [Answer: $-\frac{4}{3(x^3+2)^2} + c$]

(vii) $\int \frac{4x^3+2x}{x^4+x^2+2}dx$ [Answer: $\log_e(x^4 + x^2 + 2) + c$]

7. Integrate using partial fractions:

(i) $\int \frac{1}{(x+1)(x+3)}dx$ [Answer: $\frac{1}{2}\log_e(x+1) - \log_e(x+3) + c$]

(ii) $\int \frac{1}{x-x^3}dx$ [Answer: $\frac{1}{2}\log_e\left(\frac{x^2}{1-x^2}\right) + c$]

(iii) $\int \frac{x+2}{x^2-13x+42}dx$ [Answer: $19\log_e(x-7) - 18\log_e(x-6) + c$]

(iv) $\int \frac{x}{(x-1)(2x+1)}dx$ [Answer: $\frac{1}{6}\{2\log_e(x-1) + \log_e(2x+1)\} + c$]

(v) $\int \frac{x^2}{(x-a)(x-b)}dx$ [Ans: $x + \frac{a^2}{a-b}\log_e(x-a) - \frac{b^2}{a-b}\log_e(x-b) + c$]

8. Evaluate the following definite integrals:

(i) $\int_1^3 x^2 dx$ [Answer: $\frac{26}{3}$]

(ii) $\int_2^4 (7x^3 + x)dx$ [Answer: 426]

(iii) $\int_1^2 (x^{-2} + x)dx$ [Answer: 2]

(iv) $\int_2^3 e^{2x} dx$ [Answer: $\frac{1}{2}e^4(e^2 - 1)$]

(v) $\int_1^3 \frac{2x}{1+x^2} dx$ [Answer: ln10 – ln2]

(vi) $\int_1^2 (e^{2x} + 3x^2)dx$ [Answer: $\frac{1}{2}(e^4 - e^2 + 14)$]

(vii) $\int_1^6 \frac{60dx}{(3x+2)^2}$ [Answer: 3]

(viii) $\int_1^2 (e^{3x} + 4x^2)dx$ [Answer: $\frac{1}{3}(e^6 - e^3 + 28)$]

9. Prove that $\int_3^8 \frac{1}{x-3} dx = \log_e 5$.

10. Find the area under $f(x) = x^{1/3} + 5$ over the interval x = 1 to x = 8. [Answer: $\frac{185}{4}$]

11. Find the area bounded by the curve $f(x) = 16 - x^2$ and the x-axis. [Answer: $\frac{256}{3}$]

12. Find the area bounded by the curve $f(x) = 3x^3 - 3x$ and the straight line $h(x) = x$.

[Answer: $\frac{8}{3}$] [Hints: Area = $\left|\int_{-\frac{2}{\sqrt{3}}}^{0} [f(x) - h(x)]dx\right| + \left|\int_{0}^{\frac{2}{\sqrt{3}}} [f(x) - h(x)]dx\right|$]

13. If marginal cost $c'(x) = 8x^3$, and c(1) = 5, find the total cost function. [Answer: $c(x) = 2x^4 + 3$]

14. If marginal cost $c'(x) = 2 + x + x^2$, and c(0) = 50, find the total cost function.

[Answer: $c(x) = 2x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + 50$]

15. Let the marginal cost function (in thousand taka) of a firm is $c'(x) = 4x^2 - 6x + 1$; where x is the quantity of products, and the fixed cost of the firm is Tk. 400000, i.e., c(0) = 400000. Find the total cost of the firm for making x = 12 unit product. [Answer: Tk. 2284000]

16. If the marginal revenue function of a firm $r'(x) = 5 + 3x - x^2$ and $r(6) = 112$, find the total revenue function and the average revenue function. [Answer: revenue: $r(x) = 100 + 5x + \frac{3}{2}x^2 - \frac{1}{3}x^3$ and avg. revenue: $\frac{r(x)}{x} = \frac{100}{x} + 5 + \frac{3x}{2} - \frac{x^2}{3}$]
17. Let the marginal cost and the marginal revenue functions of a firm are $c'(x) = 0.1x^2 - 4x + 110$ and $r'(x) = 150 - x$; where x is the number of unit product. If total cost is Tk. 4000 for making 30 units product, find
    (i) fixed cost.
    (ii) condition for $r(0) = 0$
    (iii) profit function.
    (iv) number of unit production to maximize the profit.

    [Answer: (i) Tk. 1600, (ii) When no product is sold, (iii) $p(x) = 40x + \frac{3}{2}x^2 - \frac{1}{30}x^3 - 1600$ and (iv) 40]
18. Demand function is $p = 20 - 2x$; where x is the quantity. If $x = 12$, find the consumer's surplus. [Answer: 36]
19. Assume that in 1995 the annual world use of natural gas was 50 trillion cubic feet. The annual consumption of gas is increasing at a rate of 3% compounded continuously. How long will it take to use all available gas, if it is known that in 1995 there were 2200 trillion cubic feet of proven reserves? Assume that no new discoveries are made. [Answer: 28.1 years] [Hints: Find t of $\int_0^t 50e^{0.03t}dt = 2200$]
20. A bank pays interest at the rate of 6% per annum compounded continuously. Find how much should be deposited in the bank each year in order to accumulate Tk. 6000 in 3 years. [Answer: Tk. 1818.18] [Hints: Find P of $6000 = \int_0^3 Pe^{0.06t}dt$]

12

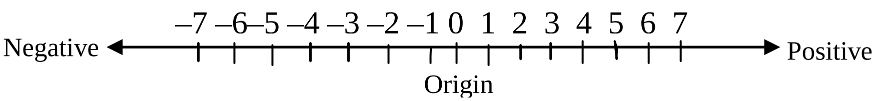
Chapter

Coordinate Geometry

Highlights:

12.1 Introduction: Coordinate geometry is the contribution of French mathematician Renatus Cartesius and so it is some time called Cartesian geometry. It is now the main branch of geometry in which two real numbers, called coordinates, are used to indicate the position of a point in a plane. The main contribution of coordinate geometry is that it has enabled the integration of algebra and geometry. For this algebraic methods are employed to represent and prove the fundamental geometrical theorems. At present coordinate geometry is considered as more powerful tool of analysis than the Euclidian geometry. For this coordinate geometry is termed as analytical geometry. Now it is very useful to solve complex business problems. In this chapter we shall discuss some of them.

12.2 Directed line: A directed line is a straight line with number units positive, zero and negative is called directed line or number line. The point of origin is the number 0. The arrow indicates its direction. On the right side of the arrow are the positive numbers and on the other side are the negative numbers. It is like a real number scale illustrated below:



Directed line

Figure – 12.1

A directed line can be horizontal normally indicated by X/OX axis and vertically indicated normally by YOY/ axis. The point where the two lines intersect each other is called the point of origin and is denoted by (0,0).

12.3 <u>Quadrants</u>: The two directed lines, when they intersect at right angles at the point of origin, divide their plane into four parts or regions. These four pasts are known as quadrants.

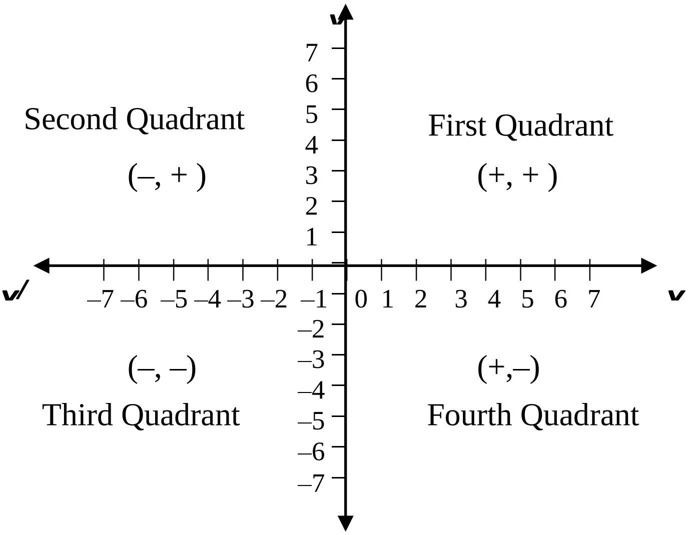


Figure – 12.2

12.4 <u>Coordinates</u>: In a two dimensional figure a point in plane has two coordinates. The exact position of the point can be located by the unit size of these coordinates. The first coordinate is known as x-coordinate and second coordinate is known as y-coordinate.

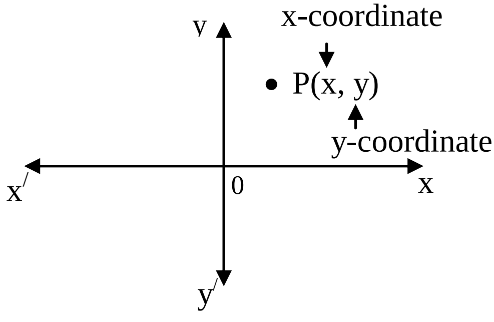


Figure – 12.3

<u>Example</u>: Plot the points (2, 3), (-5, 4), (-3, -2) and (4, -3) in the Cartesian plane.
Solution:

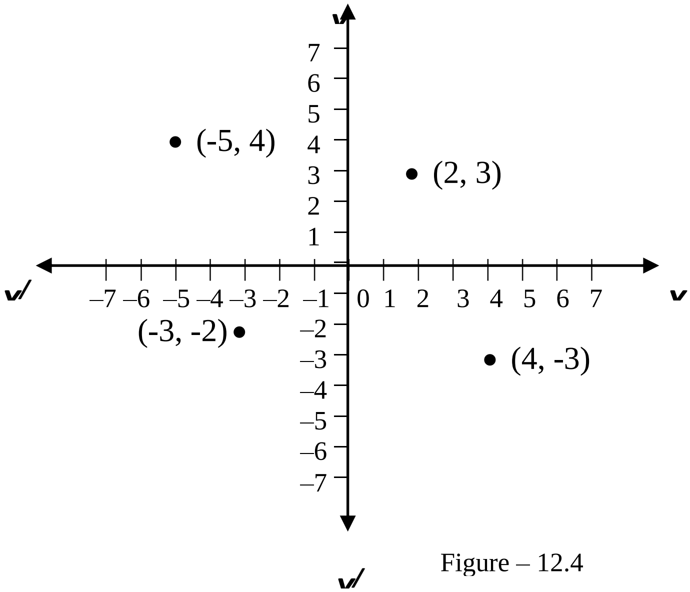


Figure – 12.4

12.5 Coordinates of mid point: We can find out the coordinates of a mid point from the coordinates of any two points using the following formula :

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points and $R(x_m, y_m)$ be mid point of them, then

$x_m = \frac{x_1 + x_2}{2}$ and $y_m = \frac{y_1 + y_2}{2}$

Example: The coordinates of the mid point of the points (-2, 5) and (6, 3) are

$\left(\frac{-2+6}{2}, \frac{5+3}{2}\right)$, i.e, (2, 4)

12.6 Distance between two points: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points. Two perpendiculars PN and QM are drawn on the axis OX from the points P and Q respectively. From Q an another perpendicular QR is drawn on PN. From the right angle triangle PQR, we have

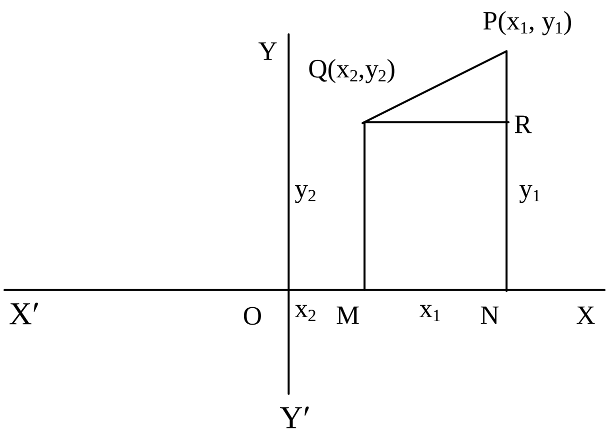


Figure – 12.5

$$PQ^2 = PR^2 + QR^2$$
$$= (PN - RN)^2 + (ON - OM)^2$$
$$= (PN - QM)^2 + (ON - OM)^2$$

$= (y_1 - y_2)^2 + (x_1 - x_2)^2$

$= (x_1 - x_2)^2 + (y_1 - y_2)^2$

PQ $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

PQ $= \sqrt{(Difference\ of\ abscissas)^2 + (Difference\ of\ ordinates)^2}$

Example: Find the distance between the points A (8, – 3) and B (– 7, 5) .
Solution: Let the distance between the two points be d

Therefore , d = √[(Difference of abscissas)^2 + (Difference of ordinates)]
$= \sqrt{[\{ 8-(-7)\}^2 + \{-3-(5)\}^2]}$
$= \sqrt{[(8+7)^2 + \{-3-5\}^2]}$
$= \sqrt{[15^2 + (-8)^2]}$
$= \sqrt{(225+64)}$
$= \sqrt{289}$
= 17.

Hence, the required distance is 17 units. (Answer)

12.7 Section formula: The coordinates of a point R(x, y) dividing a line in the ratio of $m_1 : m_2$ connecting the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \text{ and } y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

Proof: Three perpendiculars PL, RM and QN are drawn on the axis OX from the points P, R and Q respectively. From P and R two other perpendiculars PK and RT are drawn on RM and QN respectively. From the figure we see that ΔPKR and ΔRTQ are similar triangles.

So, $\frac{PK}{RT} = \frac{RK}{QT} = \frac{PR}{RQ} = \frac{m_1}{m_2}$

When, $\frac{PK}{RT} = \frac{m_1}{m_2}$

Or, $\frac{x - x_1}{x_2 - x} = \frac{m_1}{m_2}$

Or, $m_2x - m_2x_1 = m_1x_2 - m_1x$

Or, $x(m1 + m2) = m_1x_2 + m_2x_1$

So, $x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$

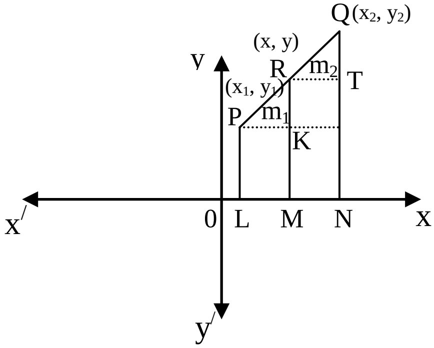


Figure – 12.6

Similarly, when $\frac{RK}{QT} = \frac{m_1}{m_2}$

Or, $\dfrac{y - y_1}{y_2 - y} = \dfrac{m_1}{m_2}$

Or, $m_2y - m_2y_1 = m_1y_2 - m_1y$

Or, $y(m1 + m2) = m_1y_2 + m_2y_1$

So, $y = \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}$

Note: If this division be external then

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} \text{ and } y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

Example: The coordinates of the point R which divides the connecting line of the points P(3, 4) and Q(-3, -4) in the ratio of 2 : 3 are

$$\left(\frac{2.(-3)+3.3}{2+3}, \frac{2.(-4)+3.4}{2+3}\right) \text{ i.e, } \left(\frac{3}{5}, \frac{4}{5}\right)$$

Example: Find the coordinates of the point which externally divides the line joining the points (4, -5) and (6, 8) in the ratio 2 **:** 1.

Solution: Let A(4, -5), B(6, 8) and the point P(x, y) divides AB externally in the ratio 2 **:** 1.

So, $x = \dfrac{2\times 6 - 1\times 4}{2-1} = 8$ and $y = \dfrac{2\times 8 - 1\times(-5)}{2-1} = 21$

Hence, the coordinates of the required point are (8, 21) (Answer)

12.8 Coordinates of the centroid of a triangle: The coordinates of the centroid of a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Proof:

Let us consider the vertices of the triangle $A(x_1,y_1)$, $B(x_2,y_2)$ and C (x_3, y_3) as shown in diagram. In the diagram median AD bisects the base BC at D with coordinates

$$D\left(\frac{x_2 + x_2}{2}, \frac{y_2 + y_2}{2}\right).$$

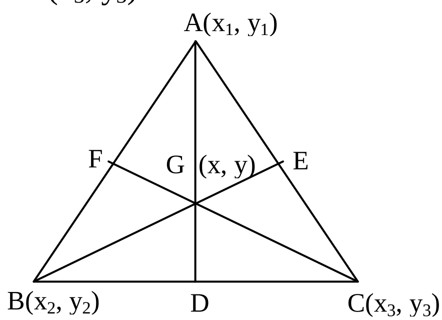


Figure – 12.7

We know that centroid is that point in a median which divides the median in the ratio 2 **:**1. Let G(x, y) be the centroid of the triangle ABC. So, G divides the median AD in the ratio 2 **:**1. Hence by the section formula, the coordinates of G are

$$x = \frac{2.\frac{x_2 + x_3}{2} + 1.x_1}{2+1} = \frac{x_1 + x_2 + x_3}{3}$$

and $$y = \frac{2.\frac{y_2 + y_3}{2} + 1.y_1}{2+1} = \frac{y_1 + y_2 + y_3}{3}$$

Therefore, the coordinates of the centroid are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

Example: The coordinates of the centroid of the triangle whose vertices are (3, 5), (5, 6) and (4, -5) are $\left(\frac{3+5+4}{3}, \frac{5+6-5}{3}\right)$ i.e., (4, 2)

12.9 Area of a triangle: The area of a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is the absolute value of $\frac{1}{2}.\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Proof:

Let, $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the coordinates of the vertices of the triangle ABC. We draw perpendiculars AL, BM and CN from A, B and C on the x-axis. It is clear from the figure,

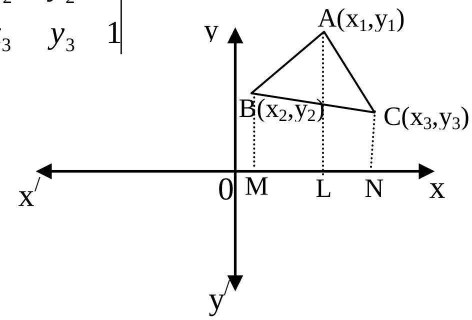


Figure – 12.8

The area of ΔABC = Area of trapezium ABML + Area of trapezium ALNC – Area of trapezium BMNC.

Since, the area of the trapezium = $\frac{1}{2}$(Sum of the parallel sides)$\times$ (Perpendicular distance between the parallel sides)

So, the area of the triangle ABC can be given as

$$\Delta ABC = \frac{1}{2}(BM + AL)ML + \frac{1}{2}(AL + CN)LN - \frac{1}{2}(BM + CN)MN$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}(x_1y_2 + x_1y_1 - x_2y_2 - x_2y_1 + x_3y_1 + x_3y_3 - x_1y_1 - x_1y_3 - x_3y_2 - x_3y_3 + x_2y_2 + x_2y_3)$$

$$= \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$$

$$= \frac{1}{2}[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}.\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad \text{(Proved)}$$

Example: Find the area of the triangle whose vertices are A(2, 3), B(5, 7) and C(-3, 4).

Solution: The area of the ΔABC = $\frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 5 & 7 & 1 \\ -3 & 4 & 1 \end{vmatrix}$

$$= \frac{1}{2}[2(7-4) - 5(3-4) - 3(3-7)]$$

$$= \frac{1}{2}[6 + 5 + 12]$$

$$= \frac{1}{2}(23)$$

= 11.5 square units (Answer)

Note: Three points must be collinear if they form a triangle whose area is zero.

Example: Show that three points A(3, 5), B(-7, 5) and C(1, 5) are collinear.

Solution: Area of the ΔABC = $\frac{1}{2}\begin{vmatrix} 3 & 5 & 1 \\ -7 & 5 & 1 \\ 1 & 5 & 1 \end{vmatrix}$

$$= \frac{1}{2}[3(5-5) + 7(5-5) + 1(5-5)]$$

$$= \frac{1}{2}.0$$

$$= 0$$

So, the points A, B and C are collinear. (Proved)

Another way: AB = $\sqrt{(3+7)^2 + (5-5)^2} = \sqrt{100+0} = \sqrt{100} = 10$

BC = $\sqrt{(-7-1)^2 + (5-5)^2} = \sqrt{64+0} = \sqrt{64} = 8$

AC = $\sqrt{(3-1)^2 + (5-5)^2} = \sqrt{4+0} = \sqrt{4} = 2$

Here, BC + AC = 8 + 2 = 10 = AB
So, the points A, B and C are collinear. (Proved)

12.10 Area of a quadrilateral: The area of a quadrilateral which vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ is

the absolute value of $\frac{1}{2}[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4]$

Example: Find the area of a quadrilateral whose vertices are A(1, 1), B(3, 5), C(4, -1) and D(2, 2).

Solution: Area of quad. $= \left| \frac{1}{2}[1.5 - 3.1 + 3.(-1) - 4.5 + 4.2 - 2.4 + 2.1 - 1.2] \right|$

$= \left| \frac{1}{2}[5 - 3 - 3 - 20 + 8 - 8 + 2 - 2] \right|$

$= \left| \frac{1}{2}(-21) \right|$

$= |-10.5|$

$= 10.5$ square units (Answer)

12.11 Straight line: The minimum distance between two distinct points is known as straight line. In the figure AB is a straight line.

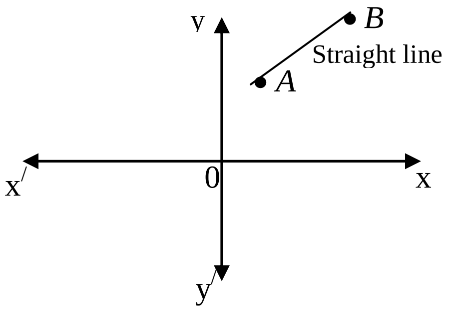


Figure – 12.9

12.12 Slope or gradient of a straight line: The slope or gradient of a straight line is the tangent of the angle formed by the line above the x – axis towards its positive direction whatever be the position of the line. If θ is the angle formed by the line and the positive direction of the x – axis, the slope of the line is

$$m = \tan\theta$$

In terms of the coordinates, the slope of the line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

The slope of a line is generally denoted by m.

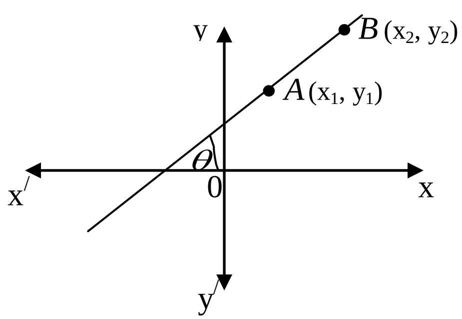


Figure – 12.10

12.13 Different forms of equations of the straight line:

1. **Equations of the coordinate axes:** The value of y ordinates of all points on the x-axis is always 0 (zero). And the value of x ordinates of all points on the y-axis is always 0 (zero). Therefore, the equation of the x-axis is $y = 0$ and
The equation of the y-axis is $x = 0$.

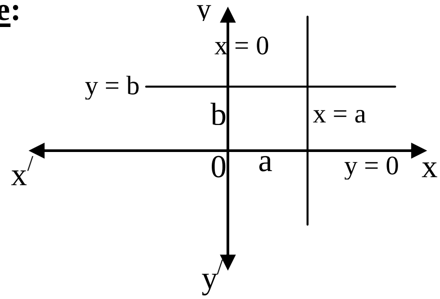


Figure – 12.11

2. **Equations of lines parallel to the coordinate axes:**
Let P(x, y) be any point on a line parallel to x-axis at a distance b from it. For any position of the point P lying on this line $y = b$.
So, the equation of this line is $y = b$.
Similarly, the equation of the line parallel to the y-axis at a distance a from it is $x = a$.

3. **Origin-slope form:** The equation of a line passing through the origin and having slope m is : $y = mx$.

4. **Slope intercept form:** The equation of the line with the slope m and an intercept on y-axis is : $y = mx + c$. This is the general equation of straight line.

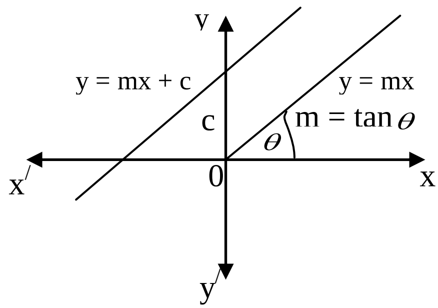


Figure – 12.12

5. **Two intercepts form:** The equation of a line having intercepts a and b on the coordinate axes is : $\frac{x}{a} + \frac{y}{b} = 1$

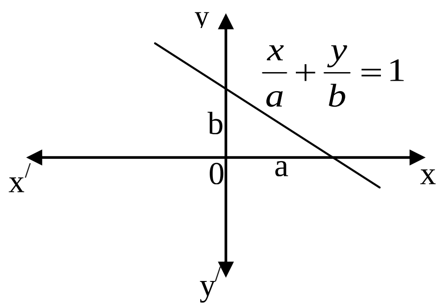


Figure – 12.13

6. **Slope-point form:** The equation of a straight line having a slope m and passing through the point (x_1, y_1) is
$$y - y_1 = m(x - x_1)$$

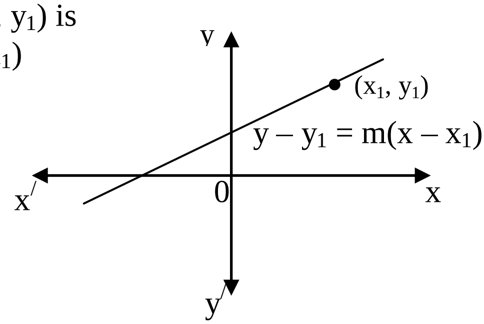


Figure – 12.14

7. **Two points form:** The equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$$

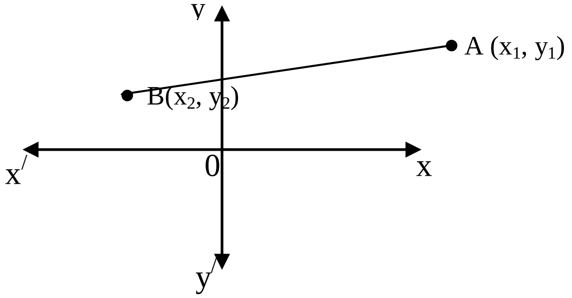


Figure – 12.15

8. **Parallel line form:** The equation of the parallel line to the line $ax + by + c = 0$ is $ax + by + k = 0$; k is a constant.
9. **Perpendicular line form:** The equation of the perpendicular line to the line $ax + by + c = 0$ is $bx - ay + k = 0$; k is a constant.

<u>Note</u>: Let m_1 and m_2 are slopes of two straight lines respectively. These two lines will be perpendicular to each other if $m_1 \times m_2 = -1$ and will be parallel to each other if $m_1 = m_2$.

12.14 <u>Circle</u>: The circle is the locus of a point which moves in such a way that its distance from a fixed point always remains constant. The fixed point is known the centre of the circle and the constant distance is called the radius of the circle.
The equation of the circle whose centre is (h, k) and the radius is a is

$$(x - h)^2 + (y - k)^2 = a^2$$

<u>Example</u>: Find the equation of the circle whose centre is (2, 3) and the radius is 5.
Solution: The required equation is $(x - 2)^2 + (y - 3)^2 = 5^2$

$$\Rightarrow \quad x^2 - 4x + 4 + y^2 - 6y + 9 = 25$$
$$\Rightarrow \quad x^2 + y^2 - 4x - 6y - 12 = 0 \qquad \text{(Answer)}$$

12.15 <u>Some worked out examples</u>:
<u>Example (1)</u>: Show that the points (6, 6), (2, 3) and (4, 7) are the vertices of a right angled triangle. [AUB-01]
Solution: Let A, B and C be the points (6, 6), (2, 3) and (4, 7) respectively, then
$AB^2 = (6 - 2)^2 + (6 - 3)^2 = 16 + 9 = 25$
$BC^2 = (2 - 4)^2 + (3 - 7)^2 = 4 + 16 = 20$
$CA^2 = (4 - 6)^2 + (7 - 6)^2 = 4 + 1 = 5$
$BC^2 + CA^2 = 20 + 5 = 25 = AB^2$
$\therefore \; AB^2 = BC^2 + CA^2$.
So, $\angle ACB = 1$ right angle

Hence, the points A(6, 6), B(2, 3) and C(4, 7) are the vertices of a right angled triangle.

<u>Example (2)</u>: Find the coordinates of the circum centre of a triangle whose coordinates are (3, -2), (4, 3) and (-6, 5). Hence find the circum radius and circumference.

Solution: Let A(3, -2), B(4, 3) and C(-6, 5) be the vertices of the triangle and P(x, y) be the circum centre. So, by definition,

$PA = PB = PC$

⇨ $PA^2 = PB^2 = PC^2$.

Now by the distance formula

$PA^2 = (x – 3)^2 + (y + 2)^2 = x^2 – 6x + 9 + y^2 + 4y + 4 = x^2 + y^2 – 6x + 4y + 13$

$PB^2 = (x – 4)^2 + (y – 3)^2 = x^2 – 8x + 16 + y^2 – 6y + 9 = x^2 + y^2 – 8x – 6y + 25$

$PC^2 = (x + 6)^2 + (y – 5)^2 = x^2 + 12x + 36 + y^2 – 10y + 25 = x^2 + y^2 + 12x – 10y + 61$

Now $PA^2 = PB^2$.

⇨ $x^2 + y^2 – 6x + 4y + 13 = x^2 + y^2 – 8x – 6y + 25$

⇨ $2x + 10y = 12$

⇨ $x + 5y = 6$ [Dividing by 2]

⇨ $x = 6 – 5y$ - - - (i)

And $PB^2 = PC^2$.

⇨ $x^2 + y^2 – 8x – 6y + 25 = x^2 + y^2 + 12x – 10y + 61$

⇨ $-20x + 4y = 36$

⇨ $-5x + y = 9$ [Dividing by 4]

⇨ $-5(6 – 5y) + y = 9$ [Using (i)]

⇨ $-30 + 25y + y = 9$

⇨ $26y = 39$

⇨ $y = \frac{39}{26}$

⇨ $y = \frac{3}{2}$

Putting the value of y in (i), we get

$$x = 6 – 5.\frac{3}{2} = -\frac{3}{2}$$

Therefore, the coordinates of the circumcentre P is $(-\frac{3}{2}, \frac{3}{2})$ (Answer)

Now, the circum radius of ΔABC, $r = \sqrt{(-\frac{3}{2}-3)^2+(\frac{3}{2}+2)^2}$ $[= PA]$

$$= \sqrt{\frac{81}{4}+\frac{49}{4}}$$

$$= \sqrt{\frac{130}{4}}$$

$$= \frac{\sqrt{130}}{2} \qquad \text{(Answer)}$$

And the circumference $= 2\pi r$

$$= 2\pi \frac{\sqrt{130}}{2}$$

$$= \pi\sqrt{130}$$

Example (3): Find the area of the triangle whose vertices are A(3, 1), B(2k, 3k), C(k, 2k) and prove that these three points will be collinear if k = – 2. [AUB-02]

Solution: Area of the Δ ABC $= \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 2k & 3k & 1 \\ k & 2k & 1 \end{vmatrix}$

$$= \frac{1}{2}[3(3k - 2k) - 2k(1 - 2k) + k(1 - 3k)]$$

$$= \frac{1}{2}(3k - 2k + 4k^2 + k - 3k^2)$$

$$= \frac{1}{2}(k^2 + 2k)$$

$$= \frac{1}{2}k(k + 2) \text{ square units}$$

Now , the points A, B, C will be collinear if the area of the triangle is zero,

i.e., $\frac{1}{2}k(k + 2) = 0$

Or, $k(k + 2) = 0$

So, $k = 0$ or, $k + 2 = 3$ => $k = -2$.

But $k = 0$ makes B and C a point.

So, $k = -2$ (Proved)

Example (4): The coordinates of the three towns A, B and C are (4, -2), (2, -2) and (6, -2). Find the distances of the towns from one to another. And prove that these three towns are situated on a straight line. [RU-80, 82 A/C]

Solution: The distance of town A and B, AB = $\sqrt{(4-2)^2 + (-2+2)^2} = \sqrt{4} = 2$ units

The distance of town B and C, BC = $\sqrt{(2-6)^2 + (-2+2)^2} = \sqrt{16} = 4$ units

The distance of town A and C, AC = $\sqrt{(6-4)^2 + (-2+2)^2} = \sqrt{4} = 2$ units

Here, AB + AC = 2 + 2 = 4 = BC

So, these three town are situated on a straight line. (Proved)

Example (5): Find the equation and the slope of the straight line joining the points (3, 5) and (2, 3). [RU-80, 82 A/C]

Solution: The equation of the line which passes through the points (3, 5) and (2, 3) is

$$y - 5 = \frac{5-3}{3-2}(x-3)$$

Or, $y - 5 = 2(x - 3)$

Or, $y = 2x - 1$ (Answer)

We know that, m is the slope of the line $y = mx + c$.

So, the slope of the line $y = 2x - 1$ is 2. (Answer)

Example (6): Prove that the first degree equation $ax + by + c = 0$ always represents a equation of a straight line, i.e., the general equation of straight line.

Proof: Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be three points on the locus represented by the equation $ax + by + c = 0$.

$\therefore$ $ax_1 + by_1 + c = 0$ --- (i)

$ax_2 + by_2 + c = 0$ --- (ii)

$ax_3 + by_3 + c = 0$ --- (iii)

Doing (i) – (ii), we have $a(x_1 - x_2) + b(y_1 - y_2) = 0$

Or, $\frac{y_1 - y_2}{x_1 - x_2} = -\frac{a}{b}$

Again doing (ii) – (iii), we have $a(x_2 - x_3) + b(y_2 - y_3) = 0$

Or, $\frac{y_2 - y_3}{x_2 - x_3} = -\frac{a}{b}$

$\therefore$ $\frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_3}{x_2 - x_3}$

Or, $(y_1 - y_2)(x_2 - x_3) = (y_2 - y_3)(x_1 - x_2)$

Or, $x_2y_1 - x_2y_2 - x_3y_1 + x_3y_2 - x_1y_2 + x_1y_3 + x_2y_2 - x_2y_3 = 0$

Or, $\frac{1}{2}\,[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)] = 0$

Or, $\frac{1}{2}.\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

That is, the area of the triangle formed by A, B and C is zero. Hence, the points A, B and C are collinear. But A, B and C are any three points on the locus of $ax + by + c = 0$. Therefore, the equation $ax + by + c = 0$ always represents a straight line. (Proved)

Example (7): Find the equation of the straight line passing through the point (4, 5) and the sum of its intercepts on the axes is 18.

Solution: Let us consider the equation of the required line be

$$\frac{x}{a}+\frac{y}{b}=1 \quad --- \quad (i)$$

This line passes through the point (4, 5), therefore, we have

$$\frac{4}{a}+\frac{5}{b}=1 \quad --- \quad (ii)$$

And $a + b = 18$ Or, $a = 18 - b$ --- (iii)

Using (iii) in (ii), we get

$$\frac{4}{18-b}+\frac{5}{b}=1$$

Or, $\frac{4b+5(18-b)}{b(18-b)}=1$

Or, $4b + 90 - 5b = 18b - b^2$

Or, $b^2 - 19b + 90 = 0$

Or, $b^2 - 10b - 9b + 90 = 0$

Or, $b(b - 10) - 9(b - 10) = 0$

Or, $(b - 10)(b - 9) = 0$

$\therefore$ $b - 10 = 0$ Or, $b = 10$

Or, $b - 9 = 0$ Or, $b = 9$

When $b = 10$, $a = 18 - 10 = 8$

And when $b = 9$, $a = 18 - 9 = 9$

So, the required equation are

$$\frac{x}{8}+\frac{y}{10}=1 \quad \text{and,} \quad \frac{x}{9}+\frac{y}{9}=1 \quad \text{(Answer)}$$

Example (8): Find the equation of the straight line passing through the point (-3, 1) and perpendicular to the line $5x - 2y + 7 = 0$. [AUB-02]

Solution: The equation of the line perpendicular to the line $5x - 2y + 7 = 0$ is

$2x + 5y + k = 0$ --- (i)

Since, line (i) passes through the point (-3, 1)

$2.(-3) + 5.1 + k = 0$

Or, $-6 + 5 + k = 0$

$\therefore$ $k = 1$

Therefore, the equation of the required line is $2x + 5y + 1 = 0$ (Answer)

Example (9): Find the equation of the line passing through (2, 5) and (5, 6). And prove that this line is perpendicular to the line passing through (-4, 5) and (-3, 2).

Solution: The equation of the line passing through the points (2, 5) and (5, 6) is

$$\frac{y-5}{5-6}=\frac{x-2}{2-5}$$

Or, $\frac{y-5}{-1}=\frac{x-2}{-3}$

Or, - 3y + 15 = - x + 2

Or, x – 3y + 13 = 0; which is the required equation. (Answer)

Let the slope of this line, $m_1 = \frac{-1}{-3}=\frac{1}{3}$

And, let the slope of the line joining the points (-4, 5) and (-3, 2),

$$m_2 = \frac{5-2}{-4-(-3)}=\frac{3}{-1}=-3$$

$\therefore$ $m_1 \times m_2 = \frac{1}{3}\times(-3)=-1$

Hence, the two lines are perpendicular to each other. (Proved)

<u>Example (10)</u>: Find the equation of the straight line passing through the point of intersection of the two lines 2x + 3y – 1 = 0 and x – 2y + 3 = 0 and intersects equal portion from both axes.

Solution: Given that 2x + 3y – 1 = 0 - - - (i)

x – 2y + 3 = 0 - - - (ii)

Doing (i) – 2×(ii), we have 7y – 7 = 0

$\therefore$ y = 1

Putting the value of y in (ii), we have

x – 2.1 + 3 = 0

$\therefore$ x = -1

So, the point of intersection of the given two straight line is (-1, 1)

Let the equation of the line which intersects equal parts from both axes be

$$\frac{x}{a}\pm\frac{y}{a}=1$$

That is, $\frac{x}{a}+\frac{y}{a}=1$ - - - (iii) Or, $\frac{x}{a}-\frac{y}{a}=1$ - - - (iv)

Since both the lines pass through (-1, 1), hence putting (-1, 1) in equation (iii), we have

$$\frac{-1}{a}+\frac{1}{a}=1 \quad \text{Or,} \quad a = 0$$

So, the line is $\frac{x}{0}+\frac{y}{0}=1$, which is meaning less.

Now, putting (-1, 1) in equation (iv), we have

$$\frac{-1}{a}-\frac{1}{a}=1 \quad \text{Or,} \quad a = -2$$

So, the equation (iv) becomes $\frac{x}{-2}-\frac{y}{-2}=1$

Or, $x - y = -2$
i.e., $x - y + 2 = 0$, which is the required equation. (Answer)

Example (11): Find the coordinates of the points at which straight line $3x - 4y + 12 = 0$ intersects the coordinates axes. What is the slope of the line? And find the length of the intersecting part. [NU-01 A/C]

Solution: The equation of the line is given by $3x - 4y + 12 = 0$ - - - (i)
When $y = 0$ then by (i), we have $x = 4$
and when $x = 0$ then by (i) we have $y = 3$
So, the coordinates of the intersecting points are (4, 0) and (0, 3) (Answer)
We can write equation (i) as follows:

$$y = \frac{3}{4}x + 3 \quad \text{- - - (ii)}$$

Comparing equation (ii) with the equation $y = mx + c$, we get the slope of the given line as follows: $m = \frac{3}{4}$ (Answer)

And the distance between the points (4, 0) and (0, 3) is

$$\sqrt{(4-0)^2 + (0-3)^2} = \sqrt{25} = 5$$

So, the length of the intersecting part = 5 units. (Answer)

Example (12): Find the equation of the circle whose centre is (4, 6) and it passes through the point (1, 1). [RU-89 A/C]

Solution: We know that the equation of a circle is

$$(x - h)^2 + (y - k)^2 = a^2 \quad \text{- - - (i)}$$

The centre of the circle is (4, 6), so, $h = 4$ and $k = 6$. Putting the value of h and k in the equation, we get

$$(x - 4)^2 + (y - 6)^2 = a^2 \quad \text{- - - (ii)}$$

Since, the circle passes through the point (1, 1),

$$(1 - 4)^2 + (1 - 6)^2 = a^2$$

⇨ $9 + 25 = a^2$

⇨ $a^2 = 34$

So, the equation of the circle is

$$(x - 4)^2 + (y - 6)^2 = 34$$

⇨ $x^2 - 8x + 16 + y^2 - 12y + 36 = 34$

⇨ $x^2 + y^2 - 8x - 12y + 18 = 0$ (Answer)

Example (13): A firm invested Tk. 20000 in a new factory that has a net return of Tk. 2000 per year. An investment of Tk. 40000 would yield a net income of Tk. 8000 per year. What is the linear relationship between investment and annual income? What would be the return on an investment of Tk. 30000? [AUB-02 MBA]

Solution: Let x and y coordinates represent the investment and the annual income respectively. Then the required linear relationship between investment and annual income is the equation of the straight line joining the points (20000, 2000) and (40000, 8000) and the equation is

$$y - 2000 = \frac{2000 - 8000}{20000 - 40000}(x - 20000)$$

Or, $y - 2000 = 0.3(x - 20000)$

Or, $y - 2000 = 0.3x - 6000$

i.e., $0.3x - y - 4000 = 0$, this is the required relationship. (Answer)

Again when the investment $x = 30000$, the annual income y can be found by putting the value of x in the above equation, i.e.,

$0.3(30000) - y - 4000 = 0$

Or, $9000 - y - 4000 = 0$

Or, $y = 5000$

So, the required income = Tk. 5000 (Answer)

Example (14): The total expense of a firm y, are partly constant and partly proportional to the number of the products x. The total expenses are Tk. 5000 when 20 products are made and Tk. 7500 when 40 products are made. [AUB-03 MBA]

(i) Find the linear relationship between x and y.
(ii) Find the variable cost for a product and the fixed cost.
(iii) What would be the total expenditure if 30 products are made?

Solution: (i) Then the required linear relationship between products and expenses is the equation of the straight line joining the points (20, 5000) and (40, 7500) and the equation is

$$y - 5000 = \frac{5000 - 7500}{20 - 40}(x - 20)$$

Or, $y - 5000 = 125(x - 20)$

Or, $y - 5000 = 125x - 2500$

i.e., $y = 125x + 2500$, this is the required relationship. (Answer)

(ii) If x represents the number of products and y represents the cost to produce x units product then the equation $y = mx + c$ means that, m is the variable cost per unit product and c is the fixed cost.

So, the equation $y = 125x + 2500$ shows that the variable cost = Tk. 125 and the fixed cost = Tk. 2500. (Answer)

(iii) When $x = 30$ units products are made, the total expenditure,

$y = \text{Tk. } [125(30) + 2500] = \text{Tk. } 6250$ (Answer)

12.16 <u>Exercise</u>:

1. Plot the points with the following coordinates: A(-5, -5), B(3, 2), C(-3, 4) and D(5, 0) in the Cartesian plane.
2. Find the coordinates of the mid-point of the join of points (-1, 5) and (7, 3) [Answer: (3, 4)]
3. Find the distance between two points (4, - 1) and (7, 3). [Answer: 5]
4. Find the distance between two points $(1+\sqrt{2}, 2)$ and $(1, 1-\sqrt{2})$. [Answer: $5+2\sqrt{2}$]
5. Find the coordinates of the point which internally divides the line joining the points (4, -5) and (6, 8) in the ratio 2 **:** 1. [Answer: $(\frac{16}{3}, \frac{11}{3})$]
6. Find the coordinates of the points which divides the connecting line of points (8, 9) and (-7, 4) internally in the ratio 2 **:** 3 and externally in the ratio 4 **:** 3. [Answer: (2, 7) and (-52, -11)]
7. Find the coordinates of the centroid of the triangle whose vertices are (3, 2), (-1, -4) and (-5, 6). [Answer: (-1, 4/3)]
8. Show that the points (1, 4), (3, -2) and (-3, 16) are collinear.
9. Show that the points A(-6, 6), B(2, 6) and C(2, 10) are the vertices of a right angled triangle.
10. If A(-3, 5), B(6, 5) and C(a, -7) are the vertices of the triangle ABC whose $\angle ABC = 1$ right angle, find the values of a. [Answer: 6]
11. Prove that the points (2a, 4a), (2a, 6a) and (2a $+\sqrt{3}$ a, 5a) are vertices of an equilateral triangle.
12. Show that the triangle whose vertices are (1, 10), (2, 1) and (-7, 0) is an isosceles triangle.
13. Show that the points (2,-2), (8, 4), (5, 7) and (-1, 1) are the vertices of a parallelogram. [NU-97]
14. If a warehouse P(x, y) is equidistant from the three markets A(4, 2), B(5, 3) and C(6, 5), find the coordinates of the warehouse. [Answer: $(\frac{1}{2}, \frac{13}{2})$]
15. If the points A(x, y), B(3, 4) and C(-2, 3) form an equilateral triangle, find the coordinates of the point A. [Answer: $(\frac{1+\sqrt{3}}{2}, \frac{7+5\sqrt{3}}{2})$ or $(\frac{1-\sqrt{3}}{2}, \frac{7-5\sqrt{3}}{2})$]
16. Find the area of the triangle whose vertices are A(2, -1), B(-3, -4) and C(0, 2). [Answer: 10.5 square units]
17. Prove that the points (a, 0), (0, b) and (1, 1) will be collinear if $\frac{1}{a} + \frac{1}{b} = 1$.
18. Find the area of the quadrilateral whose vertices are (-2, -1), (1, 3), (5, 6) and (2, 2). [Answer: 7 square units]
19. Find the equation of the straight line parallel to the x-axis and passing through (4, 5) [Answer: y = 5]

20. Find the equation of the straight line parallel to the y-axis and passing through (3, 7) [Answer: $x = 3$]
21. Find the equation of the straight line which passes through the two points (4, 5) and (3, 7) [Answer: $2x + y - 13 = 0$]
22. Find the length and the equation of the line which passes through the two points A(9, -2) and B(3, -3). And also find the coordinates of the points at which the line intersects the coordinate axes. [Answer: Length = $\sqrt{37}$, eq^n: $x - 6y - 21 = 0$, the line intersects the x-axis at (21, 0) and intersects the y-axis at (0, -7/2)]
23. In a triangle with vertices P(0, 6), Q(-2, -2) and R(4, 2), find the equation of the perpendicular bisector of the side QR. [Answer: $3x + 2y - 3 = 0$]
24. Find the equation of the straight line which is parallel to $2x - 3y - 5 = 0$ and passing through (4, 5). [Answer: $2x - 3y + 7 = 0$]
25. Find the equation of the straight line which is perpendicular to $2x + 5y - 5 = 0$ and passing through (3, -2). [Answer: $5x - 2y - 19 = 0$]
26. Find the equation of the circle whose centre is (0, 0) and it passes through the point (4, 3). [Answer: $x^2 + y^2 = 25$]
27. Find the equation of the circle whose centre is (0, 4) and it passes through the point (-3, 1). [Answer: $x^2 + y^2 - 8y - 2 = 0$]
28. Find the equation of the circle whose centre is (4, 5) and which passes through the centre of the circle $x^2 + y^2 + 4x + 6y - 12 = 0$.
    [Answer: $x^2 + y^2 - 8x - 10y - 59 = 0$]
29. As the number of units manufactured increases from 4000 to 6000, the total cost of production increases from Tk. 22000 to Tk. 30000. Find the linear relationship between the cost (y) and the number of units made (x).
    [Answer: $4x - y + 6000 = 0$] [RU-83, 87 A/C]
30. A firm invested Tk. 10 millions in a new factory that has a net return of Tk. 0.5 million per year. An investment of Tk. 20 millions would yield a net income of Tk. 2 millions per year. What is the linear relationship between investment and annual income? What would be the return on an investment of Tk. 50 millions? [Answer: $3x - 20y - 20 = 0$, Tk. 6.5 millions] [NU-02 A/C]
31. An investment of Tk. 90000 in a certain business yields an income of Tk. 8000. An investment of Tk. 50000 yields an income of Tk. 5000. What is the linear relationship between investment and income? [Answer: $3x - 40y + 50000$]
32. The total expense of a firm y, are partly constant and partly proportional to the number of the products x. The total expenses are Tk. 1040 when 12 products are made and Tk. 1600 when 20 products are made.
    a. Find the linear relationship between x and y.
    b. Find the variable cost for a product and the fixed cost.
    c. What would be the total expenditure if 15 products are made?

    [Answer: (a) $70x - y + 200 = 0$, (b) Tk. 70 & Tk. 200 and (c) Tk. 1250]

13

Chapter

Linear Programming

Highlights:

13.1 Introduction: Linear programming is a technique for determining an optimum schedule (such as maximizing profit or minimizing cost) of interdependent activities in view of the available resources. Programming is just another word for "Planning" and refers to the process of determining a particular plan of actions from amongst several alternatives. A linear programming problem with only two variables presents a simple case, for which the solution can be derived using a graphical method. We can solve any linear programming problem by simplex method. A linear programming problem may have a solution (a definite and unique solution or an infinite number of optimal solutions or an unbounded solution) or not. In this chapter we shall discuss the formulation and the solution techniques of general types of linear programming problems.

13.2 What is optimization: The fundamental problem of optimization is to arrive at the 'best' possible decision under some given circumstances. Some typical optimization problems, taken from a variety of practical field of interest. In each of these cases, we have to optimize (maximize or minimize) certain functions under certain constraints.

13.3 Summation symbol: The Greek capital letter $\sum$ (sigma) is the mathematical symbol for summation. If f(i) denotes some quantity whose value depends on the value of i, the expression

$$\sum_{i=1}^{4} i$$

is read as 'sigma i, i going from 1 to 4' and means to insert 1 for i, then 2 for i, then 3 for i, 4 for i, and sum the results. Therefore,

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10.$$

In the above example, the i under the sigma symbol is called the index of the summation. In this way,

$$\sum_{i=1}^{3}(i^2 - 1) = (1^2 - 1) + (2^2 - 1) + (3^2 - 1) = 0 + 3 + 8 = 11$$

$$\sum_{i=1}^{4} p = p + p + p + p = 4p$$

$$\sum_{i=1}^{n} p = np$$

$$\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4.$$

13.4 <u>Linear programming</u>: The general linear programming problem is to find a vector $(x_1, x_2, \ldots, x_j, \ldots, x_n)$ which minimizes or maximizes the linear form

$c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n$

subject to the linear constraints

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n = b_2$

- - - - - - - - -

$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n = b_i$

- - - - - - - - -

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n = b_m$

and $x_j \geq 0$; $j = 1, 2, 3, \ldots, n$.

where the a_{ij}, b_i and c_j are given constants and $m < n$. We shall always assume that the constraints equation have been multiplied by (–1) where necessary to make all $b_i \geq 0$, because of the variety of notations in common use, one will find the general linear programming problem stated in many forms. The more common forms are the following:

(a) Minimize $\sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$; $i = 1, 2, 3, \ldots, m$

and $x_j \geq 0$; $j = 1, 2, 3, \ldots, n$

(b) Minimize $f(X) = CX$

subject to $AX = b$

and $X \geq 0$

Where $A = (a_{ij})$, $C = (c_1, c_2, \ldots, c_n)$ is a row vector, $X = (x_1, x_2, \ldots, x_n)$ is a column vector, $b = (b_1, b_2, \ldots, b_m)$ is a column vector and 0 is an n-dimensional null column vector.

(c) Minimize CX

subject to $x_1P_1 + x_2P_2 + \ldots + x_nP_n = P_o$

and $X \geq 0$

Where P_j for $j = 1, 2, \ldots, n$ is the jth column of the matrix A and $P_o = b$.

13.5 <u>Formulation</u>: As in any other planning problem, the operations researcher must analyze the goals and the system in which the solution must operate. The complex of inter related components in a problem area, referred to by operations researchers as a 'system' is the environment of a decision, and it represents planning premises. To solve any business problem or production problem, we have to transfer it as mathematical problem. This problem transformation is known as formulation.

<u>Example</u>: (Production planning problem)

A firm manufactures 3 products A, B and C. The profit per unit sold of each product is Tk. 3, Tk. 2 and Tk. 4 respectively. The time required to manufacture one unit of each of the three products and the daily capacity of the two machines P and Q is given in the table below:

Machine	Time per unit (minutes) product			Machine capacity (minutes/day)
	A	B	C	
P	4	3	5	2000
Q	2	2	4	2500

It is required to determine the daily number of units to be manufactured for each product, so as to maximize the profit. However, the firm must manufacture at least 100 A's, 200 B's and 50 C's but no more that 150 A's. It is assumed that all the amounts produced are consumed in the market. Formulate this problem as linear programming problem.

Solution: <u>Step–1</u>: We study the situation to find the key decision to be made and in this connection looking for variables helps considerably.

<u>Step-2</u>: Select symbols for variable quantities identified in Step-1. Let the number of units of the products A, B and C manufactured daily be designated x_1, x_2 and x_3 respectively.

<u>Step-3</u>: Express feasible alternatives mathematically in terms of the variables. These feasible alternatives are those which are physically, economically and financially possible. Since it is not possible to manufacture any negative quantities, it is quite obvious that in the present situation feasible alternatives are sets of variables of x_1, x_2 and x_3 satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

<u>Step-4</u>: Identify the objective quantitatively and express it as a linear function of variables. The objective here is to maximize the profit. In view of the assumption that all the units produced are consumed in that market, it is given by the linear function

$$z = 3x_1 + 2x_2 + 4x_3$$

Step-5: Express in words the influencing factors or constraints (or restrictions) which occur generally because of the constraints on availability (resources) or requirements (demands). Express these restrictions also as linear equalities/inequalities in terms of variables. Here in order to produce x_1 units of product A, x_2 units of product B and x_3 units of product C, the total times needed on machines P and Q are given by

$$4x_1 + 3x_2 + 5x_3 \text{ and } 2x_1 + 2x_2 + 4x_3 \text{ respectively.}$$

Since the manufacturer does not have more than 2000 minutes available on machine P and 2500 minutes available on machine Q, we must have

$$4x_1 + 3x_2 + 5x_3 \leq 2000 \text{ and } 2x_1 + 2x_2 + 4x_3 \leq 2500 .$$

Also, additional restrictions are $100 \leq x_1 \leq 150$, $x_2 \geq 200$, $x_3 \geq 50$.

Hence, the manufacturer's problem can be put in the following mathematical form:

$$\begin{aligned} \text{Maximize} \quad & z = 3x_1 + 2x_2 + 4x_3 \\ \text{Subject to} \quad & 4x_1 + 3x_2 + 5x_3 \leq 2000 \\ & 2x_1 + 2x_2 + 4x_3 \leq 2500 \\ & 100 \leq x_1 \leq 150,\ x_2 \geq 200,\ x_3 \geq 50. \end{aligned}$$

Example: (Blending problem)

A firm produces an alloy having the following specifications

(i) specific gravity ≤ 0.98

(ii) chromium $\geq 8\%$

(iii) melting point $\geq 450^0$ C

Raw materials A, B and C having the properties shown in the table can be used to make the alloy:

Property	Raw material		
	A	B	C
Specific gravity	0.92	0.97	1.04
Chromium	7%	13%	16%
Melting point	440^0 C	490^0 C	480^0 C

Cost of the various raw materials per unit ton are: Tk. 90 for A, Tk. 280 for B and Tk. 40 for C. Find the proportion in which A, B and C be used to obtain an alloy of desired properties while the cost of raw materials is minimum.

Solution: Step-1: Key decision to be made is how much (percentage) or raw materials A, B and C be used for making the alloy.

: Key decision to be made is how much (percentage) or raw materials A, B and C be used for making the alloy.

Step-2: Let the percentage contents of A, B and C be x_1, x_2 and x_3 respectively.

Step-3: Feasible alternatives are sets of values of x_1, x_2 and x_3.

Step-4: Objective is to minimize the cost, i.e.,

$$\text{minimize} \quad z = 90x_1 + 280x_2 + 40x_3$$

Step-5: Constraints are imposed by the specifications required for the alloy. They are

$$0.92x_1 + 0.97x_2 + 1.04x_3 \leq 0.98$$

$7x_1 + 13x_2 + 16x_3 \geq 8$

$440x_1 + 490x_2 + 480x_3 \geq 450$

and $x_1 + x_2 + x_3 = 100$

Hence, the blending problem can be put in the following mathematical form:

Minimize $z = 90x_1 + 280x_2 + 40x_3$

Subject to $0.92x_1 + 0.97x_2 + 1.04x_3 \leq 0.98$

$7x_1 + 13x_2 + 16x_3 \geq 8$

$440x_1 + 490x_2 + 480x_3 \geq 450$

$x_1 + x_2 + x_3 = 100$

$x_1, x_2, x_3 \geq 0$

13.6 Some important definitions:

1. **Convex set:** A subset $S \subset R^n$ is said to be convex set if for each pair of point x, y in S, the line segment $[x : y] = \{ax + (1 - a)y, 0 \leq a \leq 1\}$ joining the points x and y is contained in S.

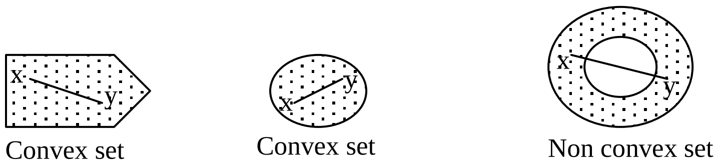


Figure – 13.1

2. **Feasible solution:** A feasible solution to a linear programming problem (LPP) is a solution which satisfies the constraints (equality or inequality constraints and the non-negativity constraints)
3. **Basic solution:** A basic solution to an LPP with m constraints in n variables is a solution obtained by setting n – m variables equal to zero and solving for the remaining m variables, provided that the determinant of the coefficients of these m variables is non zero. The m variables are called basic variables. Basic solution may or may not be feasible solution and conversely feasible solution may or may not be a basic solution.
4. **Basic feasible solution:** A basic feasible solution is a basic solution which also satisfies all basic variables are non-negative.
5. **Non degenerate basic feasible solution:** A non-degenerate basic feasible solution is a basic feasible solution with exactly m positive x_i; that is, all basic variables are positive.
6. **Optimum solution:** A optimum solution to an LPP is a feasible solution which optimizes (minimizes or maximizes) the value of the objective function.
7. **Slack variables:** Let the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i ; \quad i = 1, 2, \ldots, m$$

Then the non-negative variables x_{n+i} which satisfy

$$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i \;; \quad i = 1, 2, \ldots, m$$

are called slack variables.

Note: The variables x_{n+i} are called slack variables, because

Slack = Requirement – Production.

8. **Surplus variables:** Let the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \;; \quad i = 1, 2, \ldots, m$$

Then the non-negative variables x_{n+i} which satisfy

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i \;; \quad i = 1, 2, \ldots, m$$

are called surplus variables.

Note: The variables x_{n+i} are called surplus variables, because

Surplus = Production – Requirement.

13.7 <u>Standard form of LP problem</u>: The characteristics of the standard form of linear programming problem are

1. All the constraints are expressed in the form of equations, except the non-negativity constraints which remain inequalities (≥ 0)
2. The right hand side of each constraints equation is non-negative.
3. All the decision variables are non-negative.
4. The objective function is of the maximization or minimization type.

<u>Example</u>: Express the following LPP into standard form:

Maximize $z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$

Solution: Introducing slack variable s_1 and surplus variable s_2, the problem in the standard form can be expressed as

Maximize $z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 + s_1 = 2$

$3x_1 + 4x_2 - s_2 = 12$

$x_1, x_2, s_1, s_2 \geq 0$

13.8 <u>Graphical solution</u>: The solution of any linear programming problem with only two variables can be derived using a graphical method. This method consists of the following steps:

(1)Represent the given problem in mathematical from, i.e., formulate an LP model for the given problem.

(2) Represent the given constraints as equalities on x_1, x_2 co-ordinates plane and find the convex region formed by them.
(3) Plot the objective function.
(4) Find the vertices of the convex region and also the value of the objective function at each vertex . The vertex that gives the optimum value of the objective function gives the optimal solution to the problem.
Note: In general, a linear programming problem may have
(i) a definite and unique optional solution,
(ii) an infinite number of optimal solutions,
(iii) an unbounded solution, and
(iv) no solution.

Example: (product Allocation Problem): A factory uses three different resources for the manufacture of two different products 20 unites of the resource A, 12 units of B and 16 units of C being available. 1 unit of the first product requires 2, 2 and 4 units of the respective resources and 1 unit of the second product requires 4, 2 and 0 units of the respective resources. It is known that the first product gives a profit of two monetary units per unit and the second 3. Formulate the linear programming problem. How many units of each product should be manufactured for maximizing the profit? Solve it graphically.
Solution: Mathematical formulation of the problem:
Step-1: The key decision is to determine the number of units of the two products.
Step-2: Let x_1 units of the first product and x_2 units of the second product be manufactured for maximizing the profit.
Step-3: Feasible alternatives are the sets of the values of x_1 and x_2 satisfying $x_1 \geq 0$ and $x_2 \geq 0$, as negative number of production runs are meaningless (and thus not feasible).
Step-4: The objective is to maximize the profit realized from both the products, i.e., to maximize $z = 2x_1 + 3x_2$
Step-5: Since 1 unit of the first product requires 2, 2 and 4 units, 1 unit of the second product requires 4, 2 and 0 units of the respective resources and the units available of the three resources are 20, 12 and 16 respectively, the constraints (or restrictions) are
$2x_1 + 4x_2 \leq 20$ Or, $x_1 + 2x_2 \leq 10$
$2x_1 + 2x_2 \leq 12$ Or, $x_1 + x_2 \leq 6$
$4x_1 + 0x_2 \leq 16$ Or, $4x_1 \leq 16$
Hence the manufacturer's problem can be put in the following mathematical form:

Maximize $z = 2x_1 + 3x_2$
Subject to the constraints:
$x_1 + 2x_2 \leq 10$
$x_1 + x_2 \leq 6$
$4x_1 \leq 16$
$x_1, x_2 \geq 0$

Graphical solution of the problem:

Step-1: Construct the graph. Consider a set of rectangular Cartesian axes OX_1X_2 in the plane. Each point has coordinate of the type (x_1, x_2) and conversely every ordered pair (x_1, x_2) of real numbers determines a point in the plane.

It is clear that any point which satisfies the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant and conversely for any point (x_1, x_2) in the first quadrant, $x_1 \geq 0$ and $x_2 \geq 0$. Thus our search for the number pair (x_1, x_2) is restricted to the points of the first quadrant only.

Step-2: To graph each constraint in the first quadrant satisfying the constraints, we first treat each in equation as though it were an equation and then find the set of points in the first quadrant satisfying the constraints.

Considering, now, the first constraint as an equation, we have $x_1 + 2x_2 = 10$. Clearly, this is the equation for a straight line and any point on the straight line satisfies the inequality also. Taking into consideration the point (0, 0), we observe that $0 + 2(0) = 0 < 10$, i.e., origin also satisfies the inequality. This indicates that any point satisfying the inequality $x_1 + 2x_2 \leq 10$ lies in the first quadrant on that side of the line, $x_1 + 2x_2 = 10$ which contains the origin.

In a similar way, we see that all points satisfying the constraint $x_1 + x_2 \leq 6$ are the points in the first quadrant lying on or below the line $x_1 + x_2 = 6$.

The set of points satisfying the inequality $x_1 \leq 4$ lies on or towards the left of the line $x_1 = 4$.

Step-3: All points in the area shown shaded in figure satisfy the three constraints $x_1 + 2x_2 \leq 10$ and $x_1 + x_2 \leq 6$ and $x_1 \leq 4$ and also the non-negative restrictions $x_1 \geq 0$, $x_2 \geq 0$

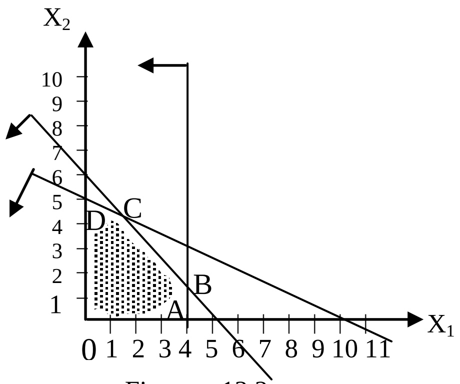


Figure – 13.2

This area is called the convex region or the solution space or the region of feasible solutions. Any point in this region is a feasible solution the given problem. The convex region OABCD is bounded by the lines $x_1 = 0$, $x_2 = 0$, $x_1 + 2x_2 = 10$, $x_1 + x_2 = 6$ and $x_1 = 4$. The five vertices of the convex region are O = (0, 0), A = (4, 0), B = (4, 2), C = (2, 4) and D = (0, 5)

Putting O = (0, 0), A = (4, 0), B = (4, 2), C = (2, 4) and D = (0, 5) in the objective function $2x_1 + 3x_2$, we find 0, 8, 14, 16 and 15 respectively. Here, 16 is the maximum value. Hence, the solution of the problem is $x_1 = 2$, $x_2 = 4$ and the maximum value of the objective function is $z = 16$.

Example: A furniture company makes tables and chairs. Each table takes 4 hours of carpentry and 2 hours in painting and varnishing shop. Each chair requires 3 hours in carpentry and 1 hours in painting and varnishing. During the current production period 240 hours of carpentry and 100 hours. of painting and varnishing time are available. Each table sold yields a profit of Tk. 420 and each chair yields a profit of Tk. 300. Determine the number of tables and chairs to be made to maximize the profit. [AUB-03]

Solution: Let x_1 = the number of tables and x_2 = the number of chairs.

∴ the profit from the tables $420x_1$ and the profit from the chairs $300x_2$

So, the total profit $420x_1 + 300x_2$ which is the objective function. We have to maximize the objective function $z = 420x_1 + 300x_2$

Required carpentry hours for tables $4x_1$ and required carpentry hours for tables $3x_2$

∴ the required carpentry hours $4x_1 + 3x_2$

Since 240 carpentry hours are available

$4x_1 + 3x_2 \leq 240$.

Similarly, for painting and varnishing, we have

$2x_1 + x_2 \leq 100$.

The non negative conditions $x_1, x_2 \geq 0$

So, the linear programming problem (LPP) of the given problem is

Maximize $z = 420x_1 + 300x_2$
Subject to the constraints:
$4x_1 + 3x_2 \leq 240$
$2x_1 + x_2 \leq 100$
$x_1, x_2 \geq 0$

<u>Graph of the problem</u>:

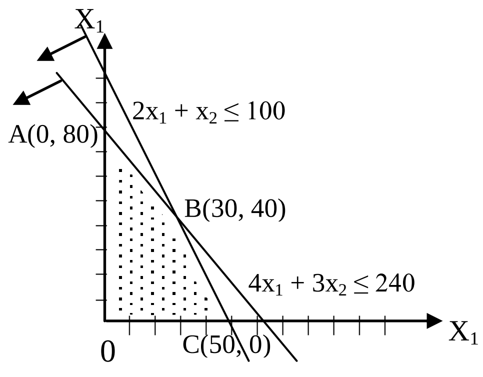


Figure – 13.3

The solution space satisfying the given constraints is shown shaded on the graph paper. Any point in the shaded region is a feasible solution to the given problem. The coordinates of the four vertices of the bounded convex region are
A (0,80), B(30,40),C(50,0) and 0(0,0)
The values of the objective function $z = 420x_1 + 300x_2$ at four vertices are 24000 at A, 24600 at B, 21000 at C and 0 at O.
Since the maximum value of the objective function is 24600 which occurs at the vertices B (30,40). Hence, the solution to the given problem $x_1 = 30$, $x_2 = 40$ and the maximum value = 24600.
Therefore, to maximize the profit, the furniture company should make 30 tables and 40 chairs.

Example: Maximize $3x_1 + 2x_2$
Subjcet to $2x_1 - x_2 \geq -2$
$x_1 + 2x_2 \leq 8$
$x_1, x_2 \geq 0$
Solution: The graph of the problem is as follows:

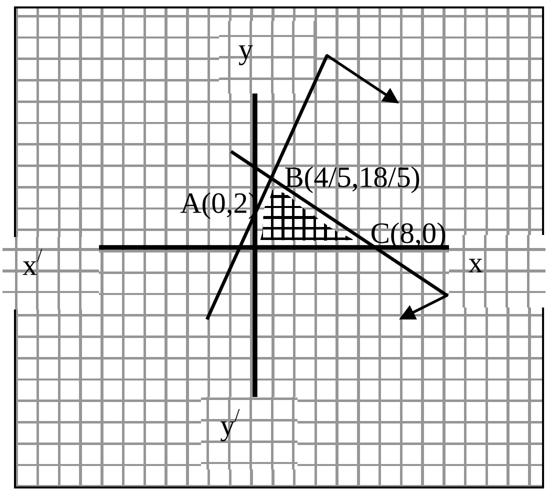


Figure – 13.4

From the graph we get the vertices A(0,2), B(4/5,18/5), C(8,0) and O(0,0). The values of the objective function at these points are
4 at A, 48/5 at B, 24 at C and 0 at O. Here, 24 is the maximum value of the objective function which occurs at the vertices C(8,0). Hence, the solution of the given problem is $x_1 = 8$, $x_2 = 0$ and max. value of z = 24 (Answer)

13.9 Simplex: A simplex is an n dimensional convex polyhedron with exactly (n +1) extreme points,

(i) If n = 0, then the convex polyhedron is a point.
(ii) If n = 1, then the convex polyhedron is a straight line.
(iii) If n = 2, then the convex polyhedron is a triangle.
(iv) If n = 3, then the convex polyhedron is tetrahedron.

Example: Minimize $z = 2x_1 - x_2 + x_3 - 5x_4 + 22x_5$

Subject to: $x_1 - 2x_4 + x_5 = 6$ (1.1)

$x_2 + x_4 - 4x_5 = 3$ (1.2)

$x_3 + 3x_4 + 2x_5 = 10$ (1.3)

$x_j \geq 0$

Solution: The initial basic feasible solution is $x_1 = 6$, $x_2 = 3$, $x_3 = 10$, $x_4 = 0$, $x_5 = 0$, with the value of the objective function for this solution by the unrestricted variable $z = 2x_1 - x_2 + x_3 = 19$. We would like to determine if a different basic feasible solution will yield a smaller value of the objective function or whether the current solution is the optimum. Note that this is equivalent to asking whether one of the non-basic variable, here x_4 and x_5, which are now set equal to zero, should be allowed to take on a positive value, if possible. From the above equation we solve for the current basic variable in terms of the non-basic variable to obtain

$$x_1 = 6 + 2x_4 - x_5 \quad --- \quad (2.1)$$
$$x_2 = 3 - x_4 + 4x_5 \quad --- \quad (2.2)$$
$$x_3 = 10 - 3x_4 - 2x_5 \quad --- \quad (2.3)$$

We next rewrite the objective function in terms of only the non basic variables by substituting for x_1, x_2 and x_3 the corresponding right-hand side expressions above to obtain

$z = 2(6 + 2x_4 - x_5) - (3 - x_4 + 4x_5) + (10 - 3x_4 - 2x_5) - 5x_4 + 22x_5$

Or, $z = 19 - 3x_4 + 14x_5$

Or, $z = 19 - (3x_4) - (-14x_5)$

With $x_4 = 0$ and $x_5 = 0$, $z = 19$, which is the value for the current basic feasible solution. We see from the last transformed expression of z that if x_4 can be made positive, the objective will decrease 3 units for each unit increase of x_4: while any positive unit increase to x_5 will increase the value of the objective function by 14 units. Since we are minimizing, it would appear to be appropriate to determine a new basic feasible solution, i.e., an extreme point solution, involving x_4 a positive level, if positive .

We next generate a new extreme point solution by replacing x_2 by x_4 to obtain the solution

$$x_1 + 2x_2 - 7x_5 = 12 \quad [(1.1) + 2(1.2)]$$
$$x_2 + x_4 - 4x_5 = 3$$
$$-3x_2 + x_3 + 14x_5 = 1 \quad [(1.3)-3(1.2)]$$

Or, $x_1 = 12 - 2x_2 + 7x_5$

$x_4 = 3 - x_2 + 4x_5$

$x_3 = 1 + 3x_2 - 14x_5$

The new basic feasible solution is $x_1 = 12$, $x_2 = 0$, $x_3 = 1$, $x_4 = 3$, $x_5 = 0$. Substituting the expressions of the basic variables x_1, x_3, x_4 in terms of the non-basic variables x_2 and x_5 into the objective function, we now have

$z = 2\,(12 - 2x_2 + 7x_5) - x_2 + (1 + 3x_2 - 14x_5) - 5\,(3 - x_2 + 4x_5) + 22x_5$
Or, $z = 10 - (-3x_2) - (-2x_5)$
From this last expression we see that any increase in the values of the non-basic variable x_2 and x_5 would increase the value of the objective function. We thus conclude that the new basic feasible solution is an optimum, with the value of the objective function of $z = 10$. Since $z = 10$ is less than $z = 19$. Hence the minimum value of the objective function is 10 .
The above process is just the direct application of the elimination procedure on the linear programming system of equations, which now includes an explicit expression of the objective function.

13.10 Development of a minimum feasible solution:
We assume that the linear programming problem is feasible that every basic feasible solution is non degenerate, and that we are given a basic feasible solution. These assumptions, as will be discussed later, are made without any loss in generality. Let the given solution be $X_0 = (x_{10}, x_{20}, \ldots, x_{mo})$ and associated set of linearly independent vectors be $\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_m$, we then have
$x_{10}\,\underline{p}_1 + x_{20}\,\underline{p}_2 + \ldots + x_{mo}\underline{p}_m = \underline{p}_0$ --- (1)
$x_{10}\underline{c}_1 + x_{20}\underline{c}_2 + \ldots + x_{mo}\underline{c}_m = \underline{z}_0$ --- (2)

Where all $x_{i0} > 0$, the c_j are cost the coefficients of the objective function, and z_0 is the corresponding value of the objective function for the given solution. Since the set $\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_m$ is linearly independent and thus forms a basis, we can express any vector from the set $\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n$ in terms of $\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_m$. Let $\underline{p}_j$ be given by
$x_{1j}\,\underline{p}_1 + x_{2j}\,\underline{p}_2 + \ldots + x_{mj}\,\underline{p}_m = p_j$; $j = 1, 2, \ldots, n$ --- (3)
and we define
$x_{1j}\,c_1 + x_2 j\,\underline{c}_2 + \ldots + x_{mj}\,c_m = z_j$; $j = 1, 2, \ldots, n$ --- (4)

Where the c_j are the cost coefficients corresponding to the $\underline{p}_j$.

Theorem 1: If for any fixed j, the condition $z_j - c_j > 0$ holds, them a set of feasible solutions can be constructed such that $z < z_0$ for any member of the set, where the lower bound of z is either finite or infinite (z is the value of the objective function for a particular member of the set of feasible solutions).
Case-1: If the lower bound is finite, a new feasible solution consisting of exactly m positive variables can be constructed whose value of the objective function is less than the value for the preceding solution.

Case-2: If the lower bound is finite, a new feasible solution consisting of exactly m +1 positive variables can be constructed whose value of the objective function can be made arbitrarily small.

Proof: The following analysis applies to the proof of both cases:

Multiplying (3) by some number θ and subtracting from (1), and similarly multiplying (4) by the some number θ and subtracting from (2), for j = 1, 2, . . ., n, we get

$$(x_{10} - \theta x_{1j})\underline{p}_1 + (x_{20} - \theta x_{2j})\underline{p}_2 + \ldots + (x_{m0} - \theta x_{mj})\underline{p}_m + \theta \underline{p}_j = \underline{p}_0 \quad --- \ (5)$$

$$(x_{10} - \theta x_{1j})c_1 + (x_{20} - \theta x_{2j})c_2 + \ldots + (x_{m0} - \theta x_{mj})c_m + \theta c_j = z_0 - \theta(z_j - c_j) \quad --- \ (6)$$

Where θc_j has been added to both sides of (6). If all the coefficients of the vectors $\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_m, \underline{p}_j$ in (5) are non negative, then we have determined a new feasible solution whose value of the objective function is by (6) $z = z_0 - \theta(z_j - c_j)$. Since the variables $x_{10}, x_{20}, \ldots, x_{m0}$ in (5) are all positive, it is clear that there is a value of $\theta > 0$ (either finite or infinite) for which the coefficients of the vectors in (5) remain positive. From the assumption that, for a fixed j, $z_j - c_j > 0$, we have

$z = z_0 - \theta (z_j - c_j) < z_0$, for $\theta > 0$.

Proof of case-1: If, for the fixed j, at least one $x_{ij} > 0$ in (3) for i = 1, 2, . . ., m, the largest value of θ for which all coefficients of (5) remain non negative is given by

$$\theta_0 = \min_i \frac{x_{i0}}{x_{ij}} > 0 \qquad \text{for } x_{ij} > 0 \quad --- \ (7)$$

Since we assumed that the problem is non degenerate, i.e., that all basic feasible solutions have m positive elements, the minimum in (7) will be obtained for a unique i. If θ_0 is substituted for θ in (5) and (6), the coefficient corresponding to this unique i will vanish. We have then constructed a new basic feasible solution consisting of $\underline{p}_j$ and m -1 vectors of the original basis. The new basis can be used as the previous one. If a new $z_j - c_j > 0$ and a corresponding $x_{ij} > 0$, another solution can be obtained which has a smaller value of the objective function. This process will continue either until all $z_j - c_j \leq 0$, or until, for some $z_j - c_j > 0$, all $x_{ij} \leq 0$. If all $z_j - c_j \leq 0$, the process terminates.

Proof of case-2: If at any stage we have, for some j, $z_j - c_j > 0$ and all $x_{ij} \leq 0$, then there is no upper bound to θ and the objective function has a lower bound of $-\infty$. We see for this case that, for any $\theta > 0$, all the coefficients of (5) are positive. We then have a feasible solution consisting of m + 1 positive elements. Hence, by taking θ large enough, the corresponding value of the objective function given by the right hand side of (6) can be made arbitrarily small.

Example: Minimize $x_2 - 3x_3 + 2x_5$

Subject to

$$\begin{aligned} x_1 + 3x_2 - x_3 \quad + 2x_5 &= 7 \\ -2x_2 + 4x_3 + x_4 &= 12 \\ -4x_2 + 3x_3 \quad + 8x_5 + x_6 &= 10 \\ x_j \geq 0, \quad j = 1, 2, \ldots, 6 & \end{aligned}$$

Solution: Our initial basis consists of $\underline{p}_1$, $\underline{p}_4$, $\underline{p}_6$ and the corresponding solution is $X_0 = (x_1, x_4, x_6) = (7, 12, 10)$

Since $c_1 = c_4 = c_6 = 0$, the corresponding value of the objective function, z_0 equals zero. $\underline{p}_3$ is selected to go into the basis, because

$$\max_j (z_i - c_i) = z_3 - c_3 = 3 > 0$$

θ is the minimum of x_{i0}/x_{i3} for $x_{i3} > 0$, that is, min. (12/4, 10/3) = 12/4 = θ_0 = 3 and hence p_4 is eliminated. We transform the tableau and obtain a new solution.

$X_0^{/} = (x_1, x_2, x_6) = (10, 3, 1)$

and the value of the objective function is –9. In the second step, since

$\max_j (z_j^{/} - c_j) = z_2^{/} - c_2 = \frac{1}{2} > 0$ and $\theta_0 = \frac{10}{5/12}$, $\underline{p}_2$ is introduced into the basis and $\underline{p}_1$ is eliminated. We transform the second-step values of tableau and obtain the solution

$X_0^{//} = (x_2, x_3, x_6) = (4, 5, 11)$

Serial	Basis	c	$\underline{p}_0$	0	1	-3	0	2	0	θ
				$\underline{p}_1$	$\underline{p}_2$	$\underline{p}_3$	$\underline{p}_4$	$\underline{p}_5$	$\underline{p}_6$	
1	$\underline{p}_1$	0	7	1	3	-1	0	2	0	
2	$\underline{p}_4$	0	12	0	-2	(4) Pivot	1	0	0	12/4=3=θ_0
3	$\underline{p}_6$	0	10	0	-4	3	0	8	1	10/3
	$z_i - c_i$		0	0	-1	3 Greatest	0	-2	0	
1	$\underline{p}_1$	0	10	1	(5/2) Pivot	0	¼	2	0	$\frac{10}{5/12}$ =4=θ_0
2	$\underline{p}_3$	-3	3	0	-1/2	1	¼	0	0	
3	$\underline{p}_6$	0	1	0	-5/2	0	-3/4	8	1	
	$z_j^{/} - c_j$		-9	0	½ Greatest	0	-3/4	-2	0	
1	$\underline{p}_2$	1	4	2/5	1	0	1/10	4/5	0	
2	$\underline{p}_3$	-3	5	1/5	0	1	3/10	2/5	0	
3	$\underline{p}_6$	0	11	1	0	0	-1/2	10	1	
	$z_j^{//} - c_j$		-11	-1/5	0	0	-4/5	-12/5	0	

with a value of the objective function equal to –11. Since $\max\left(z_j^{//} - c_j\right) \not> 0$, this solution is the minimum feasible solution.

13.11 <u>The artificial basis technique</u>: Up to this point, we have always assumed that the given linear-programming problem was feasible and contained a unit matrix that could be used for the initial basis. Although a correct formulation of a problem will usually guarantee that the problem will be feasible, many problems do not contain a unit matrix. For such problems, the method of the artificial basis is a satisfactory way to start the simplex process. This procedure also determines whether or not the problem has any feasible solutions.

The general linear programming problem is to minimize

$c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2\\
\ldots \qquad \ldots \qquad \ldots &\\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

and $x_j \geq 0$; $j = 1, 2, \ldots, n.$

For the method of the artificial basis we augment the above system as follows:

Minimize :

$c_1x_1 + c_2x_2 + \ldots + c_nx_n + wx_{n+1} + wx_{n+2} + \ldots + wx_{n+m}$

Subject to :

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} \qquad\qquad &= b_1\\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \qquad + x_{n+2} \qquad &= b_2\\
\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad &\ldots\\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \qquad\qquad + x_{n+m} &= b_m
\end{aligned}
$$

and $x_j \geq 0$; $j = 1, 2, \ldots, n, n+1, \ldots, n+m$

The quantity w is taken to be an unspecified large positive number. The vectors $\underline{p}_{n+1}, \underline{p}_{n+2}, \ldots, \underline{p}_{n+m}$ form a basis (an artificial basis) for the augmented system. Therefore, for the augmented problem, the first feasible solution is

$x_0 = (x_{n+1,\,0}, x_{n+2,\,0}, \ldots, x_{n+m,\,0}) = (b_1, b_2, \ldots, b_m) \geq 0$

$\therefore \quad x_{n+1,\,0}\,\underline{p}_{n+1} + x_{n+2,\,0}\,\underline{p}_{n+2} + \ldots + x_{n+m,\,0}\,\underline{p}_{n+m} = \underline{p}_0$ --- (1)

$w\,x_{n+1,\,0} + w\,x_{n+2,\,0} + \ldots + w\,x_{n+m,\,0} = z_0$ --- (2)

And also,

$x_{1j}\,\underline{p}_{n+1} + x_{2j}\,\underline{p}_{n+2} + \ldots + x_{mj}\,\underline{p}_{n+m} = p_j$ --- (3)

$w\,x_{1j} + w\,x_{2j} + \ldots + w\,x_{mj} = z_j$ --- (4)

Multiplying (3) by θ and then subtracting from (1) we have,

$(x_{n+1,0} - \theta\, x_{1j})\,\underline{p}_{n+1} + (x_{n+2,0} - \theta\, x_{2j})\underline{p}_{n+2} + \ldots + \ldots + (x_{n+m,0} - \theta\, x_{mj})\underline{p}_{n+m} + \theta\,\underline{p}_j = p_0$ --- (5)

Multiplying (4) by θ and then subtracting from (2) we have

$(x_{n+1,0} - \theta\, x_{1j})w + (x_{n+2,0} - \theta\, x_{2j})w + \ldots + (x_{n+m,\,0} - \theta\, x_{mj})w + \theta c_j = z_0 - \theta\,(z_j - c_j)$ --- (6)

Where, $z_j = w\sum_{i=1}^{m} x_{ij}$

$\therefore \quad (z_j - c_j) = w\sum_{i=1}^{m} x_{ij} - c_j$

For the first solution each $z_j - c_j$ will then have a w coefficient which are independent of each other. We next set up the associated computational procedure as the given table. For each j, the row free from w component and the row with w component of $z_j - c_j$ have been placed in the (m+1)st and (m+2)nd rows, respectively of that column.

We treat this table exactly like the original simplex table except that the vector introduced into the basis is associated with the largest positive element in the (m+2)nd row. For the first iteration, the vector corresponding to $\max_j \sum_{i=1}^{m} x_{ij}$ is introduced into the basis.

We continue to select a vector to be introduced into the basis, using the element in the (m+2)nd row as criterion, until either (a) all artificial vectors are eliminated from the basis or (b) no positive (m+2)nd element exists. The first alternative implies that all the elements in the (m+2)nd row equal to zero and that the corresponding basis is a feasible basis for the original problem.

Serial	Basis	c	$\underline{p}_0$	c1	c2	.	ck	.	cn	w	.	w	.	w
				$\underline{p}_1$	$\underline{p}_2$	.	$\underline{p}_k$	.	$\underline{p}_n$	$\underline{p}_{n+1}$	.	$\underline{p}_{n+l}$	.	$\underline{p}_{n+m}$
1	$\underline{p}_{n+1}$	w	$\underline{p}_{n+1,0}$	x_{11}	x_{12}	.	x_{1k}	.	x_{1n}	1	.	0	.	0
2	$\underline{p}_{n+2}$	w	$\underline{p}_{n+2,0}$	x_{21}	x_{22}	.	x_{2k}	.	x_{2n}	0	.	0	.	0
..	..	..	..	..	..	.	..	.	..	..	.	..	.	..
..	..	..	..	..	..	.	..	.	..	..	.	..	.	..
l	$\underline{p}_{n+l}$	w	$\underline{p}_{n+l,0}$	x_{l1}	x_{l2}	.	x_{lk}	.	x_{ln}	0	.	1	.	0
..	..	..	..	..	..	.	..	.	..	..	.	..	.	..
..	..	..	..	..	..	.	..	.	..	..	.	..	.	..
m	$\underline{p}_{n+m}$	w	$\underline{p}_{n+m,0}$	x_{m1}	x_{m2}	.	x_{mk}	.	x_{mn}	0	.	0	.	1
m+1			0	$-c_1$	$-c_2$	.	$-c_3$	.	$-c_n$	0	.	0	.	0
m+2			$\sum x_{n+i,0}$	$\sum x_{i1}$	$\sum x_{i2}$	.	$\sum x_{ik}$	.	$\sum x_{in}$	0	.	0	.	0

We than apply the regular simplex algorithm to determine the minimum feasible solution. If the second alternative, if the (m+2, 0) element, i.e., the artificial part of the corresponding value of the objective, is greater than zero, then the original problem is not feasible.

Example: Minimize $2x_1 + x_2$ [AUB-02]

Subject to $3x_1 + x_2 - x_3 = 3$

$4x_1 + 3x_2 - x_4 = 6$

$x_1 + 2x_2 - x_5 = 2$

and $x_j \geq 0$: j = 1, 2, . . ., 5

Solution: For finding a artificial basis we may rewrite the problem as following:

Minimize $\quad 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 + wx_6 + wx_7 + wx_8$

Subject to $\quad 3x_1 + x_2 - x_3 + x_6 = 3$

$\quad 4x_1 + 3x_2 - x_4 + x_7 = 6$

$\quad x_1 + 2x_2 - x_5 + x_8 = 2$

and $\quad x_j \geq 0 : j = 1, 2, \ldots, 8$

Using the above problem we find the following tableau

Sl.	Ba	c	P_o	2	1	0	0	0	w	w	w	θ
	sis			P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	
1	P_6	w	3	(3)	1	-1	0	0	1	0	0	$3/3=1=\theta_o$
2	P_7	w	6	4	3	0	-1	0	0	1	0	6/4
3	P_8	w	2	1	2	0	0	-1	0	0	1	2/1=2
4+1	$z_i - c_i$		0	-2	-1	0	0	0	0	0	0	
4+2			11	8 Greatest	6	-1	-1	-1	0	0	0	
1	P_1	2	1	1	1/3	-1/3	0	0		0	0	1/(1/3)=3
2	P_7	w	2	0	5/3	4/3	-1	0		1	0	2/(5/3)=6/5
3	P_8	w	1	0	(5/3)	1/3	0	-1		0	1	1/(5/3)=3/5
4+1	$z_i - c_i$		2	0	-1/3	-2/3	0	0		0	0	
4+2			3	0	10/3 Greatest	5/3	-1	-1		0	0	
1	P_1	2	4/5	1	0	-2/5	0	1/5		0		
2	P_7	w	1	0	0	(1)	-1	1		1		1/1=1
3	P_2	1	3/5	0	1	1/5	0	-3/5		0		(3/5)/(1/5)=3
4+1	$z_i - c_i$		11/5	0	0	-3/5	0	-1/5		0		
4+2			1	0	0	1 Greatest	-1	1		0		
1	P_1	2	6/6	1	0	0	-2/5	3/5				
2	P_3	0	1	0	0	1	-1	(1)				
3	P_2	1	2/5	0	1	0	1/5	-4/5				
4+1	$z_i - c_i$		14/5	0	0	0	-3/5	2/5 Greatest				
4+2			0	0	0	0	0	0				
1	P_1	2	3/5	1	0	-3/5	1/5	0				
2	P_5	0	1	0	0	1	-1	1				
3	P_2	1	6/5	0	1	4/5	-3/5	0				
4+1	$z - c_i$		12/5	0	0	-2/5	-1/5	0				

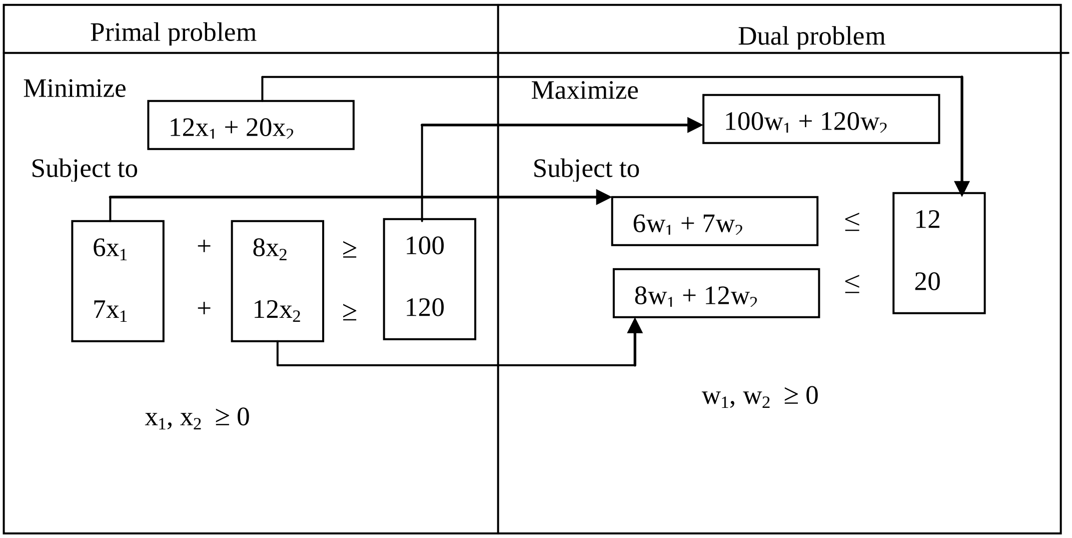
The above tableau gives us the extreme point (3/5, 5/6, 0, 0, 1).

So, the solution of the problem is

$x_1 = 3/5,\ x_2 = 5/2,\ x_3 = 0,\ x_4 = 0,\ x_5 = 1$

and the minimum value is 12/5 (Answer)

13.12 <u>Duality in linear programming problem</u>: The term duality implies that every linear programming problem whether of maximization or minimization has associated with it another linear programming problem based on the same data. The original problem is called the primal problem while the other is called its dual problem. It is important to note that in general, either problem can be considered as primal and the other as its dual. Thus, the two problems constitute the pair of dual problem.



<u>Example</u>: The primal problem:

Maximize $5x_1 - 3x_2$

Subject to $4x_1 + 5x_2 \le 45$

$3x_1 - 7x_2 \le 15$

$x_1, x_2 \ge 0$

The dual problem of the above primal problem is

Minimize $45w_1 + 15w_2$

Subject to $4w_1 + 3w_2 \ge 5$

$5w_1 - 7w_2 \ge -3$

$w_1, w_2 \ge 0$

<u>Note</u>: If either the primal or the dual problem has a finite optimum solution, then the other problem has a finite solution and the value of the objective functions are same, i.e., min. f(x) = max. g(w).

If either primal or dual has an unbounded solution then the other has no solution.

Example: Solve the following LPP and find also the solution of its dual problem.

Maximize $x_1 + x_2 + x_3$ [AUB-03]

Subject to $2x_1 + x_2 + 2x_3 \le 3$

$4x_1 + 2x_2 + x_3 \le 2$

$x_1, x_2, x_3 \ge 0$

Solution: Adding slack variables x_4, x_5 we can rewrite the problem as follows:

Maximize $x_1 + x_2 + x_3$

Subject to $2x_1 + x_2 + 2x_3 + x_4 = 3$

$4x_1 + 2x_2 + x_3 + x_5 = 2$

$x_j \ge 0;\ j = 1, 2, \ldots, 5$

Converting the problem into simplex table, we get the following tableau

Sl.	Basis	c	P_o	1	1	1	0	0	θ
				P_1	P_2	P_3	P_4	P_5	
1 2	P_4 P_5	0 0	3 2	2 4	1 2	(2) 1	1 0	0 1	$3/2 = \theta_o$ $2/1 = 2$
	$z_j - c_j$		0	-1	-1	-1	0	0	
1 2	P_3 P_5	1 0	3/2 1/2	1 3	1/2 (3/2)	1 0	1/2 -1/2	0 1	$(3/2)/(1/2) = 3$ $(1/2)/(3/2)=1/3=\theta_o$
	$z_j - c_j$		3/2	0	-1/2	0	1/2	0	
1 2	P_3 P_2	1 1	4/3 1/3	0 2	0 1	1 0	2/3 -1/3	-1/3 2/3	
	$z_j - c_j$		5/3	1	0	0	1/3	1/3	

Therefore, the extreme point solution of the primal is (1/3, 4/3) and the maximum value of the primal is 5/3.

The dual problem of the given problem is

Minimum $3w_1 + 2w_2$

Subject to $2w_1 + 4w_2 \ge 1$

$w_1 + 2w_2 \ge 1$

$2w_1 + w_2 \ge 1$

$w_1, w_2 \ge 0$

In the first table, P_4 and P_5 were in basis. In the last table the $z_j - c_j$ values of P_4 and P_5 are 1/3 and 1/3 respectively. So, the solution of the dual problem is $w_1 = 1/3$, $w_2 = 1/3$ and the minimum value of the dual is 5/3.

Note: If we solve the dual problem by simplex method, we shall get the same solution.

13.13 Some worked out examples:

Example (1): Minimize $5x_1 - 10x_2$

Subjcet to $2x_1 - x_2 \ge -2$

$x_1 + 2x_2 \le 8$

$x_1, x_2 \ge 0$

Solution: The graph of the problem is as follows:

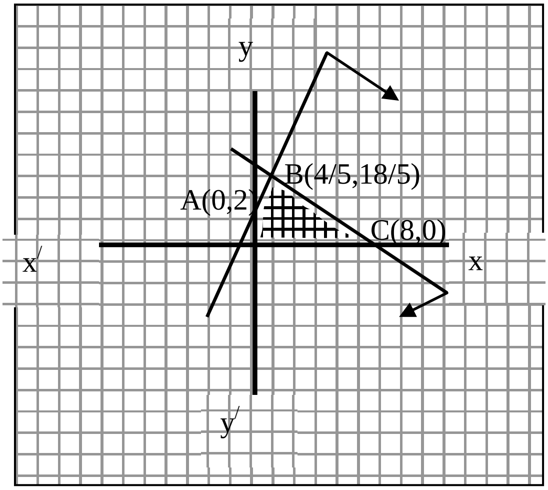


Figure – 13.5

From the graph we get the vertices A(0,2), B(4/5,18/5), C(8,0) and O(0,0). The values of the objective function at these points are – 20 at A, – 32 at B, 40 at C and 0 at O. Here, – 32 is the minimum value of the objective function which occurs at the vertices B(4/5, 18/5).
Hence, the solution of the given problem is $x_1 = 4/5$, $x_2 = 18/5$ and min. value of z = –32 (Answer)

<u>Example (2)</u>: A furniture company makes tables and chairs. Each table takes 5 hours of carpentry and 10 hours in painting and varnishing shop. Each chair requires 20 hours in carpentry and 15 hours in painting and varnishing. During the current production period 400 hours of carpentry and 450 hours. of painting and varnishing time are available. Each table sold yields a profit of $45 and each chair yields a profit of $80. Using simplex method determine the number of tables and chairs to be made to maximize the profit.

Solution: Let x_1 = the number of tables and x_2 = the number of chairs.
So, the total profit $45x_1 + 80x_2$ which is the objective function. We have to maximize the objective function $z = 45x_1 + 80x_2$
The required carpentry hours $5x_1 + 20x_2$
Since 400 carpentry hours are available
$4x_1 + 3x_2 \leq 240$.
Similarly, for painting and varnishing, we have
$10x_1 + 15x_2 \leq 450$.
The non negative conditions $x_1, x_2 \geq 0$
So, the linear programming problem (LPP) of the given problem is

Maximize $45x_1 + 80\,x_2$
Subject to $5x_1 + 20x_2 \leq 400$
$10x_1 + 15x_2 \leq 450$
$x_1, x_2 \geq 0$

We can rewrite the problem as follows:

Minimize $-45x_1 - 80x_2$
Subject to $x_1 + 4x_2 + x_3 = 80$
$2x_1 + 3x_2 + x_4 = 90$
$x_j, \geq 0$; j = 1, 2, 3, 4

Sl.	Basis	c	P_o	-45	-80	0	0	θ
				P_1	P_2	P_3	P_4	
1 2	P_3 P_4	0 0	80 90	1 2	(4) 3	1 0	0 1	$80/4 = 20 = \theta_o$ 90/3 = 30
3	$z_j - c_j$		0	45	80	0	0	
1 2	P_2 P_4	-80 0	20 30	1/4 (5/4)	1 0	1/4 -3/4	0 1	20/(1/4) = 80 $30/(5/4)=24=\vartheta_o$
3	$z_j - c_j$		-1600	25	0	-20	0	
1 2	P_2 P_1	-80 -45	14 24	0 1	1 0	2/5 -3/5	-1/5 4/5	
3	$z_j - c_j$		-2200	0	0	-5	-20	

The above tableau gives the extreme point (24, 14), i.e., $x_1 = 24$, $x_2 = 14$. So, the company will earn maximum profit $2200 if 24 tables and 14 chairs are made.

Example (3): Solve the following LPP:

$$\text{Minimize} \quad x_1 - x_2 + x_3$$
$$\text{Subject to} \quad x_1 - x_4 - 2x_6 = 5$$
$$x_2 + 2x_4 - 3x_5 + x_6 = 3$$
$$x_3 + 2x_4 - 5x_5 + 6x_6 = 5$$
$$x_i \geq 0 \; ; \; i = 1, 2, \ldots, 6$$

Solution: Form the given problem, we get the following tableau:

Sl	Basis	c	P_o	1	-1	1	0	0	0	θ
				P_1	P_2	P_3	P_4	P_5	P_6	
1 2 3	P_1 P_2 P_3	1 -1 1	5 3 5	1 0 0	0 1 0	0 0 1	-1 2 2	0 -3 -5	-2 1 6	3/1 = 3 $5/6 = \theta_o < 3$
4	$z_j - c_j$		7	0	0	0	-1	-2	3	
1 2 3	P_1 P_2 P_6	1 -1 0	20/3 13/6 5/6	1 0 0	0 1 0	1/3 -1/6 1/6	-1/3 5/3 1/3	-5/3 -13/6 -5/6	0 0 1	
4	$z_j - c_j$		9/2	0	0	-1/2	-2	1/2	0	

Since in the second step we find a positive value ½ for $z_j - c_j$ but there is no positive number above ½ . So, we can say that the feasible region is unbounded.
Therefore, the optimum solution is at infinity and also the minimum value is infinity.

Example (4): Maximize $x_1 + 2x_2 + 3x_3 - x_4$

Subject to
$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$

and $x_j \geq 0$; $j = 1, 2, 3, 4$.

Solution: Since we are always minimizing, the corresponding objective is to minimize
$-x_1 - 2x_2 - 3x_3 + x_4$

Subject to $x_1 + 2x_2 + 3x_3 = 15$
$2x_1 + x_2 + 5x_3 = 20$
$x_1 + 2x_2 + x_3 + x_4 = 10$

and $x_{ij} \geq 0$; $j = 1, 2, 3, 4.$

Since in our constraint we get only one basis vector so, we have to take two arbitrary basis vectors. For this we rewrite our problem as follows:

Minimize $-x_1 - 2x_2 - 3x_3 + x_4 + wx_5 + wx_6$

Subject to $x_1 + 2x_2 + 3x_3 + x5 = 15$
$2x_1 + x_2 + 5x_3 + x6 = 20$
$x_1 + 2x_2 + x_3 + x_4 = 10$

and $x_j \geq 0$; $j = 1, 2, . . ., 6$

Serial	Basis	c	$\underline{p}_0$	–1	–2	–3	1	w	w	θ
				$\underline{p}_1$	$\underline{p}_2$	$\underline{p}_3$	$\underline{p}_4$	$\underline{p}_5$	$\underline{p}_6$	
1	P_5	w	15	1	2	3	0	1	0	15/3=5
2	P_6	w	20	2	1	(5)	0	0	1	20/5=4=θ_0
3	P_4	1	10	1	2	1	1	0	0	10/1=10
4+1	$z_i - c_i$		10	2	4	4	0	0	0	
4+2			35	3	3	8 Greatest	0	0	0	
1	P_5	w	3	–1/5	(7/5)	0	0	1		3/(7/5)=15/7=θ_0
2	P_3	–3	4	2/5	1/5	1	0	0		4/(1/5)=20
3	P_4	1	6	3/5	9/5	0	1	0		6/(9/5)=30/9
4+1	$z_i - c_i$		–6	2/5	16/5	0	0	0		
4+2			3	–1/5	7/5 Greatest	0	0	0		
1	P_2	–2	15/7	–1/7	1	0	0			
2	P_3	–3	25/7	3/7	0	1	0			(25/7)/(3/7)=25/3
3	P_4	1	15/7	(6/7)	0	0	1			(15/7)/(6/7)=5/2=θ_0
4+1	$z_i - c_i$		–90/7	6/7	0	0	0			
4+2			0	0	0	0	0			
1	P_2	–2	5/2	0	1	0	1/6			
2	P_3	–3	5/2	0	0	1	–3/6			
3	P_1	–1	5/2	1	0	0	7/6			
4+1	$z_i - c_i$		–15	0	0	0	–1			

$\therefore$ (5/2, 5/2, 5/2, 0) is the required extreme point. The minimum value of the new objective function is –15. Hence the maximum value of our given objective function is –(–15) = 15. So, the solution of the problem is $x_1 = 5/2$, $x_2 = 5/2$, $x_3 = 5/2$, $x_4 = 0$ and the max. value is 15. (Answer)

Example (5): Solve the following LPP :

Maximize $2x_1 - 6x_2$

Subject to $3x_1 + 2x_2 \leq 6$

$x_1 - x_2 \geq -1$

$-x_1 - 2x_2 \geq 1$

$x_1, x_2 \geq 0$

Solution: We can rewrite the problem as follows:

Maximize $2x_1 - 6x_2 + 0x_3 + 0x_4 + 0x_5 + wx_6$

Subject to $3x_1 + 2x_2 + x_3 = 6$

$x_1 - x_2 + x_4 = -1$

$-x_1 - 2x_2 - x_5 + x_6 = 1$

$x_i \geq 0; \; i = 1, 2, \ldots, 6$

Sl	Basis	c	P_o	-2	6	0	0	0	w	θ
				P_1	P_2	P_3	P_4	P_5	P_6	
1	P_3	0	6	-3	2	1	0	0	0	
2	P_4	0	1	-1	1	0	1	0	0	
3	P_5	w	1	-1	-2	0	0	-1	1	
4	$z_j - c_j$		0	2	-6	0	0	0	0	Term free of w
5			1	-1	-2	0	0	-1	0	Term with w

Since all elements of fifth row are non-positive but w appears in the basis at a non-zero value 1. Hence, the problem has no feasible solution.

Example (6): Solve the following LPP:

Minimize $x_1 + 2x_2$

Subject to $x_1 - 3x_2 \leq 6$

$2x_1 + 4x_2 \geq 8$

$x_1 - 3x_2 \geq -6$

$x_1, x_2 \geq 0$

Solution: We can rewrite the problem as follows:

Minimize $x_1 + 2x_2$

Subject to $x_1 - 3x_2 \leq 6$

$2x_1 + 4x_2 \geq 8$

$-x_1 + 3x_2 \leq 6$

$x_1, x_2 \geq 0$

Introducing slack variables x_3, x_6, surplus variable x_4 and artificial variable x_5, we get the problem as follows:

Minimize $x_1 + 2x_2 + 0x_3 + 0x_4 + wx_5 + 0x_6$

Subject to $x_1 - 3x_2 + x_3 = 6$

$2x_1 + 4x_2 - x_4 + x_5 = 8$

$x_1 - 3x_2 + x_6 = -6$

$x_j \geq 0; \; j = 1, 2, \ldots, 6$

Now, we convert the problem into the simplex table as follows:

Sl	Basis	c	P_o	1	2	0	0	w	0	θ
				P_1	P_2	P_3	P_4	P_5	P_6	
1 2 3	P_3 P_5 P_6	0 w 0	6 8 6	1 2 -1	-3 (4) 3	1 0 0	0 -1 0	0 1 0	0 0 1	 8/4 = 2 = θ_o 6/3 = 2
4 5	$z_j - c_j$		0 8	-1 2	-2 4	0 0	0 0	0 0	0 0	
1 2 3	P_3 P_2 P_6	0 2 0	12 2 0	5/2 (½) -5/2	0 1 0	1 0 0	-3/4 -1/4 ¾		0 0 1	12/(5/2) = 24/5 2/(1/2) = 4 = θ_o
4 5	$z_j - c_j$		4 0	0 0	0 0	0 0	-1/2 0		0 0	

From the above table, we find the extreme point (0, 2, 12, 0, 0, 0). Therefore, the optimum solution is $x_1 = 0, x_2 = 2$ with minimum value of the objective function 4.
We can bring P_1 vector in the basis for same objective value because in 4th row of P_1 column of 2nd step we get 0 and above this 0 we get two positive elements. Taking ½ as pivot we get the following table.

Sl	Basis	c	P_o	1	2	0	0	w	0	θ
				P_1	P_2	P_3	P_4	P_5	P_6	
1 2 3	P_3 P_1 P_6	0 1 0	2 4 10	0 1 0	-5 2 5	1 0 0	½ 0 -1/2		0 0 1	
4	$z_j - c_j$		4	0	0	0	-1/2		0	

So, the second optimal solution is $x_1 = 4$, $x_2 = 0$ and $z_{max} = 4$.
There are two different optimal solutions. Hence there will exist an infinite number of optimal solutions.

13.14 <u>Exercise</u>:

1. Find the numerical values of

(i) $\sum_{a=1}^{5} a$ [Answer: 14] (ii) $\sum_{a=1}^{n} a$ if n is 6 [Answer: 20]

(iii) $\sum_{i=1}^{5} 3i$ [Answer: 45] (iv) $\sum_{p=5}^{10} p^2$ [Answer: 355]

2. Write the expanded form of

(i) $\sum_{a=1}^{5} x_a$ [Answer: $x_1 + x_2 + x_3 + x_4 + x_5$]

(ii) $\sum_{i=1}^{n} a_i x_i = b$ [Answer: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$]

3. Express in compact summation form:

   (i) $y_1 - y_2 + y_3 - y_4 + y_5 - y_6$ [Answer: $\sum_{i=1}^{6} (-)^{i-1} y_i$]

   (ii) $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$ [Answer: $\sum_{i=1}^{n} a_i x_i = c$]

4. A dietitian is planning the menu for the evening meal at a university dining hall. Three main items will be served, all having different nutritional content. The dietitian is interested to providing at least the minimum daily requirement of each of three vitamins in this one meal. The following table summarizes the vitamin content per ounce of each type of food, the cost per ounce of each food, and minimum daily requirements (MDR) for the three vitamins. Any combination of the three foods may be selected as long as the total serving size is at least 9 ounces.

Food	Vitamins			Cost per ounce, $
	1	2	3	
1	50 mg	20 mg	10 mg	0.10
2	30 mg	10 mg	50 mg	0.15
3	20 mg	30 mg	20 mg	0.12
MDR	290 mg	200 mg	210 mg	

The problem is to determine the number of ounces of each food to be included in the meal. The objective is to minimize the cost of each meal subject to satisfying minimum daily requirements of the three vitamins as well as the restriction on minimum serving size. Give the formulation of the problem.

[Answer: Minimize $z = 0.10x_1 + 0.15x_2 + 0.12x_3$

Subject to
$$50x_1 + 30x_2 + 20x_3 \geq 290$$
$$20x_1 + 10x_2 + 30x_3 \geq 200$$
$$10x_1 + 50x_2 + 20x_3 \geq 210$$
$$x_1 + x_2 + x_3 \geq 9$$
$$x_1, x_2, x_3 \geq 0]$$

5. Solve the following linear programming problems using graphical method:

(i) Maximize $z = 2x_1 + 3x_2$

Subject to
$$x_1 + x_2 \leq 30$$
$$x_2 \geq 3$$
$$x_2 \leq 12$$
$$x_1 - x_2 \geq 0$$
$$0 \leq x_1 \leq 20$$

[Answer: $x_1 = 18$, $x_2 = 12$, $z = 72$]

(ii) Maximize $z = 2x_1 - 6x_2$

Subject to
$$3x_1 + 2x_2 \leq 6$$
$$x_1 - x_2 \geq -1$$
$$-x_1 - 2x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

[Answer: No solution]

(iii) Maximize $z = 2x_1 + 3x_2$
Subject to $-x_1 + 2x_2 \leq 4$
$x_1 + x_2 \leq 6$
$x_1 + 2x_2 \leq 9$
$x_1, x_2 \geq 0$
[Answer: $x_1 = 9/2$, $x_2 = 3/2$, and $z = 27/2$]

(iv) Minimize $z = x_1 + 6x_2$
Subject to $x_1 - 3x_2 \leq 6$
$2x_1 + 4x_2 \geq 8$
$x_1 - 3x_2 \geq -6$
$x_1, x_2 \geq 0$
[Answer: Many solutions. In particular $x_1 = 0$ or 4, $x_2 = 2$ or 0 and min. value = 4]

(v) Minimize $z = 2x_1 + x_2$
Subject to $3x_1 + x_2 \geq 3$
$4x_1 + 3x_2 \geq 6$
$x_1 + 3x_2 \leq 3$
$x_1, x_2 \geq 0$
[Answer: $x_1 = 1$, $x_2 = 2/3$, and $z = 8/3$]

(vi) Maximize $z = x_1 + x_2$
Subject to $x_1 + x_2 \geq 1$
$x_1 - x_2 \leq 1$
$-x_1 + x_2 \leq 1$
$x_1, x_2 \geq 0$
[Answer: Unbounded solution]

6. Solve the following LPPs by simplex method:

(i) Minimize $z = 2x_1 + x_2$
Subject to $3x_1 + x_2 \leq 3$
$4x_1 + 3x_2 \leq 6$
$x_1 + 2x_2 \leq 2$
$x_1, x_2 \geq 0$
[Answer: $x_1 = 0$, $x_2 = 0$, and $z_{min.} = 27/2$]

(ii) Maximize $z = 2x_1 + 3x_2$
Subject to $-x_1 + 2x_2 \leq 4$
$x_1 + x_2 \leq 6$
$x_1 + 2x_2 \leq 9$
$x_1, x_2 \geq 0$
[Answer: $x_1 = 9/2$, $x_2 = 3/2$, and $z = 27/2$]

(iii) Maximize $z = 3x_1 + 6x_2 + 2x_3$
Subject to $3x_1 + 4x_2 + x_3 \leq 2$
$x_1 + 3x_2 + 2x_3 \leq 1$
$x_1, x_2, x_3 \geq 0$
[Answer: (2/5, 1/5, 0) and $z_{max.} = 12/5$]

(iv) Minimize $z = 3x_1 + 2x_2$
Subject to $2x_1 - x_2 \geq -2$
$x_1 + 2x_2 \leq 8$
$x_1, x_2 \geq 0$
[Answer: (8, 0) and $z_{min.} = 24$]

7. Solve the following LPP by simplex method and hence solve it dual problem.

(iii) Maximize $z = x_1 + x_2 + x_3$
Subject to $2x_1 + x_2 + 2x_3 \leq 3$
$4x_1 + 2x_2 + x_3 \leq 2$
$x_1, x_2, x_3 \geq 0$
[Answer: Primal: (0, 1/3, 4/3) and Max. value = 5/3,
Dual: (-1/3, -1/3) and Min. value = 5/3]

14

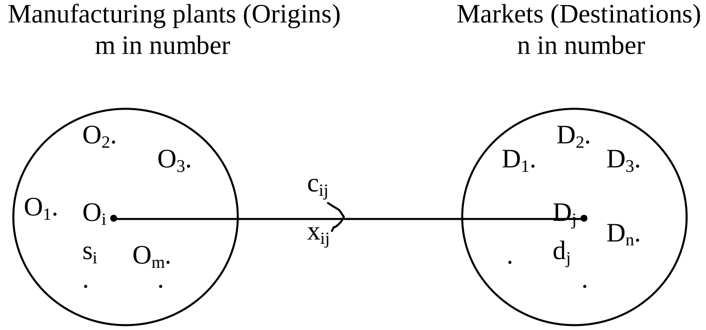
Chapter

Transportation Problem

Highlights:

14.1 Introduction: It gets its name from its application to problems involving transporting products from several origins (factories) to several destinations (markets). The two common objectives of such problems either minimize the cost of shipping n units to m destinations or maximize the profit of shipping n units to m destinations. Transportation problem is an especial type of linear programming problem. Though every linear programming problem can be solved by simplex method, there are more than one solution methods (northwest-corner method, least cost method and Vogel's approximation method) of transportation problem, which are computationally more efficient than the simplex method. Here, we will discuss only one method named northwest-corner rule in the context of some examples.

14.2 Transportation problem: The following features characterize the transportation problem:



1. m origins (or, sources) O_i with a total available quantity (supply) s_i (i =1,2,3, ...,m)
2. n destinations D_j with demand d_j (j = 1, 2, 3, . . ., n) of goods
3. a cost $c_{ij} \geq 0$ for shipping one unit from origin O_i to destination D_j.

The objective is to allocate $x_{ij} \geq 0$ units from origin O_i to destination D_j such that the restriction on the availability on the supply at each origin as well as the constraint on the demand at each destination are met, and at the same time, the total shipping cost is minimized. Then the transportation problem reduces to the following LP problem.

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{Subject to } \sum_{j=1}^{n} x_{ij} \leq s_i \ (i = 1, 2, 3, \ldots, m)$$

$$\sum_{i=1}^{m} x_{ij} \geq d_j \ (j = 1, 2, 3, \ldots, n)$$

$$x_{ij} \geq 0 \ (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$$

14.3 <u>Theorem</u>: The solution of the transportation problem is achieved only for x_{ij}'s ($1 \leq i \leq m$, $1 \leq j \leq n$) satisfying the constraints

$$\sum_{j=1}^{n} x_{ij} \leq s_i \ (i = 1, 2, 3, \ldots, m)$$

$$\sum_{i=1}^{m} x_{ij} = d_j \ (j = 1, 2, 3, \ldots, n)$$

$$x_{ij} \geq 0 \ (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$$

Proof: Let $x^* = \left(x_{ij}^*\right) \in R^{mn}$ be a solution to the transportation problem with

$$\sum x_{ij}^* = b_{j_0} > d_{j_0} \text{ for some } j_0 \in \{1, 2, 3, \ldots, \text{m}\}$$

Then there exists an i_0 such that $x_{i_0 j_0}^* > 0$.

$$\text{Let } y_{ij} = \begin{cases} x_{ij}^* \ \text{if} \ i \neq i_0,\ j \neq j_0 \\ x_{ij}^* - \varepsilon \ \text{if} \ i = i_0,\ j = j_0 \end{cases}$$

Choose $\varepsilon > 0$ sufficiently small such that

$$\sum_{i=1}^{m} y_{ij} \geq d_j$$

$$\sum_{j=1}^{n} y_{ij} \leq s_i$$

$y_{ij} \geq 0$ for all i, j

So, we get the value of the objective function, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} y_{ij} < \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}^{*}$$

it contradict the fact that x^* minimizes the total cost. Hence, complete the proof of the lemma.

Note: This lemma state that the transportation problem has solution only if the demand d_j at each destination is met exactly.

Note: Then the transportation problem reduces to the following LP problem:

Minimize Z = $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to $\sum_{j=1}^{n} x_{ij} \le s_i \;\; (i = 1, 2, 3, \ldots, m)$

$\sum_{i=1}^{m} x_{ij} = d_j \;\; (j = 1, 2, 3, \ldots, n)$

$x_{ij} \ge 0 \; (i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n)$

From the constraint conditions, we get

$$\sum_{i=1}^{m} s_i \ge \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} d_j$$

Thus, if $\sum_{j=1}^{n} d_j > \sum_{i=1}^{m} s_i$, the problem has no feasible solution, otherwise the problem has a feasible solution.

If, however, $\sum_{i=1}^{m} s_i > \sum_{j=1}^{n} d_j$, we may create a fictitious (dummy) (n +1)-st destination D_{n+1} with demand d_{n+1} and the cost $c_{i,n+1}$ given by

$d_{n+1} = \sum_{i=1}^{m} s_i - \sum_{j=1}^{n} d_j$, $c_{i,n+1} = 0$ for i = 1, 2, 3, . . ., m

That is, the demand d_{n+1} at the dummy destination D_{n+1} is $\sum_{i=1}^{m} s_i - \sum_{j=1}^{n} d_j$, and shipping cost from each of the origins to D_{n+1} is 0 (zero).

So, we can consider the following problem as transportation problem:

Minimize Z = $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$

(Primal) Subject to $\sum_{j=1}^{n} x_{ij} = s_i \;\; (i = 1, 2, 3, \ldots, m)$ - - - (i)

$\sum_{i=1}^{m} x_{ij} = d_j \;\; (j = 1, 2, 3, \ldots, n)$

$$x_{ij} \geq 0 \; (i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n)$$

We can write the problem as follows:

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{Subject to } A\underline{x} = \underline{w}$$

$$\underline{x} \geq \underline{0}$$

where,

$$A = \begin{array}{c} \\ m\left\{ \begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ n\left\{ \begin{array}{c} \\ \\ \\ \\ \end{array}\right. \end{array} \left(\begin{array}{ccccc:ccccc:c:ccccc}
1 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & 1 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & & & & & & & & & & & & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\hdashline
1 & 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & 0 & \cdots & 0 & 1 & \cdots & \cdots & 0 \\
\vdots & & & & & & & & & & & & & & & \vdots \\
0 & 0 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & 1 & \cdots & 0 & 0 & \cdots & \cdots & 1
\end{array}\right) \quad \underline{x} = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \\ x_{21} \\ \vdots \\ x_{2n} \\ x_{m1} \\ \vdots \\ x_{mn} \end{pmatrix} \quad \underline{w} = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \\ d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

$(m + n) \times (mn)$ matrix; $(m+n \times 1)$ matrix; $(mn) \times 1$ matrix

<u>Proposition</u>: The transportation problem (i)

1. has a feasible solution, namely

   $x_{ij} = s_i d_j / p$ (i = 1, 2, . . ., m; j = 1, 2, . . ., n), $p = \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$

2. has a solution of at most m + n –1 positive x_{ij}'s giving a bfs (basic feasible solution); in the non-degenerate case, every bfs has exactly m + n –1 positive x_{ij}'s, otherwise, we have the degenerate case which occurs when some partial sum of s_i's is equal to some partial sum of d_j's.
3. has every bfs integer if all the s_i's and d_j's are non-negative integers (not all of them being zero), and in particular, the problem has integer optimal solution if all s_i's and d_j's are integers.
4. has a finite minimum feasible solution.

Now, the dual problem of problem (i) is

$$\text{Maximize } Z = \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \qquad \text{- - - (ii)}$$

(Dual) Subject to $u_i + v_j \leq c_{ij}$ (i = 1, 2, . . ., m; j = 1, 2, . . ., n)

u_i, v_j are unrestricted in sign for all i, j.

We can now display the transportation problem tableau as follows:

O_i \ D_j	1	2	. . .	n	Supply	u_i
1	c_{11} x_{11}	c_{12} x_{12}	: .	c_{1n} x_{1n}	s_1	u_1
2	c_{21} x_{21}	c_{22} x_{22}	: .	c_{2n} x_{2n}	s_2	u_2
: .	: .	: .	: .	: .	: .	: .
m	c_{m1} x_{m1}	c_{m2} x_{m2}	: .	c_{mn} x_{mn}	s_m	u_m
Demand	d_1	d_2	. . .	d_n		
v_j	v_1	v_2	. . .	v_n		

We now recall that, if $\underline{x}^* = (x_{11}^*, ..., x_{1n}^*, x_{21}^*, ..., x_{2n}^*, x_{m1}^*, ..., x_{mn}^*)$ is an optimal solution of primal problem and if $\underline{w}^* = (u_1^*, u_2^*, \ldots, u_m^*, v_1^*, v_2^*, \ldots, v_n^*)$, then

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}^* = \sum_{i=1}^{m} s_i u_i^* + \sum_{j=1}^{n} d_j v_j^*$$

Also, from the complementary slackness property,

$u_i^* + v_j^* < c_{ij}$ for every optimal non-basic variable x_{ij}^*, - - - (1)

$u_i^* + v_j^* = c_{ij}$ for every optimal basic variable x_{ij}^*, - - - (2)

The equation (2) represent a system of at most m + n –1 equations in m + n variables. Therefore, some variables can be arbitrarily specified and unique values are then obtained from the remaining ones. Since, there is always at least one ‘free’ variable, we fix, by convention u_1 = 0 and then solve the resulting system of equation (2). In the non-degenerate case of the transportation problem, there would always be exactly one free variable and the choice u_1 = 0 would always lead to a unique solution of the system of equation (2).

14.4 <u>Northwest corner rule</u>: This rule is developed to solve the transportation problem by Charnes and Cooper. We discuss the rule follows:

Starting with the original $m \times n$ cost matrix together with the (m +1)-st row giving the demand d_j at each destination D_j (j = 1, 2, . . ., n) and the (n +1)-st column giving the supply s_i at each origin O_i (i = 1, 2, . . ., m), we allocate to the cell (1, 1) as many as possible without violating the constraints on supply s_1 at the origin O_1 and demand d_1 at the destination D_1. Thus

$$x_{11} = \min(s_1, d_1) = s_1 \wedge d_1$$

Three cases may now arise, namely,

(i) $s_1 > d_1$, (ii) $s_1 < d_1$ and (iii) $s_1 = d_1$.

In the first case, the demand (that is, the column) constraint is satisfied and we move to the right to the cell (1, 2) with the remaining $s_1 - d_1$ that must be allocated to the remaining cells of the first row following the same procedure, and we allocate $(s_1 - d_1) \wedge d_2$ to the cell (1, 2), that is $x_{12} = (s_1 - d_1) \wedge d_2$

In the second case, the supply (that is, the row) constraint is met, and we move to the cell (2,1) and allocate $x_{21} = (d_1 - s_1) \wedge s_2$ to this cell.

In the third case, we have degeneracy, which would be treated later.

We continue in this way, allocating as much as possible to the cell (the corresponding variable) under consideration without violating the constraints: the sum of the i-th row allocation cannot exceed s_i, the sum of the j-th column allocation cannot exceed d_j, and no allocation can be negative. Having allocated to the cell (i, j), we move to the right to the cell (i, j+1) or to the cell (i+1, j) below according as some supply or some demand remains; if the i-th row and j-th column constraints are satisfied simultaneously, we have the degenerate case. Because of the constraint that the total supply equals the total demand, it is clear that, when we finally reach the last cell (m, n), the allocation to this cell would simultaneously satisfy the m-th row and n-th column constraints.

Thus, beginning with the northwest-corner cell (1, 1), the northwest corner rule allocates to this cell (the variable x_{11}) $s_1 \wedge d_1$, satisfying either the row constraint or the column constraint, and then proceeds so as to satisfy either the row constraint or the column constraint at each step.

Alternatively, the northwest-corner rule may be viewed as initially allocating to the (1, 1) cell the number $s_1 \wedge d_1$, and then following the same rule to either the resulting $m \times (n-1)$ cost matrix or the resulting $(m-1) \times n$ cost matrix obtained from the original $m \times n$ cost matrix by crossing off the first column if $s_1 < d_1$ or the first row if $s_1 > d_1$, and then proceeding in this way successively.

Thus, in the non-degenerate case, we see that the m + n –1 variables evaluated by the northwest-corner rule constitute an initial bfs for the transportation problem. These m+n–1 variables are basic, the remaining ones are non-basic. It is to note here that, when degeneracy is present, the number of basic variables determined by the northwest-corner rule is less than m + n –1.

After finding the initial bfs by the northwest-corner rule, we now proceed to test whether this is already optimal for the problem or not. This is based on $u_i^* + v_j^* < c_{ij}$ and

$u_i^* + v_j^* = c_{ij}$. Following the convention, we set $u_1 = 0$, and then solve for u_i's and v_j's the system of the system of m + n –1 equations:
$u_i + v_j = c_{ij}$ for each basic variable x_{ij}.
Having found the u_i's and v_j's from the above system, we then calculate
$c_{ij} - u_i - v_j$ for each non-basic variable x_{ij}.
If $c_{ij} - u_i - v_j \geq 0$ for each non-basic variable x_{ij}, then the initial bfs is optimal. Otherwise, the initial bfs is not optimal and we have to improve the solution by entering some non-basic variable in the basis, and thereby making one of the current basic variables non-basic.
Now, to improve upon the initial bfs, we choose the non-basic cell (variable) with the most negative of the quantities $c_{ij} - u_i - v_j$, that is, we choose
$c_{i_0 j_0} - u_{i_0} - v_{j_0} = \min\{c_{ij} - u_i - v_j : c_{ij} - u_i - v_j < 0\}$,
and in the next iteration, the variable $x_{i_0 j_0}$ is made the basic variable in place of a current basic variable. To do so, we construct a loop consisting exclusively of the (i_0, j_0) cell and other (current) basic cells. We then allocate to the (i_0, j_0) cell as much as possible such that, after making appropriate adjustments to the other cells of the loop, the supply and demand constraints are not violated, all allocations remain non-negative, and one of the current basic variables has been reduced to 0, which, henceforth, ceases to be basic.
To see how much is to be allocated to the cell (i_0, j_0), let us first allocate an amount $\theta > 0$ to it. Let the other cells of the loop be (i_0, t), (u, t), (u, v), . . ., (z, j_0). Then, in order to preserve the supply constraint, we have to reduce the allocation to the cell (i_0, t) by the same amount. But then, we have to increase the allocation to the cell (u, t) by θ in order to maintain the corresponding demand constraint, which in turn would require that the allocation to the cell (u, v) be reduced by θ accordingly, and so on, and finally, the allocation to the cell (z, j_0) is to be reduced by θ so as not to violate the j_0-th column (demand) constraint. Therefore, if we denote by + and – respectively the situations when the allocation is to be increased and reduced by θ, then we have

cells:	(i_0, j_0)	(i_0, t)	(u, t)	(u, v)	. . .	(z, j_0)
signs:	+	–	+	–		–

Therefore, θ is to be so chosen such that the non-negativity constraints are satisfied for all the basic variables and at least one basic variable becomes 0 which then ceases to be basic; if two or more basic variables vanish simultaneously, we make one of them non-basic and keep the rest at zero level(s). The latter are so chosen that the basic cells do not form loops.
In this way, we get a new bfs. If this solution is not optimal, we continue the process of generating new bfs(s) till we get an optimal one.

Thus, the working rule of the transportation algorithm proceeds as follows:

Step 1 (Initialization): Find an initial bfs by the northwest-corner rule. In the non-degenerate case, the northwest-corner rule evaluates exactly m + n –1 positive (basic) variables.

Step 2 (Optimality): For each non-basic cell (variable), calculate the quantities $c_{ij} - u_i - v_j$. If all these quantities are non-negative, the current bfs is optimal; if some are zero, then the given problem possesses multiple solutions, and if all are strictly positive, the problem under consideration has a unique optimal solution. Otherwise (if at least one of the quantities $c_{ij} - u_i - v_j$ is negative) go to step 3.

Step 3 (Improved bfs): Choose the non-basic variable (cell) with the most negative value of $c_{ij} - u_i - v_j$. This variable (cell) would be basic in the next iteration. This is done, by constructing a loop consisting of the above non-basic cell and other basic cells. Then allocate to the non-basic cell as much as possible, making one of the current basic variable (cell) non-basic by reducing it to zero and making subsequent adjustments so as to preserve the supply and demand constraints corresponding to the rows and column involved in the loop.

Return to step 2.

After a finite number of steps, we would reach to an optimal solution to our problem.

14.4.1 Lemma: For the $m \times n$ transportation tableau, the northwest-corner rule evaluates exactly m+n–1 positive variables in the non-degenerate case.

Proof: We prove this lemma by induction method. The result is clearly true for (m, n) = (1,1), (2,1), (1,2). For (m, n) = (2,2), only one of the following two cases can arise, depending on whether

(i) $s_1 < d_1$, (ii) $s_1 > d_1$

In case (i), the allocation to the (1, 1) cell would satisfy the row constraint, and in case (ii), the allocation to the cell (1, 1) would satisfy the column constraint.

s_1		s_1
$d_1 - s_1$	$s_2 - (d_1 - s_1)$	s_2
d_1	d_2	

d_1	$s_1 - d_1$	s_1
	$d_2 - (s_1 - d_1)$	s_2
d_1	d_2	

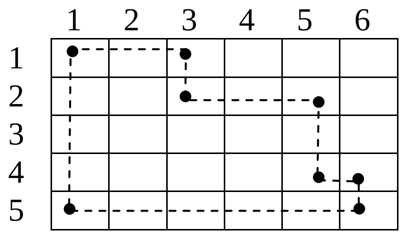
Now, since, $s_1 + s_2 = d_1 + d_2 \Rightarrow d_2 = s_2 - (d_1 - s_1)$, $s_2 = d_2 - (s_1 - d_1)$, we see that, in either case (in the absence of degeneracy), the northwest-corner rule determines three strictly positive variables, so that, the lemma is also true for (m, n) = (2, 2).

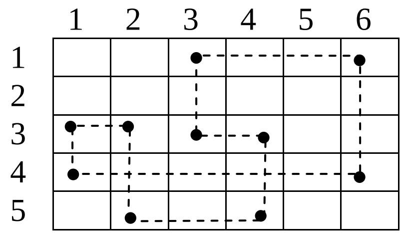
Now, after allocating to the (1,1) cell, the northwest-corner rule applies to either the $m \times (n-1)$ cost matrix or to the $(m-1) \times n$ cost matrix. Making the induction hypothesis that the lemma holds for each of these cost matrices, in either of the above two cases, the northwest-corner rule evaluates (m + n –2) + 1 = m + n –1 strictly positive variables for the original $m \times n$ cost matrix. This completes the proof by induction.

14.5 <u>Loop</u>: A loop is a sequence of cells in the original $m \times n$ transportation tableau for the transportation problem such that

1. each pair of consecutive cells lie in either the same row or the same column,
2. no three (or more) consecutive cells lie in the same row or the same column,
3. the first and last cells of the sequence lie in the same row or the same column,
4. no cell appears more than once in the sequence.

<u>Example of loops</u>: Two valid loops are shown in the figure below, each starting form the cell (1, 3). The loop in the first figure is obtained by the sequence of cells.
(1, 3), (1, 1), (5, 1), (5, 6), (4, 6), (4, 5), (2, 5), (2, 3)
and that in the second figure is formed by the sequence of cells
(1, 3), (1, 6), (4, 6), (4, 1), (3, 1), (3, 2), (5, 2), (5, 4), (3, 4), (3, 3)





We note that, in forming a loop starting form any cell, we need an old number of cells (excluding the initial cell).
From the construction of the initial bfs by the northwest-corner rule, it is clear that no loop can exist connecting the basic cells (for which the allocations are positive).

<u>Example</u>: A production company has four factories in different places to produce its products and three market places, where these products are sold. Transport cost of per unit product from each factory to every market are shown below:

D_j / O_i	1	2	3	Supply s_i
1	2	7	4	5
2	3	3	1	8
3	5	4	7	7
4	1	6	2	14
Demand d_j	7	9	18	

Find the optimal transport system that fulfills the supplies of factories and demands of markets with minimum transport cost. [AUB-2003 MBA]

Solution: To solve the problem by northwest corner rule, we form table (1) as follows: Firstly, in the northwest corner cell (1, 1), we allot minimum of factory-1's supply and market-1's demand, that is, min. $\{s_1 = 5,\ d_1 = 7\} = 5$. This allocation fulfills the first supply constraint but first demand constraint. So, secondly, we allot cell (2, 1) minimum of market-1's remaining demand and factory-2's supply; that is min. {7 – 5, 8} = 2. It fulfills the first demand constraint but second supply constraint. So, we go to cell (2, 2). Last allocation 14 in cell (4, 3) satisfies all constraints.

(1)

O_i \ D_j	1	2	3	Supply s_i	u_i
1	[2] 5	[7] (5)	[4] (-1)	5	0
2	[3] 2	[3] 6	[1] (-5)	8	1
3	[5] (1)	[4] 3	[7] 4	7	2
4	[1] (2)	[6] (7)	[2] 14	14	-3
Demand d_j	7	9	18		
v_j	2	2	5		

We always take $u_1 = 0$ and then calculate other u_i's and v_j's using $u_i + v_j = c_{ij}$ for only basic cells in the above table. As for example $v_1 = c_{11} - u_1 = 2 - 0 = 2$ and $u_2 = c_{21} - v_1 = 3 - 2 = 1$. Then we calculate x_{ij} for non-basic variable using $c_{ij} - u_i - v_j$ and put in parentheses. As for example $(x_{12}) = (c_{12} - u_1 - v_2) = (7 - 0 - 2) = (5)$ and $(x_{13}) = (c_{13} - u_1 - v_3) = (4 - 0 - 5) = (-1)$. There are negative numbers in parentheses in the non-basic cells. So, the above table is not optimal. We search the smallest negative number in non-basic cells to build up a loop for getting the next tableau. Here, – 5 is the smallest number in the non-basic cell (2, 3). So, we form a loop starting from the non-basic cell (2, 3) and goes through the basic cells (2, 2), (3, 2), (3, 3). In the next table, we make cell (2, 3) basic allotting the respective max. {6, 3, 4} = 4 which satisfies the supply and demand constraints. Then we deduce 4 from 6 in cell (2, 2), reduce 3 in cell (3, 2) by 4 and deduce 4 from 4 in cell (3, 3). Now allot of cell (3, 3) becomes 0, that is, this cell is a non-basic cell in second table. And all other basic cells are unchanged.

(2)

D_j / O_i	1	2	3	Supply s_i	u_i
1	[2] 5	[7] (5)	[4] (4)	5	0
2	[3] 2	[3] 2	[1] 4	8	1
3	[5] (1)	[4] 7	[7] (5)	7	2
4	[1] (-3)	[6] (2)	[2] 14	14	2
Demand d_j	7	9	18		
v_j	2	2	0		

In the same way as in table (1) we calculate u_i's, v_j's and xij's for non-basic cells in table (2). Forming a loop through cells (4, 1), (4, 3), (2, 3) and (2, 1) we build up the following table.

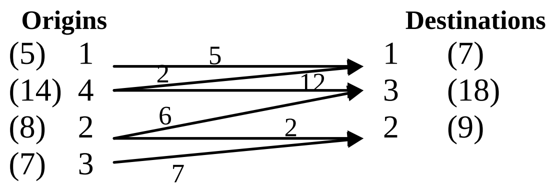
(3)

D_j / O_i	1	2	3	Supply s_i	u_i
1	[2] 5	[7] (2)	[4] (1)	5	0
2	[3] (3)	[3] 2	[1] 6	8	-2
3	[5] (4)	[4] 7	[7] (5)	7	-1
4	[1] 2	[6] (2)	[2] 12	14	-1
Demand d_j	7	9	18		
v_j	2	5	3		

Since, in the non-basic cells all are non-negative in the parentheses, hence table (3) is optimal table. From table (3) we find the following optimal solution:

$$\underline{x}^* = \begin{pmatrix} x_{11}^* & x_{12}^* & x_{13}^* \\ x_{21}^* & x_{22}^* & x_{23}^* \\ x_{31}^* & x_{32}^* & x_{33}^* \\ x_{41}^* & x_{42}^* & x_{43}^* \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & 7 & 0 \\ 2 & 0 & 12 \end{pmatrix}$$

Or,



with the required minimum cost

$$Z^* = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}^* = 2 \times 5 + 3\times 2 + 1\times 6 + 4 \times 7 + 1 \times 2 + 2 \times 12 = 76$$

From the optimal table, we also get the solution of the dual problem as follows:

$$\underline{u}^* = (u_1^*, u_2^*, u_3^*, u_4^*) = (0, -2, -1, -1)$$

$$\underline{v}^* = (v_1^*, v_2^*, v_3^*) = (2, 5, 3)$$

so that,

$$Z^* = \sum_{i=1}^{m} s_i u_i^* + \sum_{j=1}^{n} d_j v_j^* = 0 - 16 - 7 -14 + 14 + 45 + 54 = 76 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}^*$$

14.6 <u>Degeneracy case</u>: If from the beginning some sequential partial some of the supplies s_i's equals some sequential partial sum of the demand d_j's, then we have degeneracy in the initial bfs. In this case, the northwest-corner rule the basic variables whose number is less than m+n-1. Even if we start with the non-degenerate initial bfs, we may have degeneracy in subsequent iterations. This last situation can arise when in the process of making a non-basic variable, at least two basic variables in the loop reduce to 0 (zero) simultaneously (so that, at least two current basic variables have values exactly equal to that by which the non-basic variable in the loop increases).

These degenerate cases are dealt with by the ε-perturbation method, due to Dantzig. In the ε-perturbation method, the values of s_i's and d_j's are perturbed so that the modified problem reads as

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{Subject to } \sum_{j=1}^{n} x_{ij} = s_i + \varepsilon \quad (i = 1, 2, 3, \ldots, m) \qquad \text{- - - (i)}$$

$$\sum_{i=1}^{m} x_{ij} = d_j \ \ (j = 1, 2, 3, \ldots, n)$$

$$\sum_{i=1}^{m} x_{in} = d_n + m\,\varepsilon$$

$$x_{ij} \geq 0 \ (i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n)$$

Where ε > 0 is a small quantity and after the final solution is obtained, we set ε = 0. Thus, the modified problem is obtained from the original one by increasing each supply s_i by the small quantity ε > 0, keeping the first n-1 demands d_1, d_2, . . ., d_{n-1} unchanged and changing the n-th demand d_n to d_n+mε in order to keep the total supply equal to the total demand. We then proceed in the same way as that for the non-degenerate case, and after reaching the optimal solution to the modified problem, we finally set ε = 0 to get the optimal solution to the original problem.

Another way of dealing with the degenerate case is to allocate values 0 to the additional variables and treating these variables as basic so that the total number of basic variables equals m+n-1, some with positive valves and the rest with values 0 assigned to them. The choice is arbitrary, to a point: basic variables cannot form loops, and in case of tie, preference is given to variables with the lowest associated costs, but it is not necessary. We then proceed in the same way as that for the non-degenerate case.

Example: Solve the transportation problem with the following 3×3 cost matrix, where the supplies s_1, s_2 and s_3 and the demands d_1, d_2 and d_3 are as indicated.

O_i \ D_j	1	2	3	Supply s_i
1	8	7	3	60
2	3	8	9	70
3	11	3	5	80
Demand d_j	50	80	80	

[AUB-2002 MBA]

Solution: Here, since $s_1 + s_2 = d_1 + d_2$, we have the degenerate case, and the northwest-corner rule evaluated the following initial bfs:

(1)

O_i \ D_j	1	2	3	Supply s_i	u_i
1	[8] 50	[7] 10	[3]	60	
2	[3]	[8] 70	[9]	70	
3	[11]	[3]	[5] 80	80	
Demand d_j	50	80	80		
v_j					

To resolve degeneracy, we apply the ε-perturbation method, and the modified tableau is given below, where each supply is increased by ε and only the demand d_3 is increased by 3ε, ε > 0 being a small quantity.

(2)

O_i \ D_j	1	2	3	Supply s_i	u_i
1	[8] 50	[7] 10+ε	[3] (-5)	60+ε	0
2	[3] (-6)	[8] 70-ε	[9] 2ε	70+ε	1
3	[11] (6)	[3] (-1)	[5] 80+ε	80+ε	-3
Demand d_j	50	80	80 +3ε		
v_j	8	7	8		

(3)

O_i \ D_j	1	2	3	Supply s_i	u_i
1	[8] (6)	[7] 60+ε	[3] (-5)	60+ε	0
2	[3] 50	[8] 20-ε	[9] 2ε	70+ε	1
3	[11] (12)	[3] (-1)	[5] 80+ε	80+ε	-3
Demand d_j	50	80	80 +3ε		
v_j	2	7	8		

(4)

O_i \ D_j	1	2	3	Supply s_i	u_i
1	[8] (6)	[7] 60-ε	[3] 2ε	60+ε	0
2	[3] 50	[8] 20+ε	[9] (5)	70+ε	1
3	[11] (7)	[3] (-6)	[5] 80+ε	80+ε	2
Demand d_j	50	80	80 +3ε		
v_j	2	7	3		

(5)

D_j / O_i	1	2	3	Supply s_i	u_i
1	[8] (12)	[7] (6)	[3] 60+ε	60+ε	0
2	[3] 50	[8] 20+ε	[9] (-1)	70+ε	7
3	[11] (13)	[3] 60-ε	[5] 20+2ε	80+ε	2
Demand d_j	50	80	80 +3ε		
v_j	-4	1	3		

(6)

D_j / O_i	1	2	3	Supply s_i	u_i
1	[8] (11)	[7] (6)	[3] 60+ε	60+ε	0
2	[3] 50	[8] (1)	[9] 20+ε	70+ε	6
3	[11] (12)	[3] 80	[5] ε	80+ε	2
Demand d_j	50	80	80 +3ε		
v_j	-3	1	3		

From the last tableau (6), we see that we have reached the optimal solution of the modified problem. Now, setting ε = 0, we get the optimal solution of the original problem, which is given in the 3×3 matrix form as well as schematically below:

$$\underline{x}^* = \begin{pmatrix} 0 & 0 & 60 \\ 50 & 0 & 20 \\ 0 & 80 & 0 \end{pmatrix}$$, with minimum transport cost

$$Z^* = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}^* = 3\times60 + 3\times50 + 9\times20 + 3\times80 = 750$$

If we want to avoid the ε-perturbation method of resolving degeneracy we have to make one more variable basic in initial tableau by assigning the value 0 to a non-basic cell such that this cell does not constitute a loop with other basic cells of the tableau. The method is as follows:

(1)

D_j / O_i	1	2	3	Supply s_i	u_i
1	8 50	7 10	3 (-6)	60	0
2	3 (-6)	8 70	9 (-1)	70	1
3	11 (7)	3 0	5 80	80	-4
Demand d_j	50	80	80		
v_j	8	7	9		

(2)

D_j / O_i	1	2	3	Supply s_i	u_i
1	8 50	7 (6)	3 10	60	0
2	3 (-12)	8 70	9 (-1)	70	7
3	11 (1)	3 10	5 70	80	2
Demand d_j	50	80	80		
v_j	8	1	3		

(3)

D_j / O_i	1	2	3	Supply s_i	u_i
1	8 (12)	7 (6)	3 60	60	0
2	3 50	8 20	9 (-1)	70	7
3	11 (13)	3 60	5 20	80	2
Demand d_j	50	80	80		
v_j	-4	1	3		

(4)

D_j / O_i	1	2	3	Supply s_i	u_i
1	8 (11)	7 (5)	3 60	60	0
2	3 50	8 0	9 20	70	6
3	11 (13)	3 80	5 (1)	80	1
Demand d_j	50	80	80		
v_j	-3	2	3		

Therefore, we get the optimal solution from the final tableau (4) as before.

Example: There are three television production factories and four market places at Dinajpur. Supply of each firm, demand of each market per day and transport cost from each factory to every market are given below:

D_j / O_i	1	2	3	4	Supply s_i
1	10	7	3	6	3
2	1	6	8	3	5
3	7	4	5	3	7
Demand d_j	3	2	6	4	

Find the optimal transport system with minimum transport cost. [AUB-2003 MBA Production Mgt.]

Solution: Here, since $s_1 = 3 = d_1$, the problem is of degeneracy type. Calculating the initial bfs by the northwest-corner rule putting 0 in basic cell (1, 2), we can construct the the following tableau:

(1)

D_j / O_i	1	2	3	4	Supply s_i	u_i
1	10 3	7 0	3 (-6)	6 (-1)	3	0
2	1 (-8)	6 2	8 3	3 (-3)	5	-1
3	7 (1)	4 (1)	5 3	3 4	7	4
Demand d_j	3	2	6	4		
v_j	10	7	9	7		

(2)

D_j / O_i	1	2	3	4	Supply s_i	u_i
1	10 1	7 2	3 (-14)	6 (-9)	3	0
2	1 2	6 (8)	8 3	3 (-3)	5	-9
3	7 (9)	4 (9)	5 3	3 4	7	-12
Demand d_j	3	2	6	4		
v_j	10	7	17	15		

(3)

D_j / O_i	1	2	3	4	Supply s_i	u_i
1	10 (14)	7 2	3 1	6 (5)	3	0
2	1 3	6 (-6)	8 2	3 (-3)	5	5
3	7 (9)	4 (-5)	5 3	3 4	7	2
Demand d_j	3	2	6	4		
v_j	-4	7	3	1		

(4)

D_j / O_i	1	2	3	4	Supply s_i	u_i
1	10 (15)	7 (6)	3 3	6 (5)	3	0
2	1 3	6 2	8 0	3 (-3)	5	5
3	7 (10)	4 (1)	5 3	3 4	7	2
Demand d_j	3	2	6	4		
v_j	-5	1	3	1		

(5)

D_j / O_i	1	2	3	4	Supply s_i	u_i
1	10 (11)	7 (3)	3 3	6 (5)	3	0
2	1 3	6 2	8 (3)	3 0	5	2
3	7 (6)	4 (-2)	5 3	3 4	7	2
Demand d_j	3	2	6	4		
v_j	-1	4	3	1		

(6)

D_j / O_i	1	2	3	4	Supply s_i	u_i
1	10 (11)	7 (5)	3 3	6 (5)	3	0
2	1 3	6 (2)	8 (3)	3 2	5	2
3	7 (6)	4 2	5 3	3 2	7	2
Demand d_j	3	2	6	4		
v_j	-1	2	3	1		

The tableau (6) is optimal, since all $\Delta_{ij} = c_{ij} - u_i - v_j \geq 0$ for all non-basic cells. Therefore, the optimal solution is: Markets: 1 2 3 4

$$\underline{x}^* = \text{Factories: } \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{pmatrix} 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 2 & 3 & 2 \end{pmatrix},$$

$Z^* = 3\times3 + 1\times3 + 3\times2 + 4\times2 + 5\times3 + 3\times2 = 47$ (Answer)

14.7 <u>Multiple solutions</u>: We have already pointed out that, in the final tableau corresponding to the optimal solution of the transportation problem

$$c_{ij} - u_i - v_j = \begin{cases} non-negative \ corresponding \ to \ a \ non-basic \ cell \\ 0 \ corresponding \ to \ a \ basic \ cell \end{cases}$$

If $c_{i_0 j_0} - u_{i_0} - v_{j_0} = 0$ for some non-basic cell (i_0, j_0), then we have multiple solutions to the given problem, and another optimal solution is obtained by making the (i_0, j_0) cell basic in the next iteration by considering a loop consisting exclusively of the current non-basic (i_0, j_0) cell and other basic cells in the tableau.

Example: A production firm of computer monitor has four factories (origins) and six showrooms (destinations) in the city of New York. Supply of each factory, demand of each showroom and transport cost of a monitor from each factory to every showroom are as follows:

O_i \ D_j	1	2	3	4	5	6	Supply s_i
1	9	12	9	6	9	10	5
2	7	3	7	7	5	5	6
3	6	5	9	11	3	11	2
4	6	8	11	2	2	10	9
Demand d_j	4	4	6	2	4	2	

Find the optimal transportation systems that minimize total transport cost. [AUB-2003]

Solution: We fist find the (non-degenerate) initial bfs by the northwest corner rule and then form successively the following tableau:

O_i \ D_j	1	2	3	4	5	6	s_i	u_i
1	[9] 4	[12] 1	[9] (-7)	[6] (-1)	[9] (2)	[10] (-5)	5	0
2	[7] (7)	[3] 3	[7] 3	[7] (9)	[5] (7)	[5] (-1)	6	-9
3	[6] (4)	[5] (0)	[9] 2	[11] (11)	[3] (3)	[11] (3)	2	-7
4	[6] (2)	[8] (1)	[11] 1	[2] 2	[2] 4	[10] 2	9	-5
d_j	4	4	6	2	4	2		
v_j	9	12	16	7	7	15		

O_i \ D_j	1	2	3	4	5	6	s_i	u_i
1	[9] 4	[12] (7)	[9] 1	[6] (6)	[9] (9)	[10] (10)	5	0
2	[7] (0)	[3] 4	[7] 2	[7] (9)	[5] (7)	[5] (-1)	6	-2
3	[6] (-3)	[5] (0)	[9] 2	[11] (11)	[3] (3)	[11] (3)	2	0
4	[6] (-5)	[8] (1)	[11] 1	[2] 2	[2] 4	[10] 2	9	2
d_j	4	4	6	2	4	2		
v_j	9	5	9	0	0	8		

D_j \ O_i	1	2	3	4	5	6	s_i	u_i
1	[9] 3	[12] (7)	[9] 2	[6] (1)	[9] (4)	[10] (-3)	5	0
2	[7] (0)	[3] 4	[7] 2	[7] (4)	[5] (2)	[5] (-6)	6	-2
3	[6] (-3)	[5] (0)	[9] 2	[11] (6)	[3] (-2)	[11] (-2)	2	0
4	[6] 1	[8] (6)	[11] (5)	[2] 2	[2] 4	[10] 2	9	-3
d_j	4	4	6	2	4	2		
v_j	9	5	9	5	5	13		

D_j \ O_i	1	2	3	4	5	6	s_i	u_i
1	[9] 1	[12] (7)	[9] 4	[6] (1)	[9] (4)	[10] (3)	5	0
2	[7] (0)	[3] 4	[7] 0	[7] (4)	[5] (2)	[5] 2	6	-2
3	[6] (-3)	[5] (0)	[9] 2	[11] (6)	[3] (-2)	[11] (4)	2	0
4	[6] 3	[8] (6)	[11] (5)	[2] 2	[2] 4	[10] (6)	9	-3
d_j	4	4	6	2	4	2		
v_j	9	5	9	5	5	7		

D_j \ O_i	1	2	3	4	5	6	s_i	u_i
1	[9] (8)	[12] (7)	[9] 5	[6] (4)	[9] (7)	[10] (3)	5	0
2	[7] (3)	[3] 4	[7] 0	[7] (7)	[5] (5)	[5] 2	6	-2
3	[6] 1	[5] (0)	[9] 1	[11] (9)	[3] (1)	[11] (4)	2	0
4	[6] 3	[8] (3)	[11] (2)	[2] 2	[2] 4	[10] (3)	9	0
d_j	4	4	6	2	4	2		
v_j	6	5	9	2	2	7		

Since in tableau (5) $\Delta_{ij} = c_{ij} - u_i - v_j \geq 0$ in non-basic cell, the table is optimal. Therefore, the optimal solution vector $\underline{x}^*$ and the associated minimum cost Z^* are given by

$$\text{Showrooms: } \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

$$\underline{x}_1^* = \text{Factories: } \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 2 & 4 & 0 \end{pmatrix}$$

$$Z^* = 9\times5 + 3\times4 + 5\times2 + 6\times1 + 9\times1 + 6\times3 + 2\times2 + 2\times4 = 112$$

From tableau (5) above, we observe that the cell (3, 2) is non-basic but $c_{32} - u_3 - v_2 = 0$. This shows that the given transportation problem has multiple optimal solution. One more integral optimal solution is obtained by making the cell (3, 2) basic in the next iteration. Constructing a loop through the non-basic cell (3, 2) shown in tableau (5), we then get the following tableau:

D_j / O_i	1	2	3	4	5	6	s_i	u_i
1	9 (3)	12 (7)	9 5	6 (4)	9 (7)	10 (3)	5	0
2	7 (3)	3 3	7 1	7 (7)	5 (5)	5 2	6	-2
3	6 1	5 1	9 (0)	11 (9)	3 (1)	11 (4)	2	0
4	6 3	8 (3)	11 (2)	2 2	2 4	10 (3)	9	0
d_j	4	4	6	2	4	2		
v_j	6	5	9	2	2	7		

Hence, the other integral optimal solution to the given problem is

$$\text{Showrooms: } \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

$$\underline{x}_1^* = \text{Factories: } \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 2 & 4 & 0 \end{pmatrix}$$

$$Z^* = 9\times5 + 3\times3 + 7\times1 + 5\times2 + 6\times1 + 5\times1 + 6\times3 + 2\times2 + 2\times4 = 112$$

<u>Remark</u>: In this example, if the variables are not restricted to non-negative integers only, then the given problem has infinite number of optimal solutions, namely,

$$\underline{x}^* = \lambda \underline{x}_1^* + (1-\lambda)\underline{x}_2^* = \begin{pmatrix} 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3+\lambda & 1-\lambda & 0 & 0 & 2 \\ 1 & 1-\lambda & \lambda & 0 & 0 & 0 \\ 3 & 0 & 0 & 2 & 4 & 0 \end{pmatrix}, \ \lambda \in [0,1]$$

14.8 <u>When total supply exceeds total demand</u>: In the transportation problem in which total supply exceeds total demand, we have to create a dummy destination to which the cost of shipping is zero from each origin. The initial bfs is found by the northwest-corner rule and the same procedure is applied to the modified problem. In problems with excess supply, it is clear that not all the items can be shipped and some would remain at some origin(s). In the optimal solution of the modified problem, the number(s) of items in the dummy is to be interpreted as those remaining excess at the respective origin(s).

<u>Example</u>: Consider the transportation problem with three origins and four destinations, where the transportation cost from each origin to each of the destinations, the supply at each origin, and the demand at each destination are given in the following tableau.

D_j / O_i	1	2	3	4	Supply s_i
1	15	24	11	12	500
2	25	20	14	16	600
3	12	16	22	13	500
Demand d_j	300	350	350	400	

Find the optimal transport system with minimum cost.

Solution: Here, total supply $s_1 + s_2 + s_3 = 500 + 600 + 500 = 1600$
And total demand $= d_1 + d_2 + d_3 + d_4 = 300+350 + 350 + 400 = 1400$
Total supply = 1600 > 1400 = total demand.
Therefore, we have to introduce a dummy destination to which the cost of transportation from each of the origins is zero. Then finding the initial bfs by the northwest-corner rule, we proceed to form the following tableau successively.

D_j / O_i	1	2	3	4	Dummy	s_i	u_i
1	[15] 300	[24] 200	[11] (-7)	[12] (-8)	[0] (-7)	500	0
2	[25] (14)	[20] 150	[14] 350	[16] 100	[0] (-3)	600	-4
3	[12] (4)	[16] (-1)	[22] (11)	[13] 300	[0] 200	500	-7
d_j	300	350	350	400	200		
v_j	15	24	18	20	7		

D_j \ O_i	1	2	3	4	Dummy	s_i	u_i
1	[15] 300	[24] 100	[11] (-7)	[12] 100	[0] (1)	500	0
2	[25] (14)	[20] 250	[14] 350	[16] (8)	[0] (5)	600	-4
3	[12] (-4)	[16] (-9)	[22] (3)	[13] 300	[0] 200	500	1
d_j	300	350	350	400	200		
v_j	15	24	18	12	-1		

D_j \ O_i	1	2	3	4	Dummy	s_i	u_i
1	[15] 300	[24] (9)	[11] (2)	[12] 200	[0] (1)	500	0
2	[25] (5)	[20] 250	[14] 350	[16] (-1)	[0] (-4)	600	5
3	[12] (-4)	[16] 100	[22] (12)	[13] 200	[0] 200	500	1
d_j	300	350	350	400	200		
v_j							

D_j \ O_i	1	2	3	4	Dummy	s_i	u_i
1	[15] 100	[24] (5)	[11] (-2)	[12] 400	[0] (-3)	500	0
2	[25] (9)	[20] 250	[14] 350	[16] (3)	[0] (-4)	600	1
3	[12] 200	[16] 100	[22] (12)	[13] (4)	[0] 200	500	-3
d_j	300	350	350	400	200		
v_j	15	19	13	12	3		

D_j \ O_i	1	2	3	4	Dummy	s_i	u_i
1	[15] 100	[24] (5)	[11] (-2)	[12] 400	[0] (1)	500	0
2	[25] (9)	[20] 150	[14] 350	[16] (3)	[0] 200	600	1
3	[12] 200	[16] 300	[22] (12)	[13] (4)	[0] (4)	500	-3
d_j	300	350	350	400	200		
v_j	15	19	13	12	-1		

D_j \ O_i	1	2	3	4	Dummy	s_i	u_i
1	15 (2)	24 (7)	11 100	12 400	0 (3)	500	0
2	25 (9)	20 150	14 250	16 (1)	0 200	600	3
3	12 300	16 200	22 (12)	13 (2)	0 (4)	500	-1
d_j	300	350	350	400	200		
v_j	13	17	11	12	-3		

From the final tableau (6), the optimal solution vector and the associated minimum cost for the given transportation problem are

$$\underline{x}^* = \begin{pmatrix} 0 & 0 & 100 & 400 \\ 0 & 150 & 250 & 0 \\ 300 & 200 & 0 & 0 \end{pmatrix}$$

$Z^* = 11\times100 + 12\times400 + 20\times150 + 14\times250 + 12\times300 + 16\times200 = 19200.$

We also note that out of total available supply of 600 at the origin O_2, only 400 units are shipped under the optimal policy of transportation to minimize the total shipping cost, and 200 units remain unshipped at O_2.

14.9 <u>Maximization problem</u>: If the problem is to maximize the objective function, so that the problem is

$$\text{Maximize Z} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} = -\sum_{i=1}^{m}\sum_{j=1}^{n}(-c_{ij})x_{ij} \rightarrow \text{minimize}$$

$$\text{Subject to } \sum_{j=1}^{n} x_{ij} = s_i \ \ (i = 1, 2, 3, \ldots, m) \qquad \text{- - - (i)}$$

$$\sum_{i=1}^{m} x_{ij} = d_j \ \ (j = 1, 2, 3, \ldots, n)$$

$$x_{ij} \geq 0 \ (i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n)$$

then the optimal solution vector may be obtained by following one of the methods given below:

1. The method same as that for minimization problem with the single exception in the optimality criterion: The current bfs is optimal if $\Delta_{ij} = c_{ij} - u_j - v_j \le 0$ for all non-basic cells (i, j); otherwise, the current bfs is improved in next iterations.
2. The transportation tableau starts with the costs $-c_{jj}$, and the initial bfs is determined with the optimality criterion as minimization problem. The current bfs is optimal if $\Delta_{ij} = -c_{ij} - u_i - v_j \ge 0$ for all non-basic cell (i, j); otherwise the non-basic cell (i_0, j_0) with the most negative value of Δ_{ij} is chosen to become basic in the next iteration.

<u>Example</u>: A computer production firm has two factories to produce their products and three markets to sell the products. By selling a computer, the firm earns different amount of profit from different markets. The following tableau shows the daily supply of each factory, daily demand of each market and profit (in thousand taka) per computer at each market.

D_j / O_i	1	2	3	Supply s_i
1	4	4	9	25
2	3	5	8	20
Demand(d_j)	18	16	11	

Find the optimal transport system to maximize the total profit satisfying the supply and demand constraints. [AUB-2001 MBA]

Solution: <u>First Method</u>:

D_j / O_i	1	2	3	Supply s_i	u_i
1	4 18	4 7	9 (2)	25	0
2	3 (-2)	5 9	8 11	20	1
Demand d_j	18	16	11		
v_j	4	4	7		

Here, Δ_{ij} = c_{ij}- u_i- v_j, since, Δ_{13} > 0 and maximum of all Δ_{ij} >0 is 2 in the cell (1, 3).
Hence, we have to make the cell (1,3) as basic cell

D_j / O_i	1	2	3	Supply s_i	u_i
1	4 18	4 (-2)	9 7	25	0
2	3 (0)	5 16	8 4	20	-1
Demand d_j	18	16	11		
v_j	4	6	9		

Since Δ_{ij} = $c_{ij} - u_i - v_j \le 0$ for all non-basic cell (i, j), hence, the table is optimal.

The optimal solution is $\underline{x}^* = \begin{pmatrix} 18 & 0 & 7 \\ 0 & 16 & 4 \end{pmatrix}$,

and the total profit is $Z^* = (4\times 18 + 9\times 7 + 5\times 16 + 8\times 4)$ thousand taka
= 247 thousand taka
= 247000 taka

The problem has multiple solution, since $\Delta_{21} = 0$ for non-basic cell.

<u>Second Method</u>:

D_j / O_i	1	2	3	Supply s_i	u_i
1	[-4] 18	[-4] 7	[-9] (-2)	25	0
2	[-3] (2)	[-5] 9	[-8] 11	20	-1
Demand d_j	18	16	11		
v_j	-4	-4	-7		

Here we use
$-c_{ij} = u_i + v_j$
$\Delta_{ij} = -c_{ij} - u_i - v_j$

D_j / O_i	1	2	3	Supply s_i	u_i
1	[-4] 18	[4] (2)	[-9] 7	25	0
2	[-3] (0)	[-5] 16	[-8] 4	20	1
Demand d_j	18	16	11		
v_j	-4	-6	-9		

Since $\Delta_{ij} = c_{ij} - u_i - v_j \geq 0$ for all non-basic cell (i, j), hence the last tableau is optimal. The optimal solution and the maximum profit are

$\underline{x}^* = \begin{pmatrix} 18 & 0 & 7 \\ 0 & 16 & 4 \end{pmatrix}$,

And $Z^* = [-\{(-4)\times 18 + (-4)\times 7 + (-5)\times 16 + (-8)\times 4\}]$ thousand taka
= 247 thousand taka
= 247000 taka.

14.10 <u>Exercises</u>:

1. Define transportation problem.
2. When does a degeneracy case arise in a transportation problem?
3. Discuss northwest corner rule to solve transportation problem.
4. How can we understand to have multiple solutions of a transportation problem?

5. Solve the following transportation problem:

O_i \ D_j	1	2	3	4	s_i
1	14	9	18	6	11
2	10	11	7	16	13
3	25	20	11	34	19
d_j	6	10	12	15	

[Answer: $\underline{x}^* = \begin{pmatrix} 0 & 0 & 0 & 11 \\ 6 & 3 & 0 & 4 \\ 0 & 7 & 12 & 0 \end{pmatrix}$, $Z^* = 479$]

6. Solve the following transportation problem:

O_i \ D_j	1	2	3	4	s_i
1	10	5	6	7	25
2	8	2	7	6	25
3	9	3	4	8	50
d_j	15	20	30	35	

[Answer: $\underline{x}^* = \begin{pmatrix} 0 & 0 & 0 & 25 \\ 15 & 0 & 0 & 10 \\ 0 & 20 & 30 & 0 \end{pmatrix}$ Or, $\begin{pmatrix} 0 & 0 & 0 & 25 \\ 0 & 15 & 0 & 0 \\ 15 & 5 & 30 & 0 \end{pmatrix}$, $Z^* = 535$]

7. Solve the following transportation problem:

O_i \ D_j	1	2	3	4	s_i
1	14	9	18	6	11
2	10	11	7	16	13
3	25	20	11	34	19
d_j	6	10	12	15	

[Answer: $\underline{x}^* = \begin{pmatrix} 0 & 0 & 0 & 11 \\ 6 & 3 & 0 & 4 \\ 0 & 7 & 12 & 0 \end{pmatrix}$, $Z^* = 479$]

8. There are three rice mills and four wholesales market places where rice is sold at Gazipur. Supply of each rice mill, demand of each market and transport cost of per ton rice from each mill to every market in U.S $ are given in the following table.

O_i \ D_j	1	2	3	4	Supply (in ton)
1	$3	$1	$2	$2	6
2	$5	$2	$5	$6	4
3	$6	$4	$8	$8	8
Demand (in ton)	4	6	4	4	

Find the optimal transport system with minimum total transport cost.

[Answer: $\underline{x}^* = \begin{pmatrix} 0 & 0 & 2 & 4 \\ 0 & 2 & 2 & 0 \\ 4 & 4 & 0 & 0 \end{pmatrix}$ (in ton), $Z^* = \$66$]

9. Solve the following three origins and four destinations transportation problem:

O_i \ D_j	1	2	3	4	s_i
1	19	30	50	10	7
2	70	30	40	60	9
3	40	8	70	20	18
d_j	5	8	7	14	

[Answer: $\underline{x}^* = \begin{pmatrix} 5 & 0 & 0 & 2 \\ 0 & 2 & 7 & 0 \\ 0 & 6 & 0 & 12 \end{pmatrix}$, $Z^* = 743$]

10. Solve the following transportation problem as follows:

O_i \ D_j	1	2	3	Supply s_i
1	2	3	2	10
2	4	1	3	7
Demand d_j	6	2	5	

[Answer: $\underline{x}^* = \begin{pmatrix} 6 & 0 & 4 \\ 0 & 2 & 1 \end{pmatrix}$, $Z^* = 25$]

11. A furniture firm has two factories to produce cots and three markets to sell the cots. By selling a cot, the firm earns different amount of profit from different markets. The following tableau shows the daily supply of each factory, daily demand of each market and profit (in hundred taka) per cot at each market.

O_i \ D_j	1	2	3	Supply s_i
1	2	3	2	5
2	4	1	3	7
Demand d_j	2	6	4	

Find the optimal transport system to maximize the total daily profit satisfying the supply and demand constraints.

[Answer: $\underline{x}^* = \begin{pmatrix} 0 & 5 & 0 \\ 2 & 1 & 4 \end{pmatrix}$, $Z^* = 3600$ taka]

15

Chapter

Assignment Problem

Highlights:

15.1 Introduction: A special case of the transportation model is the assignment problem. This problem is appropriate in a situation, which involves the assignment of resources to tasks with minimum cost or maximum profit (e.g., assign n persons to n different tasks or jobs). Also in production and human resource management, it is very necessary to study assignment problem.

15.2 Assignment problem: There are n jobs, which must be performed, and n persons available with c_{ij} being the cost (for example, the training cost) if the i-th person (i = 1, 2, 3, . . ., n) is assigned to the j-th job (j = 1, 2, 3, . . ., n). The problem is to assign the n persons to n jobs on a one-one basis (so that, for example, if the i-th person is assigned to the j-th job then it is unavailable for each of the remaining persons still left unassigned) such that the total cost is minimized. Let

$$x_{ij} = \begin{cases} 1, \ if \ \ person \ i \ is \ assigned \ to \ \ job \ \ j \\ 0, \ otherwise \end{cases}$$

Then the linear programming (LP) formulation of the assignment problem is

Minimize: $Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$

Subject to $\sum_{j=1}^{n} x_{ij} = 1 \ (i = 1, 2, 3, \ldots, n)$

$$\sum_{i=1}^{n} x_{ij} = 1 \ (j = 1, 2, 3, \ldots, n)$$

$x_{ij} \geq 0$ for i, j = 1, 2, 3, . . ., n

From this formulation, we see that the assignment problem is a particular case of the transportation problem.

15.3 <u>Algorithm of the Hungarian method</u>: The Hungarian method works solely on the cost matrix C = (c_{ij}). The method would be justified later.

Start with the cost matrix

$$C = (c_{ij}) = \begin{pmatrix} c_{11} & c_{12} & c_{13} & . & . & . & c_{1n} \\ c_{21} & c_{22} & c_{23} & . & . & . & c_{2n} \\ c_{31} & c_{32} & c_{33} & . & . & . & c_{3n} \\ \cdots & & \cdots & & \cdots & & \\ c_{n1} & c_{n2} & c_{n3} & . & . & . & c_{nn} \end{pmatrix}$$

<u>Step-1</u>: (Initialization): Make the transformations $c_{ij}^{/} = c_{ij} - \min_{j}\{c_{ij}\}$, i = 1, 2, 3, . . ., n; $c_{ij}^{//} = c_{ij}^{/} - \min_{i}\{c_{ij}^{/}\}$, $j = 1, 2, 3, \ldots, n$. That is, for each row i, find the smallest element and subtract it from each element in that row, thereby getting the new cost matrix $C^{/} = (c_{ij}^{/})$. Next, for each column j of the matrix $C^{/} = (c_{ij}^{/})$, find the smallest element and then subtract it from each element of that column to get the modified cost matrix $C^{//} = (c_{ij}^{//})$. The cost matrix $C^{//}$ will have at least one zero in each row and each column.

GO TO step-2.

<u>Step-2</u>: (Optimality Criterion): In any modified cost matrix, determine whether there exists a feasible assignment involving only zero costs, that is, determine whether the modified cost matrix has n zero entries no two of which lie in the same column. If such an assignment exists, it is optimal for the given problem. These zeros are boxed in the final table.

<u>Step-3</u>: (Iterative Step): Cover all zeros in the modified cost matrix with as few horizontal and vertical lines as possible, each horizontal line covering the zeros of a row and each vertical line covering the zeros of a column. The total number of lines in this minimal covering would, of course, be less than n (this number smallest number in the cost matrix not covered by a line. Subtract this number from every element (of the cost matrix) not covered by a line and add it to every element covered by two lines (both horizontal and vertical lines), or equivalently, the minimum cost element is to be subtract from each element of an uncovered row (including the element of that row covered by a vertical line, if any) and then added to every covered column.

RETURN TO step-2.

After a finite number of iterations, the number of minimal covering lines would be exactly equal to n, the order of the cost matrix C, which would give the optimal assignment corresponding to the zero costs of the modified cost matrix such that no two (or more) zeros would be in the same row or column.

Remark-1: If the objective is to maximize the total profit given by some profit matrix C = (c_{ij}) (where c_{ij} is now interpreted as the profit obtained when the i-th person is assigned to the j-th job; i, j = 1, 2, 3, . . ., n) then the above procedure may be applied to the matrix – C = (– c_{ij}) to get the optimal assignment. Alternatively, we may step Step-1 of the above algorithm as follows:

We make the transformations

$$c_{ij}^{/} = \max_{i,j}\{c_{ij}\} - c_{ij},\ c_{ij}^{//} = c_{ij}^{/} - \min_{j}\{c_{ij}^{/}\},\ c_{ij}^{///} = c_{ij}^{//} - \min_{i}\{c_{ij}^{//}\};\ \text{i, j = 1, 2, 3, . . ., n;}$$

that is, we subtract each element of the profit matrix C = (c_{ij}) from the largest element of C. From the resulting matrix $C^{/}$ we get the matrix $C^{///}$ by first subtracting from each row of the matrix $C^{/}$ the smallest element of that row, thereby getting the matrix $C^{//}$, and then subtracting from each column of $C^{//}$ the smallest element of that column. Starting with the modified matrix $C^{///}$, we now follow the procedure of Steps-2 and Step-3 of the original algorithm.

Remark-2: In the case of multiple solutions, there will be more than n number zeros in the optimal cost matrix. This is the necessary condition but not the sufficient condition of multiple solutions.

15.4 <u>The dual of the assignment problem</u>: The LP form of the assignment problem is

Minimize: $Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$

Subject to $\sum_{j=1}^{n} x_{ij} = 1\ (i = 1, 2, 3, \ldots, n)$ (Primal)

$\sum_{i=1}^{n} x_{ij} = 1\ (j = 1, 2, 3, \ldots, n)$

$x_{ij} \geq 0$ for i, j = 1, 2, 3, . . ., n

The dual of the assignment problem is

Maximize: $z = \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j$ (Dual)

Subject to $u_i + v_j \leq c_{ij}$; $i, j = 1, 2, 3, \ldots, n$

u_i, v_j unrestricted in sign for i, j = 1, 2, 3, . . ., n.

15.5 <u>Justification of Hungarian method</u>: If $\underline{x}$ is any feasible solution of the primal problem, then using the equality constraints of the primal problem, we get

$$Z - z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{i=1}^{n} u_i - \sum_{j=1}^{n} v_j$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} - \left(\sum_{i=1}^{n} u_i\right)\left(\sum_{j=1}^{n} x_{ij}\right) - \left(\sum_{j=1}^{n} v_j\right)\left(\sum_{i=1}^{n} x_{ij}\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} (c_{ij} - u_i - v_j)x_{ij}$$

Now, if $(\underline{u}, \underline{v})$ is also feasible, from the inequality constraint of the dual, we get

$C_{ij} \equiv c_{ij} - u_i - v_j \geq 0$ for all i, j = 1, 2, 3, . . ., n.

$$\Longrightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}x_{ij} \geq 0$$

for all feasible $\underline{x}$ (satisfying the non-negativity constraint of the primal problem).

Now, for any feasible $\underline{x}$ and any $(\underline{u}, \underline{v})$,

$$Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} (C_{ij} + u_i + v_j)x_{ij},$$

so that, $\underline{x^*}$ is optimal for the primal problem if and only if it is optimal for the problem

$$\text{Minimize: } \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} (c_{ij} - u_i - v_j)x_{ij}$$

$$\text{Subject to } \sum_{j=1}^{n} x_{ij} = 1 \; (i = 1, 2, 3, \ldots, n) \qquad \text{(A)}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \; (j = 1, 2, 3, \ldots, n)$$

$$x_{ij} \geq 0 \quad \text{for } i, j = 1, 2, 3, \ldots, n$$

because both problems have same constraints and both objective functions are linear functions with positive signs and positive coefficients.

Now, let $\underline{x}$ and $(\underline{u}, \underline{v})$ be such that

$C_{ij}x_{ij} \equiv (c_{ij} - u_i - v_j)x_{ij} = 0$ for i, j = 1, 2, 3, . . ., n.

If $\underline{x}$ and $(\underline{u}, \underline{v})$ are both feasible, then (since Z – z = 0) they are also optimal for the primal problem and the dual problem respectively. Therefore, the optimal (minimum) value of the objective function in (A) is 0 (zero) for the optimal solution $\underline{x^*}$ and $(\underline{u^*}, \underline{v^*})$ of the primal problem and the dual problem respectively with

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}^* = \sum_{i=1}^{n} u_i^* + \sum_{j=1}^{n} v_j^* \qquad \text{(B)}$$

Now, the optimal solution $\underline{x}^*$ of the primal problem has exactly one 1 in each row and each column with zeros elsewhere in the remaining components. There, if we subtract u_i^* from every element of the i-th row (i = 1, 2, 3, . . ., n) and then subtract v_j^* from every element of the j-th column (j = 1, 2, 3, . . ., n), we get and equivalent (with the same optimal solution as the original primal problem) problem for which the optimal value is 0.

The Hungarian method, starting with the given primal problem with the cost matrix $C = (c_{ij})$ proceeds to find an equivalent problem with the cost matrix (C_{ij}) which yields the minimum optimal value 0.

Step-1 of the Hungarian algorithm gives an equivalent problem with the modified cost matrix with at least one zero in each row and each column. In the modified cost matrix, a zero-cost assignment; if feasible, gives the minimum value 0, and hence, it must be optimal. That is, the optimality criterion in Step-2 of the algorithm. Otherwise, the optimal assignment has not yet been found and we go to Step-3, which is a procedure of redistributing the zeros and introducing more zeros. Here, we subtract the minimum uncovered element, say c, from every element of an uncovered row and add c to element of a covered column. Thus, we get the modified matrix with

(1) at least one more zero entry,
(2) old zeros covered by a single (horizontal/vertical) line being retained,
(3) the rest of old zeros being replaced by c.

But these operations are equivalent to subtracting $\frac{c}{2}$ from each uncovered row and each uncovered column and then adding $\frac{c}{2}$ to each covered row and each covered column, thereby giving the modified cost matrix of an equivalent problem.

Here, we mention about a simple test, which would indicate whether the zeros in any cost matrix are well distributed to give the optimal assignment (as in Step-2). We draw the minimum number of horizontal/vertical lines through row(s) /column(s) so as to cover all the zeros of the corresponding cost matrix. If this number is equal to the order of the cost matrix, the optimal solution has been found and is given as follows:

In the final modified cost matrix, we identify the positions of the zeros lying in different rows and columns (so that no such zeros can lie in the same row or in the same column); the optimal $\underline{x}^*$ is then found by making the corresponding components 1 and other components all zeros, or equivalently, we may express the optimal assignment in the permutation symbol introduced in the above examples.

<u>Remark</u>: We have only considered the n-person n-job problem. The more general m-person n-job may be treated as follows:

(1) if m > n and we require that all jobs be performed, we introduce m – n dummy jobs each of which may be done by each person at a cost 0,

(2) if m < n and we require all persons to be assigned some jobs, we introduce n – m dummy persons, each capable of performing every job at zero cost.

We may then apply the Hungarian method to the modified problem.

15.6 Some worked out examples:

Example: Solve the following assignment problem of training cost:

Persons \ Jobs	1	2	3	4	5	6
1	9	22	58	11	19	27
2	43	78	72	50	63	48
3	41	28	91	37	45	33
4	74	42	27	49	39	32
5	36	11	57	22	25	18
6	3	56	53	31	17	28

[AUB-2002 BBA]

Solution: We first identify the smallest element in each row and then subtract from each element of a row the corresponding smallest element. Next, we identify the smallest element in each column of the modified cost matrix and subtract this number from each element of that column. We then have

						Row min.
9	22	58	11	19	27	9
43	78	72	50	63	48	43
41	28	91	37	45	33	28
74	42	27	49	39	32	27
36	11	57	22	25	18	11
3	56	53	31	17	28	3

==>

	0	13	49	2	10	18
	0	35	29	7	20	5
	13	0	63	9	17	5
	47	15	0	22	12	5
	25	0	46	11	14	7
	0	53	50	28	14	25
Column min.	0	0	0	2	10	5

=>

							Subtract:
	0	13	49	0	0	13	
	0	35	29	5	10	0	4
	13	0	63	7	7	0	4
	47	15	0	20	2	0	
	25	0	46	9	4	2	4
	0	53	50	26	4	20	4
Add:	4	4				4	

We thus get the above modified matrix. The next problem is to find a minimal cover, that is, the minimum number of horizontal and vertical lines covering all the zeros of the modified matrix given above. One such minimal cover is shown above. Since the minimum number of horizontal/vertical lines is 5, which is less than the order of the original cost matrix (that is, 6), we have to find the minimum of the uncovered elements of the modified cost matrix. This number is found to be 4. Next, we subtract 4 from all uncovered elements and add 4 to every twice-covered element. These procedures are done below.

$$\Rightarrow \begin{pmatrix} 4 & 17 & 49 & \boxed{0} & 0 & 17 \\ 0 & 35 & 25 & 1 & 6 & \boxed{0} \\ 13 & \boxed{0} & 59 & 3 & 3 & 0 \\ 51 & 19 & \boxed{0} & 20 & 2 & 4 \\ 25 & 0 & 42 & 5 & \boxed{0} & 2 \\ \boxed{0} & 53 & 46 & 22 & 0 & 20 \end{pmatrix}$$

Since the number of minimum zero covered lines of the above matrix is 6, equal to the order of the original matrix, hence this matrix is optimal. From the last matrix above, we get the following two optimal assignments:

$$\begin{matrix} Persons \\ Jobs \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 2 & 3 & 5 & 1 \end{pmatrix} ; \quad \begin{matrix} Persons \\ Jobs \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 6 & 3 & 2 & 5 \end{pmatrix}$$

with the minimum cost

$$Z^* = 11 + 48 + 28 + 27 + 25 + 3 = 142 = 11 + 43 + 33 + 27 + 11 + 17.$$

Example: A production company produced four types of products in four firms. It has recruited four semi skilled managers as need. They have a little knowledge in different products. To assign them to their actual posts, company pre-determined their training cost in taka as follows:

$$\begin{matrix} & & \text{Firms:} & 1 & 2 & 3 & 4 \end{matrix}$$

$$\text{Managers:} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 2000 & 5000 & 4000 & 4000 \\ 0 & 9000 & 3000 & 7000 \\ 1000 & 8000 & 8000 & 9000 \\ 9000 & 4000 & 1000 & 6000 \end{pmatrix}$$

Find their optimal assignment under minimum training cost.[AUB-03MBA(Production mgt.)]

Solution: The given training cost matrix is to be as follows dividing every element of the given matrix by 1000:

$$\begin{pmatrix} 2 & 5 & 4 & 4 \\ 0 & 9 & 3 & 7 \\ 1 & 8 & 8 & 9 \\ 9 & 4 & 1 & 6 \end{pmatrix} \begin{matrix} \text{Row min.} \\ 2 \\ 0 \\ 1 \\ 1 \end{matrix}$$

Subtracting row minimum from every element of that row, we get

$$\Longrightarrow \begin{pmatrix} 0 & 3 & 2 & 2 \\ 0 & 9 & 3 & 7 \\ 0 & 7 & 7 & 8 \\ 8 & 3 & 0 & 5 \end{pmatrix}$$

Column min. 0 3 0 2

Subtracting column minimum from every element of that column, we get

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 6 & 3 & 5 \\ 0 & 4 & 4 & 6 \\ 8 & 0 & 0 & 3 \end{pmatrix}$$

Since, we can cover all zeros by 3 horizontal/vertical lines, which is less than 4, the order of the cost matrix; so, the above matrix is not optimal. Here, minimum uncovered element is 3. Subtracting minimum uncovered element 3 from every uncovered elements and adding to all twice-covered elements, we get

$$\begin{pmatrix} 3 & 0 & 2 & \boxed{0} \\ 0 & 3 & \boxed{0} & 2 \\ \boxed{0} & 1 & 4 & 3 \\ 11 & \boxed{0} & 0 & 3 \end{pmatrix}$$

This is the optimal matrix because there is no way to cover all zeros by less than 4 horizontal/vertical lines. This optimal matrix gives the following optimal assignment:

$$\begin{matrix} Managers: \\ Firms: \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

And the minimum training cost = Tk. (4 + 3 + 1 + 4)1000 = Tk. 12000

<u>Example</u>: Five wagons are available at five stations, which are required at five other stations. The distances from each of the first set of stations to each of the second set are given below in miles:

$$\text{Stations:}\quad \begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 10 & 5 & 9 & 18 & 11 \\ 13 & 19 & 6 & 12 & 14 \\ 3 & 2 & 4 & 4 & 5 \\ 18 & 9 & 12 & 17 & 15 \\ 11 & 6 & 14 & 19 & 10 \end{pmatrix} \end{array}$$

Stations:

How should the wagons be transported so as to minimize the total mileage covered?

Solution: The above problem may be viewed as an assignment problem where each wagon at the first set of five stations is to be assigned to one and only one station of the second set with the objective of minimizing total miles covered by them. Now, subtracting from every element of any row the corresponding row minimum number and then subtracting from every element of any column of the modified matrix the corresponding column minimum number, and finally, finding the minimal cover (by covering all the zeros of the final modified matrix by the minimum number of horizontal/vertical lines each passing through a row/column), we get

$$\begin{pmatrix} 10 & 5 & 9 & 18 & 11 \\ 13 & 19 & 6 & 12 & 14 \\ 3 & 2 & 4 & 4 & 5 \\ 18 & 9 & 12 & 17 & 15 \\ 11 & 6 & 14 & 19 & 10 \end{pmatrix} \begin{array}{c} \text{Row min.} \\ 5 \\ 6 \\ 2 \\ 9 \\ 6 \end{array} \implies \begin{pmatrix} 5 & 0 & 4 & 13 & 6 \\ 7 & 13 & 0 & 6 & 8 \\ 1 & 0 & 2 & 2 & 3 \\ 9 & 0 & 3 & 8 & 6 \\ 5 & 0 & 8 & 13 & 4 \end{pmatrix}$$

$$\text{Column min.}\quad 1 \quad 0 \quad 0 \quad 2 \quad 3$$

$$\implies \begin{pmatrix} 4 & 0 & 4 & 11 & 3 \\ 6 & 13 & 0 & 4 & 5 \\ 0 & 0 & 2 & 0 & 0 \\ 8 & 0 & 3 & 6 & 3 \\ 4 & 0 & 8 & 11 & 1 \end{pmatrix}$$

From the last tableau, we see that the minimum number of horizontal and vertical lines covering all the zero entries is 3, that is, less than 5, the order of the matrix. The minimum uncovered element is 1. In the next iteration, we subtract 1 from every uncovered element and add 1 to every twice-covered element. We then get

$$\begin{pmatrix} 3 & 0 & 4 & 10 & 2 \\ 5 & 13 & 0 & 3 & 4 \\ 0 & 1 & 3 & 0 & 0 \\ 7 & 0 & 3 & 5 & 2 \\ 3 & 0 & 8 & 10 & 0 \end{pmatrix}$$

The minimum uncovered number is now 3, and we subtract 3 from every uncovered element and add 3 to every twice-covered element, getting

$$\begin{pmatrix} \boxed{0} & 0 & 4 & 7 & 2 \\ 2 & 13 & \boxed{0} & 0 & 4 \\ 0 & 4 & 7 & \boxed{0} & 3 \\ 4 & \boxed{0} & 3 & 2 & 2 \\ 0 & 0 & 8 & 7 & \boxed{0} \end{pmatrix}$$

Since the number of minimum zero covered lines of the above matrix is 5, equal to the order of the original matrix, hence this matrix is optimal. From the last matrix above, we get the following optimal solution to the problem:

$$\begin{matrix} \textit{Stations in first set}: \\ \textit{Stations in second set}: \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}$$

That is, the wagons at the 1st, 2nd, 3rd, 4th and 5th stations are to be transported respectively to the 1st, 3rd, 4th, 2nd and 5th stations of the second set so as to minimize the total mileage covered which is

10 + 6 + 4 + 9 + 10 = 39 miles.

<u>Example</u>: Four salesmen are to be assigned to four districts, one person exactly in one district. Estimates of sales revenue in U.S $ for each salesman are as follows:

Salesmen \ Districts:	1	2	3	4
1	320	350	400	280
2	400	250	300	220
3	420	270	340	300
4	25	390	410	350

Find the optimal assignment of each of the four salesmen to one of the four districts that maximizes total sales revenue. [AUB-2002 MBA, 2003 BBA]

Solution: <u>First Method</u>:

We start with the following matrix:

$$\begin{pmatrix} -320 & -350 & -400 & -280 \\ -400 & -250 & -300 & -220 \\ -420 & -270 & -340 & -300 \\ -250 & -390 & -410 & -350 \end{pmatrix}$$

Subtracting from every element of any row the corresponding row minimum number, and then subtracting from every element of any column the corresponding column minimum number, we get successively

$$\begin{pmatrix} 80 & 50 & 0 & 120 \\ 0 & 150 & 100 & 180 \\ 0 & 150 & 80 & 120 \\ 160 & 20 & 0 & 60 \end{pmatrix} \Longrightarrow \begin{pmatrix} 80 & 30 & 0 & 60 \\ 0 & 130 & 100 & 120 \\ 0 & 130 & 80 & 60 \\ 160 & 0 & 0 & 0 \end{pmatrix}$$

In the last matrix, the minimum uncovered element is 60. We now subtract 60 from every uncovered element and add 60 to every twice-covered element, getting

$$\begin{pmatrix} 140 & 30 & \boxed{0} & 60 \\ \boxed{0} & 70 & 40 & 60 \\ 0 & 70 & 20 & \boxed{0} \\ 220 & \boxed{0} & 0 & 0 \end{pmatrix}$$

The last tableau gives the optimal solution:

$$\begin{matrix} Salesmen: \\ Districts: \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

with the maximum total revenue: $(400+400+300+390) = $1490.

<u>Second Method</u>:

Since the largest element in the original 4×4 matrix is 420, subtracting each element of the original matrix from 420, we get the first matrix of the following two, and the second is obtained by subtracting from each row the corresponding row minimum number.

$$\begin{pmatrix} 100 & 70 & 20 & 140 \\ 20 & 170 & 120 & 200 \\ 0 & 150 & 80 & 120 \\ 170 & 30 & 10 & 70 \end{pmatrix} \Longrightarrow \begin{pmatrix} 80 & 50 & 0 & 120 \\ 0 & 150 & 10 & 180 \\ 0 & 150 & 80 & 120 \\ 160 & 20 & 0 & 60 \end{pmatrix}$$

Subtracting each column minimum from corresponding column, we get

$$\begin{pmatrix} 80 & 30 & 0 & 60 \\ 0 & 130 & 100 & 120 \\ 0 & 130 & 80 & 60 \\ 160 & 0 & 0 & 0 \end{pmatrix}$$

In the last matrix, the minimum uncovered element is 60. We now subtract 60 from every uncovered element and add 60 to every twice-covered element, getting

$$\begin{pmatrix} 140 & 30 & \boxed{0} & 60 \\ \boxed{0} & 70 & 40 & 60 \\ 0 & 70 & 20 & \boxed{0} \\ 220 & \boxed{0} & 0 & 0 \end{pmatrix}$$

The last tableau gives the optimal solution:

$$\begin{matrix} \textit{Salesmen}: \\ \textit{Districts}: \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

with the maximum total revenue: \$(400+400+300+390) = \$1490.

<u>Remark</u>: If the objective function of the above example is multiplied by some constant $\alpha > 0$, then the modified problem has the same optimal solution as that example. This observation allows us to start with the cost matrix

$$\begin{pmatrix} 32 & 35 & 40 & 28 \\ 40 & 25 & 30 & 22 \\ 42 & 27 & 34 & 30 \\ 25 & 39 & 41 & 35 \end{pmatrix}$$ with smaller entries in above example.

<u>Example</u>: A batch of four jobs can be assigned to five different machines. The set-up time of each job on each machine is given in the following table

$$\begin{array}{ll} & \text{Machines: } \quad 1 \quad\; 2 \quad\; 3 \quad\; 4 \quad\; 5 \\ \text{Jobs: } \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \end{pmatrix} \end{array}$$

Find an optimal assignment of jobs to machines, which will minimize the total set-up time.
Solution: Here, the number of jobs is 4 while the number of (and the machine to which is dummy is assigned under the optimal policy would remain idle for the problem under consideration). Then the matrix corresponding to the modified problem is

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 12 & 11 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We now subtract from each row the corresponding row minimum and then we subtract from each element of any column the corresponding column-minimum (which is 0 for each column for the above matrix). Next, we cover all the zeros of the resulting matrix by the minimum number of horizontal/vertical lines, as shown in the second matrix below. Since we can cover all the zeros by 3 such lines, we now subtract the minimum of the uncovered numbers, that is, 1, from every element in every uncovered row (including those elements of that row covered by a vertical line only) and add 1 to every covered column (or, equivalently, we subtract 1 from every uncovered element adding 1 to every twice-covered element).

$$\begin{array}{c|c} & \text{Subtract} \\ \begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 2 \\ 7 \\ 5 \\ 7 \\ 0 \end{matrix} \\ \hline & 21 \end{array} \quad \Longrightarrow \quad \begin{array}{rc|c} & \begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \\ \hline \text{Subtract:} & \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} & 0 \end{array}$$

$$\begin{array}{rc|c} & & \text{Subtract} \\ & \begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ \end{matrix} \\ \hline \text{Add:} & \begin{matrix} 1 & & & 1 & 1 \end{matrix} & 1 \end{array}$$

Finally, we get

$$\begin{pmatrix} 8 & 8 & 1 & \boxed{0} & 6 \\ \boxed{0} & 3 & 2 & 7 & 5 \\ 0 & \boxed{0} & 3 & 7 & 9 \\ 6 & 7 & 3 & 3 & \boxed{0} \\ 1 & 0 & \boxed{0} & 1 & 1 \end{pmatrix}$$

From the above matrix, we see that the machine 3 (to which is assigned the dummy job) would remain idle, and the optimal solution is

$$\underline{x}^* = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ or, symbolically } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 5 \end{pmatrix} \begin{matrix} : Jobs \\ : Machines \end{matrix}$$

with the minimum set-up time: 2 + 7 + 6 + 7 = 22.

We note that

$$\sum_{i=1}^{5} u_i^* + \sum_{j=1}^{5} v_j^* = 21 + 1 = 22 = \text{total minimum set-up cost,}$$

(More precisely, $u_1^* = 2 + 1 = 3$, $u_2^* = 7 + 1 = 8$, $u_3^* = 5 + 1 = 6$, $u_4^* = 7 + 1 = 8$, $u_5^* = 0$,
$v_1^* = 0 - 1 = -1$, $v_2^* = 0$, $v_3^* = 0$, $v_4^* = 0 - 1 = -1$, $v_5^* = 0 - 1 = 1$

$$\therefore \sum_{i=1}^{5} u_i^* + \sum_{j=1}^{5} v_j^* = 3 + 8 + 6 + 8 + 0 - 1 + 0 + 0 - 1 - 1 = 22$$

which satisfies the result of (B) in the previous theorem).

15.7 <u>Exercise</u>:

1. Define assignment problem.
2. Discuss Hungarian algorithm to solve assignment problem.
3. Is Hungarian algorithm accurate to solve assignment problem? And discuss your answer.
   Or, discuss the justification of Hungarian algorithm.
4. Solve the following assignment problems with cost matrices by the Hungarian method:

$$\text{(i)} \begin{pmatrix} 2 & 5 & 4 & 4 \\ 0 & 9 & 3 & 7 \\ 1 & 8 & 8 & 9 \\ 9 & 4 & 1 & 6 \end{pmatrix} \quad \text{(ii)} \begin{pmatrix} 3 & 2 & 4 & 2 \\ 5 & 2 & 4 & 8 \\ 6 & 1 & 3 & 5 \\ 1 & 2 & 8 & 9 \end{pmatrix} \quad \text{(iii)} \begin{pmatrix} 14 & 13 & 17 & 14 \\ 16 & 15 & 16 & 15 \\ 18 & 14 & 20 & 17 \\ 20 & 13 & 15 & 18 \end{pmatrix}$$

(iv) $\begin{pmatrix} 1 & 5 & 5 & 2 & 3 \\ 1 & 5 & 3 & 1 & 4 \\ 4 & 1 & 0 & 2 & 0 \\ 1 & 4 & 5 & 3 & 3 \\ 2 & 5 & 4 & 6 & 6 \end{pmatrix}$ (v) $\begin{pmatrix} 2 & 2 & 5 & 7 & 4 & 2 \\ 1 & 2 & 5 & 7 & 3 & 2 \\ 6 & 4 & 4 & 2 & 1 & 3 \\ 3 & 5 & 8 & 7 & 1 & 2 \\ 4 & 5 & 3 & 2 & 6 & 4 \\ 3 & 2 & 2 & 7 & 4 & 3 \end{pmatrix}$

[Answer: (i) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}, Z^* = 12$ (ii) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}, Z^* = 8$ (iii) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}, Z^* = 58$
(iv) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 2 & 3 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ $Z^* = 10$ (v) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 6 & 4 & 3 \end{pmatrix}, Z^* = 10$]

5. The director of data processing for a consulting firm wants to assign four programming tasks to four of her programmers. She has estimated the total number of days each programmer would take if assigned each of the programs as follows:

Programmers \ Tasks:	1	2	3	4
1	80	200	150	170
2	150	160	120	100
3	220	190	160	300
4	250	150	120	90

Determine the optimal assignment of programmers to programming tasks if the objective is to minimize the total number of days to complete all four tasks.

[Answer: $\begin{matrix} \text{Programmers}: \\ \text{Tasks}: \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$, with minimum number of days = 480]

6. A production company has six production firms in six different countries. It has recruited six high level officers from many countries for these firms. To assign them to their actual firms, company pre-determine the total of their adaptability training cost and production system training cost in taka as follows:

Officers \ Firms	1	2	3	4	5	6
1	10000	25000	12000	20000	18000	15000
2	15000	20000	13000	18000	25000	20000
3	25000	15000	10000	19000	30000	21000
4	7000	35000	14000	17000	10000	18000
5	17000	30000	9000	25000	18000	15000
6	9000	10000	20000	14000	20000	10000

Determine the optimal assignment of officers to different firms if the objective is to minimize the total training cost.

[Answer: $\begin{matrix} Officers: \\ Firms: \end{matrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{pmatrix}$ and minimum training cost = Tk. 72,000]

7. A production company has six production firms in six different countries. It has recruited six high level officers from many countries for these firms. To assign them to their actual firms, company pre-determine the daily profit in U.S $ earned by them as follows:

Officers \ Firms	1	2	3	4	5	6
1	10000	25000	12000	20000	18000	15000
2	15000	20000	13000	18000	25000	20000
3	25000	15000	10000	19000	30000	21000
4	7000	35000	14000	17000	10000	18000
5	17000	30000	9000	25000	18000	15000
6	9000	10000	20000	14000	20000	10000

Determine the optimal assignment of officers to different firms if the objective is to maximize the total profit.

[Answer: $\begin{matrix} Officers: \\ Firms: \end{matrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 1 & 2 & 4 & 3 \end{pmatrix}$ and maximum profit = $1,45,000]

8. Introducing the appropriate dummies, solve the following assignment problems of cost matrices.

(i) $\begin{pmatrix} 18 & 24 & 28 & 32 \\ 10 & 15 & 19 & 22 \\ 8 & 13 & 17 & 19 \end{pmatrix}$ (ii) $\begin{pmatrix} 65 & 73 & 63 & 57 \\ 67 & 70 & 65 & 58 \\ 68 & 72 & 69 & 55 \\ 67 & 75 & 70 & 59 \\ 71 & 69 & 75 & 57 \\ 69 & 71 & 66 & 59 \end{pmatrix}$

[Answer: (i) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ Or, $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $Z^* = 50$ (ii) $\begin{pmatrix} 1 & 2 & 3 & 5 \\ 1 & 3 & 4 & 2 \end{pmatrix}$, $Z^* = 254$]

Trigonometry

Highlights:

A1.1 Introduction: Trigonometry is the branch of Mathematics which deals with the measurement of angles. It is the most powerful tool of mathematics. But till now it has no mentionable application in business section. We attached this section in this book for more interested students.

A1.2 Trigonometric ratios: The basic measurement of the trigonometric ratios from a right angled triangle is as follows:

$\sin\theta = \dfrac{perpendicular}{hypotenuse}$, $\quad \text{cosec}\,\theta = \dfrac{hypotenuse}{perpendicular}$

$\cos\theta = \dfrac{base}{hypotenuse}$, $\quad \sec\theta = \dfrac{hypotenuse}{base}$

$\tan\theta = \dfrac{perpendicular}{base}$, $\quad \cot\theta = \dfrac{base}{perpendicular}$

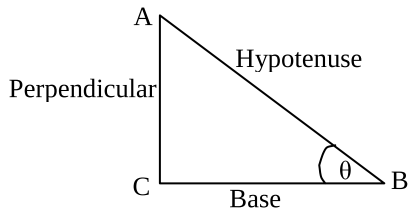


Figure A1.1

From the above measurement of the trigonometric ratios, we get

$\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Note: Since hypotenuse always greater than or equal to perpendicular and base,

$\sin\theta \le 1$, $\quad \cos\theta \le 1$, $\quad \text{cosec}\,\theta \ge 1$ $\quad$ and $\quad \sec\theta \ge 1$.

A1.3 Fundamental relations:

(i) $\sin^2\theta + \cos^2\theta = 1$, (ii) $\sec^2\theta + \tan^2\theta = 1$, (iii) $\text{cosec}^2\theta + \cot^2\theta = 1$

(iv) $\sin(-\theta) = -\sin\theta$, (v) $\cos(-\theta) = \cos\theta$, (vi) $\tan(-\theta) = -\tan\theta$

For multiple angles:

(a) $\sin(A + B) = \sin A\cos B + \cos A\sin B$, (b) $\sin(A - B) = \sin A\cos B - \cos A\sin B$

(c) $\cos(A + B) = \cos A\cos B - \sin A\sin B$, (d) $\cos(A - B) = \cos A\cos B + \sin A\sin B$

(e) $\sin 2\theta = 2\sin\theta.\cos\theta$, (f) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$,

(g) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}.\cos\dfrac{C-D}{2}$, (h) $\sin C - \sin D = 2\cos\dfrac{C+D}{2}.\sin\dfrac{C-D}{2}$

(i) $\cos C + \cos D = 2\cos\frac{C+D}{2}.\cos\frac{C-D}{2}$, (j) $\cos C - \cos D = 2\sin\frac{C+D}{2}.\sin\frac{D-C}{2}$

A1.4 <u>Trigonometric ratios of some standard angles</u>:

Angle	0^0	30^0	45^0	60^0	90^0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
$\text{cosec}\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

<u>Example</u>: If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, prove that $m^2 - n^2 = 4\sqrt{mn}$.

Solution: L.H.S $= m^2 - n^2$

$= (\tan\theta + \sin\theta)^2 - (\tan\theta - \sin\theta)^2$

$= 4\tan\theta.\sin\theta$

$= 4\sqrt{(\tan\theta\sin\theta)^2}$

$= 4\sqrt{\tan^2\theta(1-\cos^2\theta)}$

$= 4\sqrt{\tan^2\theta - \sin^2\theta}$

$= 4\sqrt{(\tan\theta+\sin\theta)(\tan\theta-\sin\theta)}$

$= 4\sqrt{mn}$ = R.H.S (Proved)

<u>Example</u>: Find the value of A: $2\sin^2A - 5\cos A + 1 = 0$; $A < 90^0$.

Solution: Given that $2\sin^2A - 5\cos A + 1 = 0$

Or, $2(1-\cos^2A) - 5\cos A + 1 = 0$

Or, $2 - 2\cos^2A - 5\cos A + 1 = 0$

Or, $2\cos^2A + 5\cos A - 3 = 0$

Or, $2\cos^2A - \cos A + 6\cos A - 3 = 0$

Or, $2\cos A(\cos A - 1) + 3(\cos A - 1) = 0$

Or, $(\cos A - 1)(2\cos A + 3) = 0$

Or, $\cos A - 1 = 0$ Or, $\cos A = 1 = \cos 0^0$, So, $A = 0^0$.

And $2\cos A + 3 = 0$ Or, $\cos A = -\frac{3}{2}$ [Not acceptable]

Therefore, $A = 0^0$. (Answer)

A1.5 <u>Inverse trigonometric ratios</u>: If $\sin\theta = x$, $\theta = \sin^{-1}x = \sin^{-1}(\sin\theta)$. Similarly, $\theta = \cos^{-1}(\cos\theta) = \tan^{-1}(\tan\theta)$. Again if $\theta = \sin^{-1}x$, $x = \sin\theta = \sin(\sin^{-1}x)$. $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$, $\sec^{-1}x$ and $\cot^{-1}x$ are known as inverse trigonometric ratios.

A1.6 Limit: In the chapter of limit and continuity, we have seen the fundamental formulae of limit. To find the limit of trigonometric functions the following formula is very important.

$$\underset{x\to 0}{Lim}\frac{\sin x}{x} = \underset{x\to 0}{Lim}\frac{\tan x}{x} = \underset{x\to 0}{Lim}\ \cos x = 1$$

The proof of this formula is beyond the scope of this book.

Example: Evaluate the limit $\lim_{x\to 0}\frac{\tan x}{2x}$.

Solution: $\lim_{x\to 0}\frac{\tan x}{2x} = \lim_{x\to 0}\frac{1}{2}\left(\frac{\tan x}{x}\right)$

$= \frac{1}{2}\lim_{x\to 0}\left(\frac{\tan x}{x}\right)$ $\quad [\because \lim_{x\to 0} kf(x) = k\lim_{x\to 0} f(x)]$

$= \frac{1}{2}.1$ $\quad [\because \lim_{x\to 0}\frac{\tan x}{x} = 1]$

$= \frac{1}{2}$ (Answer)

Example: Evaluate the limit $\lim_{x\to \pi}\frac{1+\cos x}{(\pi - x)^2}$.

Solution: Let x = π + h. So $h\to 0$ as $x\to\pi$.

$\lim_{x\to\pi}\frac{1+\cos x}{(\pi - x)^2} = \lim_{h\to 0}\frac{1+\cos(\pi+h)}{\{\pi-(\pi+h)\}^2}$

$= \lim_{h\to 0}\frac{1-\cosh}{h^2}$ [We know that, cos(π + x) = – cosx]

$= \lim_{h\to 0}\frac{2\sin^2 h/2}{h^2}$ $\quad [\because 1 - \cos 2x = 2\sin^2 x]$

$= \lim_{h\to 0}\frac{2\sin^2 h/2}{4(h/2)^2}$

$= \frac{1}{2}\lim_{h\to 0}\left(\frac{\sin h/2}{h/2}.\frac{\sin h/2}{h/2}\right)$

$= \frac{1}{2}\lim_{h/2\to 0}\left(\frac{\sin h/2}{h/2}\right)\times\lim_{h/2\to 0}\left(\frac{\sin h/2}{h/2}\right)$ $\quad [\because h/2\to 0 \text{ as } h\to 0]$

$= \frac{1}{2}\times 1\times 1$

$= \frac{1}{2}$ (Answer)

A1.7 <u>Differentiation</u>: We have learnt fundamental theorems on differentiation and various differentiation techniques in chapter 10. Here we shall learn how to differentiate a trigonometric function.

<u>Example</u>: Find the differential coefficient of sin x by first principle.

Solution: Here, f(x) = sin x, f(x + h) = sin(x + h)

So, $\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$

$= \lim_{h\to 0}\frac{2\sin\frac{x+h-x}{2}.\cos\frac{x+h+x}{2}}{h}$ $[\because \sin C - \sin D = 2\sin\frac{D-C}{2}.\cos\frac{C+D}{2}]$

$= \lim_{h\to 0}\frac{2\sin\frac{h}{2}.\cos(x+\frac{h}{2})}{h}$

$= \lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} \times \lim_{h\to 0}\cos(x+\frac{h}{2})$

$= 1.\cos x$ $[\because \lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} = 1$ *and* $\lim_{h\to 0}\cos(x+\frac{h}{2}) = \cos x]$

= cos x (Answer)

In the similar way we find the following results:

(i) $\frac{d}{dx}$(sin x) = cos x

(ii) $\frac{d}{dx}$(cos x) = - sin x

(iii) $\frac{d}{dx}$(tan x) = $\sec^2$ x

(iv) $\frac{d}{dx}$(cot x) = - cosec^2 x

(v) $\frac{d}{dx}$(sec x) = sec x. tan x

(vi) $\frac{d}{dx}$(cosec x) = - cosec x. cot x

(vii) $\frac{d}{dx}$(sin mx) = m cos mx

(viii) $\frac{d}{dx}$(cos mx) = - m sin mx

(ix) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(x) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(xi) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xii) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(xiii) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(xiv) $\frac{d}{dx}$(cosec-1 x) $= -\frac{1}{x\sqrt{x^2-1}}$

The above results are used as formula.

<u>Example</u>: Find the differential coefficient of sin(ax + c).

Solution: Let u = ax + c

$$\therefore \quad \frac{d}{dx}[\sin(ax + c)] = \frac{d}{du}(\sin u).\ \frac{d}{dx}(ax + c)$$
$$= \cos u.\ a$$
$$= a \cos u$$
$$= a \cos(ax + c) \qquad \text{(Answer)}$$

Example: Find the $\frac{dy}{dx}$ of $y = \log(\sin x^2)$ [NU – 01 A/C]

Solution: $\frac{dy}{dx} = \frac{d}{dx}[\log(\sin x^2)]$

$$= \frac{1}{\sin x^2} . \frac{d}{dx}(\sin x^2)$$
$$= \frac{1}{\sin x^2} .\cos x^2 . \frac{d}{dx}(x^2)$$
$$= \frac{\cos x^2}{\sin x^2} .2x$$
$$= 2x \cot x^2 \qquad \text{(Answer)}$$

Example: Differentiate with respect to x: $\frac{x+\sin x}{1+\cos x}$. [NU-00 A/C]

Solution: Let $y = \frac{x+\sin x}{1+\cos x}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x+\sin x}{1+\cos x}\right)$$
$$= \frac{(1+\cos x)\frac{d}{dx}(x+\sin x) - (x+\sin x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$
$$= \frac{(1+\cos x)(1+\cos x) - (x+\sin x)(-\sin x)}{(1+\cos x)^2}$$
$$= \frac{1+2\cos x+\cos^2 x + x\sin x + \sin^2 x}{(1+\cos x)^2}$$
$$= \frac{1+2\cos x + x\sin x + 1}{(1+\cos x)^2} \qquad [\because \sin^2 x + \cos^2 x = 1]$$
$$= \frac{2+2\cos x + x\sin x}{(1+\cos x)^2} \qquad \text{(Answer)}$$

<u>Example</u>: If $y = e^{m\sin^{-1}x}$, then show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$ [RU-90]

Solution: Given that $y = e^{m\sin^{-1}x}$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(e^{m\sin^{-1}x})$

Or, $\frac{dy}{dx} = e^{m\sin^{-1}x}\frac{d}{dx}(m\sin^{-1}x)$

Or, $\frac{dy}{dx} = y.\, m\frac{1}{\sqrt{1-x^2}}$

Or, $\sqrt{1-x^2}\,\frac{dy}{dx} = my$

Or, $(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2y^2$ [Doing square]

Again differentiating with respect to x, we get

$$\frac{d}{dx}[(1-x^2)\left(\frac{dy}{dx}\right)^2] = \frac{d}{dx}(m^2y^2)$$

Or, $(1-x^2)\frac{d}{dx}[\left(\frac{dy}{dx}\right)^2] + \left(\frac{dy}{dx}\right)^2.\frac{d}{dx}(1-x^2) = m^2\frac{d}{dx}(y^2)$

Or, $(1-x^2).2\frac{dy}{dx}.\frac{d}{dx}(\frac{dy}{dx}) + \left(\frac{dy}{dx}\right)^2(-2x) = m^2.2y.\frac{dy}{dx}$

Or, $2(1-x^2)\frac{dy}{dx}.\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 2m^2y\frac{dy}{dx}$

Or, $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = m^2y$ [Dividing by $2\frac{dy}{dx}$]

So, $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$ (Proved)

A1.8 <u>Integration</u>: We know that the integration is the anti-differentiation. Here we shall learn how to integrate a trigonometric function.

<u>Example</u>: We know that, $\frac{d}{dx}(\sin x) = \cos x$

So, $\int\cos x\, dx = \sin x + c$; c is the integral constant.

In the similar way we find the following results:

(i) $\int \sin x\, dx = -\cos x + c$ (ii) $\int \cos x\, dx = \sin x + c$

(iii) $\int \sin mx\, dx = -\frac{1}{m}\cos mx + c$ (iv) $\int \cos mx\, dx = \frac{1}{m}\sin mx + c$

(v) $\int \sec^2 x\, dx = \tan x + c$ (vi) $\int \mathrm{cosec}^2 x\, dx = -\cot x + c$

(vii) $\int \sec x.\tan x\, dx = \sec x + c$ (viii) $\int \mathrm{cosec}\, x.\cot x\, dx = -\mathrm{cosec}\, x + c$

(ix) $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$ (x) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + c$

(xi) $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$ (xii) $\int \tan x\, dx = \log(\sec x) + c$

(xiii) $\int \cot x\, dx = \log(\sin x) + c$

Example: Evaluate $\int \frac{1}{1+\sin x} dx$

Solution: Let $I = \int \frac{1}{1+\sin x} dx$

$$= \int \frac{1-\sin x}{(1+\sin x)(1-\sin x)} dx = \int \frac{1-\sin x}{1-\sin^2 x} dx = \int \frac{1-\sin x}{\cos^2 x} dx$$

$$= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx = \int (\sec^2 x - \sec x.\tan x) dx$$

$$= \int \sec^2 x\, dx - \int \sec x.\tan x\, dx = \tan x - \sec x + c$$ (Answer)

Example: Evaluate $\int \frac{1}{\sqrt{1-x^2}} dx$ [NU-97]

Solution: Let $I = \int \frac{1}{\sqrt{1-x^2}} dx$

$$= \sin^{-1} x + c \qquad [\because \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + c$$

Example: Evaluate $\int_0^{\frac{\pi}{2}} x \sin x\, dx$ [RU-90]

Solution: Here, $\int x \sin x\, dx = x\int \sin x\, dx - \int [\frac{d}{dx}(x).\int \sin x\, dx] dx$

$$= -x\cos x - \int 1.(-\cos x)\, dx$$

$$= -x\cos x + \int \cos x\, dx = -x\cos x + \sin x$$

So, $\int_0^{\frac{\pi}{2}} x\sin x\, dx = \left|\sin x - x\cos x\right|_0^{\frac{\pi}{2}} = (\sin\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{\pi}{2}) - \sin 0 - 0.\cos 0$

$= (1 - \frac{\pi}{2}.0) - 0 = 1$ (Answer)

A1.9 <u>Exercise</u>:

1. Prove the following relations:

   (i) $\frac{\tan\theta}{\sec\theta - 1} + \frac{\tan\theta}{\sec\theta + 1} = 2\cos ec\theta$, (ii) $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \cos ec\theta + \cot\theta$

   (iii) $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

2. Find the value of A when $A < 90^0$:

   (i) $2\sin^2 A = 3\cos A$ [Answer: 60^0]

   (ii) $5\ \text{cosec}^2 A - 7\cot A\ \text{cosec}\ A - 2 = 0$ [Answer: 60^0]

3. Evaluate the following limits:

   (i) $\lim_{x\to 0}\frac{\tan x}{7x}$ (ii) $\lim_{x\to 0}\frac{5\tan x}{x}$ (iii) $\lim_{x\to 0} x\sin x$ (iv) $\lim_{x\to\pi}\frac{\sin x}{\pi - x}$ (v) $\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{(\frac{\pi}{2} - x)}$

   [Answer: (i) $\frac{1}{7}$, (ii) 5, (iii) 0, (iv) 1, (v) $\frac{1}{2}$]

4. Differentiate the following functions:

   (i) $x^4 \tan x$, (ii) $\sin x .\cos x$, (iii) $\sin 6x$, (iv) $\sin ax.\ e^{-bx}$ (v) $x^3 \cos x$,
   (vi) $\log(\sec x + \tan x)$, (vii) $(5x^3 + \sin^2 x)^{1/4}$, (viii) $\sin\{\ln(1 + x^2)\}$
   [Answer: (i) $x^3(4\tan x + x\sec^2 x)$, (ii) $\cos 2x$, (iii) $6\cos 6x$,
   (iv) $e^{-bx}(a\cos ax - b\sin ax)$, (v) $x^2(3\cos x - x\sin x)$, (vi) $\sec x$,

   (vii) $\frac{15x^2 + \sin 2x}{4\sqrt[4]{(5x^2 + \sin^2 x)^3}}$, (viii) $\frac{2x\cos\{\ln(1+x^2)\}}{1+x^2}$]

5. Evaluate the following integrals:

   (i) $\int(\sin 5x + \cos 3x)\,dx$ [Answer: $\frac{1}{3}\sin 3x - \frac{1}{5}\cos 5x + c$]

   (ii) $\int(\sec x.\tan x - 3\ \text{cosec}^2 x)\,dx$ [Answer: $\sec x + 3\cot x + c$]
   (iii) $\int\sec x(\sec x - \tan x)\,dx$ [Answer: $\tan x - \sec x + c$]

6. Evaluate the following integrals:

   (i) $\int_0^{\frac{\pi}{2}}\sin^2 x\,dx$ [Answer: $\frac{\pi}{4}$]

   (ii) $\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin x}dx$ [Answer: 1]

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